

KEY
TO
ACADEMIC ALGEBRA
AND
ADVANCED ALGEBRA

BY
WILLIAM J. MILNE, PH.D., LL.D.
PRESIDENT OF NEW YORK STATE NORMAL COLLEGE, ALBANY, N.Y.



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KEY TO AC. AND ADV. ALGEBRA.

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KEY TO ACADEMIC ALGEBRA

DEFINITIONS AND NOTATION

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When $a = 5$, $b = 3$, $c = 10$, $m = 4$, and $n = 1$:

2. $10 a = 10 \cdot 5 = 50.$
3. $2 ab = 2 \cdot 5 \cdot 3 = 30.$
4. $3 cm = 3 \cdot 10 \cdot 4 = 120.$
5. $6 bc = 6 \cdot 3 \cdot 10 = 180.$
10. $am^4 = 5 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 1280.$
11. $(ab)^2 = (ab)(ab) = (5 \cdot 3)(5 \cdot 3) = 15 \cdot 15 = 225.$
12. $a^2b^2 = aabb = 5 \cdot 5 \cdot 3 \cdot 3 = 225.$
13. $\sqrt{2acn} = \sqrt{2 \cdot 5 \cdot 10 \cdot 1} = \sqrt{10 \cdot 10} = 10.$
14. $3 b^2cn^2 = 3 \cdot 3 \cdot 3 \cdot 10 \cdot 1 \cdot 1 = 270.$
15. $a^2 - b^2 = 5 \cdot 5 - 3 \cdot 3 = 25 - 9 = 16.$
16. $(a - b)^2 = (5 - 3)^2 = 2^2 = 2 \cdot 2 = 4.$
17. $(n + 1)^5 = (1 + 1)^5 = 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.$
18. $n^5 + 1 = 1^5 + 1 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 + 1 = 1 + 1 = 2.$
19. $\sqrt[3]{4 ac^2m} = \sqrt[3]{4 \cdot 5 \cdot 10 \cdot 10 \cdot 4} = \sqrt[3]{2 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 2 \cdot 2} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \times 5 \cdot 5 \cdot 5 \times 2 \cdot 2 \cdot 2} = 2 \times 5 \times 2 = 20.$
20. $\frac{c + 2m}{c - 2m} = \frac{10 + 2 \cdot 4}{10 - 2 \cdot 4} = \frac{10 + 8}{10 - 8} = \frac{18}{2} = 9.$
21. $c + \frac{2m}{c - 2m} = 10 + \frac{2 \cdot 4}{10 - 2 \cdot 4} = 10 + \frac{8}{10 - 8} = 10 + 4 = 14.$
22. $a^b c = 5^3 \cdot 10 = 5 \cdot 5 \cdot 5 \cdot 10 = 1250.$
23. $m^{a-b} = 4^{5-3} = 4^2 = 4 \cdot 4 = 16.$
24. $(bm)^{a-b} = (bm)^{5-3} = (bm)^2 = (bm)(bm) = (3 \cdot 4)(3 \cdot 4) = 12 \cdot 12 = 144.$

SUBTRACTION

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31. $4a + b - \{x + 4a + b - 2y - (x + y)\}$
 $= 4a + b - \{x + 4a + b - 2y - x - y\}$
 $= 4a + b - \{4a + b - 3y\}$
 $= 4a + b - 4a - b + 3y = 3y.$

32. $ab - \{ab + ac - a - (2a - ac) + (2a - 2ac)\}$
 $= ab - \{ab + ac - a - 2a + ac + 2a - 2ac\}$
 $= ab - \{ab - a\}$
 $= ab - ab + a = a.$
33. $a + \{y - \{5 + 4a - (6y + 3)\} - (7y - 4a - 1)\}$
 $= a + \{y - \{5 + 4a - 6y - 3\} - 7y + 4a + 1\}$
 $= a + \{-6y - \{2 + 4a - 6y\} + 4a + 1\}$
 $= a + \{-6y - 2 - 4a + 6y + 4a + 1\}$
 $= a + [-1] = a - 1.$
34. $4m - [p + 3n - (m + n) + 3 - (6p - 3n - 5m)]$
 $= 4m - [p + 3n - m - n + 3 - 6p + 3n + 5m]$
 $= 4m - [-5p + 5n + 4m + 3]$
 $= 4m + 5p - 5n - 4m - 3 = 5p - 5n - 3.$
35. $a + 2b + (14a - 5b) - \{6a + 6b - (5a - 4a - 4b)\}$
 $= a + 2b + 14a - 5b - \{6a + 6b - (5a - 4a + 4b)\}$
 $= 15a - 3b - \{6a + 6b - (a + 4b)\}$
 $= 15a - 3b - \{6a + 6b - a - 4b\}$
 $= 15a - 3b - \{5a + 2b\}$
 $= 15a - 3b - 5a - 2b = 10a - 5b.$
36. $12a - \{[4 - 3b - (6b + 3c)] + b - 8 - (5a - 2b - 6)\}$
 $= 12a - \{[4 - 3b - 6b - 3c] + b - 8 - 5a + 2b + 6\}$
 $= 12a - \{4 - 3b - 6b - 3c + b - 8 - 5a + 2b + 6\}$
 $= 12a - \{2 - 6b - 3c - 5a\}$
 $= 12a - 2 + 6b + 3c + 5a$
 $= 17a + 6b + 3c - 2.$
37. $a + b - \{-[a + b - (c + x)] - [3a - (c - x + a) - b] + 4a\}$
 $= a + b - \{-[a + b - c - x] - [3a - c + x - a - b] + 4a\}$
 $= a + b - \{-a - b + c + x - 2a + c - x + b + 4a\}$
 $= a + b - \{a + 2c\}$
 $= a + b - a - 2c = b - 2c.$
38. $x^3 - [x^2 - (1 - x)] - \{1 + [x^2 - (1 - x) + x^3]\}$
 $= x^3 - x^2 + (1 - x) - 1 - [x^2 - (1 - x) + x^3]$
 $= x^3 - x^2 + 1 - x - 1 - x^2 + (1 - x) - x^3$
 $= -2x^2 - x + (1 - x)$
 $= -2x^2 - x + 1 - x = 1 - 2x - 2x^2.$
39. $4 - \{[5y - (3 - 2x - 2)] - [x + (5y - x + 3)]\}$
 $= 4 - [5y - (3 - 2x - 2)] + [x + (5y - x + 3)]$
 $= 4 - 5y + (3 - 2x - 2) + x + (5y - x + 3)$
 $= 4 - 5y + 3 - 2x - 2 + x + 5y - x + 3$
 $= 7 - 2x - 2 + x - x + 3$
 $= 7 - 2x + 2 + x - x - 3 = 6 - 2x.$
40. $ab - \{5 + x - (b + c - ab + x)\} + [x - (b - c - 7)]$
 $= ab - \{5 + x - b - c + ab - x\} + [x - b + c + 7]$
 $= ab - 5 + b + c - ab + x - b + c + 7 = 2 + 2c + x.$
41. $a^2 - b^2 - \{ad + a^2 - (x + a^2 - b^2) - b^2\} + 5ad - (x + 3ad)$
 $= a^2 - b^2 - \{ad + a^2 - x - a^2 + b^2 - b^2\} + 5ad - x - 3ad$
 $= a^2 - b^2 - ad + x + 2ad - x = a^2 - b^2 + ad.$

$$\begin{aligned}
 42. \quad & a - (b - c) - [a - \{b - c - (b + c - a) + (a + b) + (c - a)\}] \\
 & = a - b + c - [a - \{b - c - b - c + a + a - b + c - a\}] \\
 & = a - b + c - [a + b + c - a] = a - b + c - b - c = a - 2b.
 \end{aligned}$$

$$\begin{aligned}
 43. \quad & -\{3ax - [5xy - 3z] + z - (4xy + [6z + 7ax] + 3z)\} \\
 & = -\{3ax - 5xy + 3z + z - (4xy + 6z + 7ax + 3z)\} \\
 & = -\{3ax - 5xy + 4z - 4xy - 9z - 7ax\} \\
 & = -\{-4ax - 9xy - 5z\} = 4ax + 9xy + 5z.
 \end{aligned}$$

$$\begin{aligned}
 44. \quad & 1 - x - \{1 - x - [1 - x - (1 - x) - (x - 1)] - x + 1\} \\
 & = 1 - x - \{1 - x - [1 - x - 1 + x - x + 1] - x + 1\} \\
 & = 1 - x - \{1 - x - [-x + 1] - x + 1\} \\
 & = 1 - x - \{1 - x + x - 1 - x + 1\} \\
 & = 1 - x - \{-x + 1\} = 1 - x + x - 1 = 0.
 \end{aligned}$$

$$\begin{aligned}
 45. \quad & 1 - x - \{1 - [x - 1 + (x - 1) - (1 - x) - x] + 1 - x\} \\
 & = 1 - x - \{1 - [x - 1 + x - 1 - 1 + x - x] + 1 - x\} \\
 & = 1 - x - \{1 - 2x + 3 + 1 - x\} \\
 & = 1 - x - 5 + 3x = 2x - 4.
 \end{aligned}$$

$$\begin{aligned}
 46. \quad & x - [-\{-(x) + x\} - 2x] \\
 & = x - [-\{+x + x\} - 2x] \\
 & = x - [-2x - 2x] \\
 & = x + 4x = 5x.
 \end{aligned}$$

$$\begin{aligned}
 47. \quad & (a - b) - \{-a - (b - a) + (a - b)\} \\
 & = a - b + a + (b - a) - (a - b) \\
 & = 2a - b + b - a - a + b = b.
 \end{aligned}$$

$$\begin{aligned}
 48. \quad & a - 7 - [-\{-a - (-a - \overline{a - 3})\}] \\
 & = a - 7 - [-\{-a - (-a - a + 3)\}] \\
 & = a - 7 - [-\{-a + 2a - 3\}] \\
 & = a - 7 - [-a + 3] \\
 & = a - 7 + a - 3 = 2a - 10.
 \end{aligned}$$

$$\begin{aligned}
 49. \quad & a - x - [-\{a + (x - a) - (x - 4a)\}] \\
 & = a - x - [-\{a + x - a - x + 4a\}] \\
 & = a - x - [-4a] \\
 & = a - x + 4a = 5a - x.
 \end{aligned}$$

$$\begin{aligned}
 50. \quad & 5xy - [-\{(y^2 - xy) - (xy - \overline{y^2 - 2xy})\}] \\
 & = 5xy - [-\{y^2 - xy - (xy - y^2 + 2xy)\}] \\
 & = 5xy - [-\{y^2 - xy - 3xy + y^2\}] \\
 & = 5xy - [-2y^2 + 4xy] \\
 & = 5xy + 2y^2 - 4xy = xy + 2y^2.
 \end{aligned}$$

$$\begin{aligned}
 51. \quad & 2a - [a - \{b - (\overline{3b - 2a - b})\}] - (b - a) \\
 & = 2a - a + \{b - (\overline{3b - 2a - b})\} - b + a \\
 & = 2a + b - (3b - 2a - b) - b \\
 & = 2a - 3b + 2a - b \\
 & = 2a - 3b + 2a - b = 4a - 4b.
 \end{aligned}$$

$$\begin{aligned}
 52. \quad & a - [-\{(m - a) - \{a - (m - \overline{2m + 6a})\}]\} \\
 & = a - [-m + a - \{a - (m - 2m - 6a)\}] \\
 & = a - [-m + a - \{a - (-m - 6a)\}] \\
 & = a - [-m + a - \{a + m + 6a\}] \\
 & = a - [-m + a - 7a - m] \\
 & = a + 2m + 6a = 7a + 2m.
 \end{aligned}$$

53. $a - \{ -b - (c - d) \} + a - [-b + \{ -2c - (d - e) \}]$
 $= 2a - \{ -b - c + d \} - [-b + \{ -2c - d + e \}]$
 $= 2a + b + c - d - [-b - 2c - d + e]$
 $= 2a + b + c - d + b + 2c + d - e$
 $= 2a + 2b + 3c - e.$
54. $a^2 + 5 - [2ab - \{ - (7 - 3ab) - \overline{ab + 2a^2 - z} \} - (3a - z)]$
 $= a^2 + 5 - [2ab - \{ -7 + 3ab - ab - 2a^2 + z \} - 3a + z]$
 $= a^2 + 5 - [2ab + 7 - 2ab + 2a^2 - z - 3a + z]$
 $= a^2 + 5 - 7 - 2a^2 + 3a = -a^2 + 3a - 2.$
55. $2x + (3y - \{ 2x - [y + 4x - (3y - x)] - 2y \} - \overline{x - y})$
 $= 2x + 3y - \{ 2x - [y + 4x - 3y + x] - 2y \} - x + y$
 $= 2x + 3y - \{ 2x - [-2y + 5x] - 2y \} - x + y$
 $= x + 4y - \{ 2x + 2y - 5x - 2y \}$
 $= x + 4y + 3x = 4x + 4y.$
56. $1 - \{ - \{ - [- (a - \overline{a - 1}) - 3] - 2 \} - a \} - [a - (a - 1)]$
 $= 1 + \{ - \{ - [- (a - \overline{a - 1}) - 3] - 2 \} + a - a + (a - 1) \}$
 $= 1 + \{ - \{ - (a - a + 1) - 3 \} - 2 \} + a - 1$
 $= a + \{ - [+ 2a - 1 - 3] - 2 \}$
 $= a + \{ - 2a + 4 - 2 \} = a - 2a + 2 = 2 - a.$

MULTIPLICATION

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29. Let $2x$ = number of votes A received.
 Then, $x + 500$ = number of votes B received.
 By the 1st condition, $2x + (x + 500) = 8000$;
 $\therefore x = 2500$,
 $2x = 5000$,
 whence, $x + 500 = 3000$.
 and
 Hence, A received 5000 votes and B 3000 votes.
 1st condition: $5000 + 3000 = 8000$.
 2d condition: 3000 is 500 more than $\frac{1}{2}$ of 5000.

30. Let $3x$ = number of dollars A had.
 Then, $x + 10$ = number of dollars B had.
 By the 1st condition, $3x - (x + 10) = 40$.
 $3x - x - 10 = 40$;
 $\therefore x = 25$,
 $3x = 75$,
 and $x + 10 = 35$.
 Hence, A had \$75 and B \$35.
 1st condition: \$75 is \$40 more than \$35.
 2d condition: \$35 is \$10 more than $\frac{1}{3}$ of \$75.

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31. Let x = number of years in Mary's age.
 Then, $3(x + 1)$ = number of years in her mother's age
 By the 1st condition, $3(x + 1) - x = 25$.
 $3x + 3 - x = 25$;
 $\therefore x = 11$,
 and $3(x + 1) = 36$.

Hence, Mary is 11 years old and her mother is 36.

1st condition: 11 years is 25 years less than 36 years.

2d condition: $11 + 1$, or 12 years, is $\frac{1}{3}$ of 36 years.

32. Let x = number of marbles Clarence has.

Then, $3x - 3$ = number of marbles John has.

By the 2d condition, $x + 3x - 3 = 41$;

$$\therefore x = 11,$$

and $3x - 3 = 30$.

Hence, John has 30 marbles and Clarence has 11.

1st condition: $(30 + 3)$ marbles = 3 times 11 marbles.

2d condition: $(11 + 30)$ marbles = 41 marbles.

33. Let $2x$ = number of cents the second had.

Then, $x - 50$ = number of cents the first had.

By the 1st condition, $2x + x - 50 = 820$;

$$\therefore x = 290,$$

whence, $2x = 580$,

and $x - 50 = 240$.

Hence, the first boy had \$2.40 and the second had \$5.80.

1st condition: \$5.80 + \$2.40 = \$8.20.

2d condition: \$2.40 is 50 cents less than $\frac{1}{2}$ of \$5.80.

34. Let x = the number.

Then, $3(2x - 5) = 5x - 9$.

$$6x - 15 = 5x - 9;$$

$$\therefore x = 6, \text{ the number.}$$

Condition $3(2 \cdot 6 - 5) = 5 \cdot 6 - 9$;

that is $21 = 21$.

35. Let $5x$ = number of years in B's age.

Then, $3x$ = number of years in A's age.

Since 8 years ago A's age was $(3x - 8)$ years and B's $(5x - 8)$ years,

by the 2d condition, $2(3x - 8) = 5x - 8$.

$$6x - 16 = 5x - 8;$$

$$\therefore x = 8,$$

whence, $3x = 24$,

and $5x = 40$.

Hence, A is 24 years old and B is 40.

1st condition: 24 years = $\frac{3}{5}$ of 40 years.

2d condition: $(24 - 8)$ years = $\frac{1}{2}$ of $(40 - 8)$ years.

36. Let x = number of years in A's age

Then, $x + 2 = 2(x - 2)$.

$$x + 2 = 2x - 4;$$

$$\therefore x = 6.$$

Hence, A is 6 years old.

Condition: $(6 + 2)$ years = 2 times $(6 - 2)$ years.

37. Let x = number of hours occupied in making the trip.

Then, $x - 3$ = number of hours the first rode,

and $x - 1$ = number of hours the second rode.

Since the first wheelman rode 10 miles an hour, the number of miles from A to B may be expressed by $10(x - 3)$; since the second rode 8 miles an

hour, the distance may be expressed also by $8(x - 1)$. Equating these two expressions, $10(x - 3) = 8(x - 1)$,
or $10x - 30 = 8x - 8$;

$$\therefore x = 11, \text{ the number of hours,}$$

whence, $10(x - 3) = 80$, the number of miles.

1st condition: The first wheelman makes the trip of 80 miles by riding 10 miles an hour for 8 hours and resting 3 hours.

2d condition: The second wheelman makes the trip of 80 miles by riding 8 miles an hour for 10 hours and resting 1 hour.

3d condition: They make the trip in equal times, the first in $(8 + 3)$ hours, the second in $(10 + 1)$ hours.

33. Let $2x =$ number of sheep left in 1st flock,
and $3x =$ number of sheep left in 2d flock.
Therefore, $2x + 100 =$ number of sheep in 1st flock at first,
and $3x + 20 =$ number of sheep in 2d flock at first.

By the 1st condition, $3x + 20 = 2x + 100$;

$$\therefore x = 80,$$

whence, $2x + 100 = 260$, number in 1st flock,

and $3x + 20 = 260$, number in 2d flock.

1st condition: There were 260 sheep in each flock.

2d condition: $260 - 100$, or 160, is to $260 - 20$, or 240, as 2 is to 3.

39. Let $x =$ number of apples at 5 cents each.

Then, $17 - x =$ number of apples at 3 cents each.

By the 1st condition, $5x + 3(17 - x) = 61$,

$$5x + 51 - 3x = 61;$$

$$\therefore x = 5,$$

and $17 - x = 12$.

Hence, Mary bought 5 apples at 5 cents each and 12 apples at 3 cents each

1st condition: 5 apples + 12 apples = 17 apples.

2d condition: 5 times 5 cents + 12 times 3 cents = 61 cents.

40. Let $x =$ number of years in son's age.

Then, $2x =$ number of years in father's age.

By the 2d condition, $3(x - a) = 2x - a$,

$$3x - 3a = 2x - a;$$

$$\therefore x = 2a,$$

and $2x = 4a$.

Hence, George is $2a$ years old and his father $4a$ years old.

1st condition: $2a$ years is $\frac{1}{2}$ of $4a$ years.

2d condition: $(2a - a)$ years is $\frac{1}{3}$ of $(4a - a)$ years.

41. Let $x =$ number of feet in width of rug.

Then, $x + 3 =$ number of feet in length of rug,

$x + 4 =$ number of feet in width of floor,

and $x + 7 =$ number of feet in length of floor.

By the 3d condition,

$$(x + 4)(x + 7) - x(x + 3) = 172.$$

$$x^2 + 11x + 28 - x^2 - 3x = 172;$$

$$\therefore x = 18,$$

whence, $x + 4 = 22$,

and $x + 7 = 25$.

Hence, the floor is 25 feet long and 22 feet wide.

1st condition: The rug is 21 feet long and 18 feet wide.

2d condition: The floor is 25 feet long and 22 feet wide.

3d condition: (25×22) sq. ft. $-(21 \times 18)$ sq. ft. = 172 sq. ft.

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$$23. (-5n - 2b)(-5n + 2b) = (-5n)^2 - (2b)^2 = 25n^2 - 4b^2.$$

$$24. (-x - 2y)(-x + 2y) = (-x)^2 - (2y)^2 = x^2 - 4y^2.$$

$$25. (-5 - 3m)(-5 + 3m) = (-5)^2 - (3m)^2 = 25 - 9m^2.$$

$$\begin{aligned} 30. & (a + x - y)(a - x + y) \\ &= [a + (x - y)][a - (x - y)] \\ &= a^2 - (x - y)^2 \\ &= a^2 - x^2 + 2xy - y^2. \end{aligned}$$

$$\begin{aligned} 36. & (y + c + d)(y + c - d) \\ &= [(y + c) + d][(y + c) - d] \\ &= (y + c)^2 - d^2 \\ &= y^2 + 2cy + c^2 - d^2. \end{aligned}$$

$$\begin{aligned} 31. & (x + c - d)(x - c + d) \\ &= [x + (c - d)][x - (c - d)] \\ &= x^2 - (c - d)^2 \\ &= x^2 - c^2 + 2cd - d^2. \end{aligned}$$

$$\begin{aligned} 37. & (a + x + y)(a + x - y) \\ &= [(a + x) + y][(a + x) - y] \\ &= (a + x)^2 - y^2 \\ &= a^2 + 2ax + x^2 - y^2. \end{aligned}$$

$$\begin{aligned} 32. & (r + p - q)(r - p + q) \\ &= [r + (p - q)][r - (p - q)] \\ &= r^2 - (p - q)^2 \\ &= r^2 - p^2 + 2pq - q^2. \end{aligned}$$

$$\begin{aligned} 38. & (x^2 + 2x + 1)(x^2 + 2x - 1) \\ &= [(x^2 + 2x) + 1][(x^2 + 2x) - 1] \\ &= (x^2 + 2x)^2 - 1^2 \\ &= x^4 + 4x^3 + 4x^2 - 1. \end{aligned}$$

$$\begin{aligned} 33. & (r + p + q)(r - p - q) \\ &= [r + (p + q)][r - (p + q)] \\ &= r^2 - (p + q)^2 \\ &= r^2 - p^2 - 2pq - q^2. \end{aligned}$$

$$\begin{aligned} 39. & (x^2 + 2x - 1)(x^2 - 2x + 1) \\ &= [x^2 + (2x - 1)][x^2 - (2x - 1)] \\ &= (x^2)^2 - (2x - 1)^2 \\ &= x^4 - (4x^2 - 4x + 1) \\ &= x^4 - 4x^2 + 4x - 1. \end{aligned}$$

$$\begin{aligned} 34. & (x + b + n)(x - b - n) \\ &= [x + (b + n)][x - (b + n)] \\ &= x^2 - (b + n)^2 \\ &= x^2 - b^2 - 2bn - n^2. \end{aligned}$$

$$\begin{aligned} 40. & (x^2 + 3x - 2)(x^2 - 3x + 2) \\ &= [x^2 + (3x - 2)][x^2 - (3x - 2)] \\ &= (x^2)^2 - (3x - 2)^2 \\ &= x^4 - (9x^2 - 12x + 4) \\ &= x^4 - 9x^2 + 12x - 4. \end{aligned}$$

$$\begin{aligned} 35. & (y + c + d)(y - c - d) \\ &= [y + (c + d)][y - (c + d)] \\ &= y^2 - (c + d)^2 \\ &= y^2 - c^2 - 2cd - d^2. \end{aligned}$$

$$\begin{aligned} 41. & (x^2 + 3x + 2)(x^2 - 3x + 2) \\ &= [(x^2 + 2) + 3x][(x^2 + 2) - 3x] \\ &= (x^2 + 2)^2 - (3x)^2 \\ &= x^4 + 4x^2 + 4 - 9x^2 \\ &= x^4 - 5x^2 + 4. \end{aligned}$$

$$\begin{aligned} 42. & (m^4 - 2m^2 + 1)(m^4 + 2m^2 + 1) \\ &= [(m^4 + 1) - 2m^2][(m^4 + 1) + 2m^2] \\ &= (m^4 + 1)^2 - (2m^2)^2 \\ &= m^8 + 2m^4 + 1 - 4m^4 \\ &= m^8 - 2m^4 + 1. \end{aligned}$$

$$\begin{aligned} 43. & (2x + 3y - 4z)(2x + 3y + 4z) \\ &= [(2x + 3y) - 4z][(2x + 3y) + 4z] \\ &= (2x + 3y)^2 - (4z)^2 \\ &= 4x^2 + 12xy + 9y^2 - 16z^2. \end{aligned}$$

$$\begin{aligned} 44. & (2x^2 - xy + 3y^2)(2x^2 + xy - 3y^2) \\ &= [2x^2 - (xy - 3y^2)][2x^2 + (xy - 3y^2)] \\ &= (2x^2)^2 - (xy - 3y^2)^2 \\ &= 4x^4 - (x^2y^2 - 6xy^3 + 9y^4) \\ &= 4x^4 - x^2y^2 + 6xy^3 - 9y^4. \end{aligned}$$

$$\begin{aligned} 45. & (x^2 + xy + y^2)(x^2 - xy + y^2) \\ &= [(x^2 + y^2) + xy][(x^2 + y^2) - xy] \\ &= (x^2 + y^2)^2 - (xy)^2 \\ &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= x^4 + x^2y^2 + y^4. \end{aligned}$$

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46. $[(a + b) + (c + d)][(a + b) - (c + d)]$
 $= (a + b)^2 - (c + d)^2$
 $= a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$
47. $(a + b + x + y)(a + b - x - y)$
 $= (a + b)^2 - (x + y)^2$
 $= a^2 + 2ab + b^2 - x^2 - 2xy - y^2.$
48. $(a + b + m - n)(a + b - m + n)$
 $= (a + b)^2 - (m - n)^2$
 $= a^2 + 2ab + b^2 - m^2 + 2mn - n^2.$
49. $(x - m + y - n)(x - m - y + n)$
 $= (x - m)^2 - (y - n)^2$
 $= x^2 - 2mx + m^2 - y^2 + 2ny - n^2.$
50. $(p - q + r + s)(p - q - r - s)$
 $= (p - q)^2 - (r + s)^2$
 $= p^2 - 2pq + q^2 - r^2 - 2rs - s^2.$
51. $(a - m - b - n)(a + m - b + n)$
 $= (a - b - m - n)(a - b + m + n)$
 $= (a - b)^2 - (m + n)^2$
 $= a^2 - 2ab + b^2 - m^2 - 2mn - n^2.$
52. $(a + x + b - y)(a - x + b + y)$
 $= (a + b + x - y)(a + b - x + y)$
 $= (a + b)^2 - (x - y)^2$
 $= a^2 + 2ab + b^2 - x^2 + 2xy - y^2.$
54. $31 \times 29 = (30 + 1)(30 - 1) = 30^2 - 1^2 = 900 - 1 = 899.$
55. $42 \times 38 = (40 + 2)(40 - 2) = 1600 - 4 = 1596.$
56. $69 \times 71 = (70 - 1)(70 + 1) = 4900 - 1 = 4899.$
57. $48 \times 52 = (50 - 2)(50 + 2) = 2500 - 4 = 2496.$
58. $57 \times 63 = (60 - 3)(60 + 3) = 3600 - 9 = 3591.$
59. $95 \times 85 = (90 + 5)(90 - 5) = 8100 - 25 = 8075.$
60. $98 \times 102 = (100 - 2)(100 + 2) = 10000 - 4 = 9996.$
61. $99 \times 101 = (100 - 1)(100 + 1) = 10000 - 1 = 9999.$
62. $95 \times 105 = (100 - 5)(100 + 5) = 10000 - 25 = 9975.$
65. $98^2 = (98 + 2)(98 - 2) + 2^2 = 100 \cdot 96 + 4 = 9604.$
66. $96^2 = (96 + 4)(96 - 4) + 4^2 = 100 \cdot 92 + 16 = 9216.$
67. $93^2 = (93 + 7)(93 - 7) + 7^2 = 100 \cdot 86 + 49 = 8649.$
68. $97^2 = (97 + 3)(97 - 3) + 3^2 = 100 \cdot 94 + 9 = 9409.$
69. $58^2 = (58 + 2)(58 - 2) + 2^2 = 60 \cdot 56 + 4 = 3364.$
70. $49^2 = (49 + 1)(49 - 1) + 1^2 = 50 \cdot 48 + 1 = 2401.$
71. $87^2 = (87 + 13)(87 - 13) + 13^2 = 100 \cdot 74 + 169 = 7569.$
72. $79^2 = (79 + 1)(79 - 1) + 1^2 = 80 \cdot 78 + 1 = 6241.$
73. $68^2 = (68 + 2)(68 - 2) + 2^2 = 70 \cdot 66 + 4 = 4624.$
74. $129^2 = (129 + 9)(129 - 9) + 9^2 = 138 \cdot 120 + 81 = 16641.$

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35. $(\overline{a+b} + 5)(\overline{a+b} + 2) = (a+b)^2 + 7(a+b) + 10$
 $= a^2 + 2ab + b^2 + 7a + 7b + 10.$
36. $(\overline{a-b} - 4)(\overline{a-b} + 10) = (a-b)^2 + 6(a-b) - 40$
 $= a^2 - 2ab + b^2 + 6a - 6b - 40.$
37. $(\overline{x+y} - 1)(\overline{x+y} + 2) = (x+y)^2 + 1(x+y) - 2$
 $= x^2 + 2xy + y^2 + x + y - 2.$
38. $(\overline{x-y} - 2)(\overline{x-y} - 8) = (x-y)^2 - 10(x-y) + 16$
 $= x^2 - 2xy + y^2 - 10x + 10y + 16.$
39. $(\overline{x^2+x} - 1)(\overline{x^2+x} + 3) = (x^2+x)^2 + 2(x^2+x) - 3$
 $= x^4 + 2x^3 + x^2 + 2x^2 + 2x - 3$
 $= x^4 + 2x^3 + 3x^2 + 2x - 3.$
40. $(\overline{2m+n} - 3)(\overline{2m+n} + 4) = (2m+n)^2 + 1(2m+n) - 12$
 $= 4m^2 + 4mn + n^2 + 2m + n - 12.$

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5. Selling price of grain, hay, and potatoes, $(a+b+c)$ dollars.

Amount deducted, $(x+y)$ dollars.

Hence, B owed A $[(a+b+c) - (x+y)]$ dollars.

Since this was to be taken from 100 dollars, the amount due in return was

$$100 - [(a+b+c) - (x+y)], \text{ dollars}$$

$$= 100 - (a+b+c) + (x+y), \text{ dollars.}$$

11. a times b miles, or ab miles.

If he stops c of the a hours to rest, he will ride $(a-c)$ hours, and hence $(a-c)$ times b miles, or $b(a-c)$ miles.

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17. The length of each side of the square part of the field will be that of the shorter side of the field, or b rods; hence, the area of the square part will be b^2 square rods.

Since the area of the square part is b^2 square rods and the area of the whole field is ab square rods, the area of the other part is $(ab - b^2)$ square rods.

22. ab = number of miles he rode the first a hours. After the first $(a+c)$ hours he rode $(b+2)$ miles an hour for $10 - (a+c)$ hours; hence, during this interval, he rode $(10-a-c)(b+2)$ miles, or $(10b - ab - bc + 20 - 2a - 2c)$ miles; therefore, the whole distance, or x miles, is equal to

$$(ab + 10b - ab - bc + 20 - 2a - 2c) \text{ miles}$$

$$= (10b - bc + 20 - 2a - 2c) \text{ miles.}$$

23. am = number of miles he rode the first a hours. $3(m-5)$ = number of miles he rode the next 3 hours. $b(m+2)$ = number of miles he rode the last b hours. Hence he rode in all $(am + 3m - 15 + bm + 2b)$ miles. If $a = 4$, $b = 2$, and $m = 10$, the whole distance is $(4 \cdot 10 + 3 \cdot 10 - 15 + 2 \cdot 10 + 2 \cdot 2)$ miles, or 79 miles; and the whole time occupied in making the journey is $(a + 3 + b)$ hours = $(4 + 3 + 2)$ hours, or 9 hours

DIVISION

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$$\begin{array}{r}
 12. \quad \begin{array}{r} a^2m^4 - 4am^3 + 3m^2 \\ a^2m^4 - am^3 \\ \hline -3am^3 + 3m^2 \\ -3am^3 + 3m^2 \end{array} \quad \left| \begin{array}{r} am - 1 \\ am^3 - 3m^2 \end{array} \right.
 \end{array}$$

TEST.—Let $a = 1$, $m = 2$. Then, the dividend becomes $16 - 32 + 12$, or -4 , the divisor $2 - 1$, or 1 , and the quotient $8 - 12$, or -4 . Since $-4 \div 1 = -4$, it may be assumed that the quotient is correct.

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$$\begin{array}{r}
 17. \quad \begin{array}{r} ax^3 - a^2x^2 - bx^2 + b^2 \\ ax^3 \quad \quad - bx^2 \\ \hline -a^2x^2 \quad \quad + b^2 \\ -a^2x^2 + abx \\ \hline \quad \quad - abx + b^2 \\ \quad \quad - abx + b^2 \end{array} \quad \left| \begin{array}{r} ax - b \\ x^2 - ax - b \end{array} \right.
 \end{array}$$

TEST.—Let $a = 1$, $b = 1$, $x = 5$. Then, the dividend becomes $125 - 25 - 25 + 1$, or 76 , the divisor $5 - 1$, or 4 , and the quotient $25 - 5 - 1$, or 19 . Since $76 \div 4 = 19$, it may be assumed that the quotient is correct.

$$\begin{array}{r}
 22. \quad \begin{array}{r} a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4 \\ a^4 - 2a^3x + a^2x^2 \\ \hline -2a^3x + 5a^2x^2 - 4ax^3 \\ -2a^3x + 4a^2x^2 - 2ax^3 \\ \hline \quad \quad a^2x^2 - 2ax^3 + x^4 \\ \quad \quad a^2x^2 - 2ax^3 + x^4 \end{array} \quad \left| \begin{array}{r} a^2 - 2ax + x^2 \\ a^2 - 2ax + x^2 \end{array} \right.
 \end{array}$$

TEST.—Let $a = 1$, $x = 10$. Then, the dividend becomes $1 - 40 + 600 - 4000 + 10000$, or 6561 , the divisor $1 - 20 + 100$, or 81 , and the quotient also 81 . Since $6561 \div 81 = 81$, it may be assumed that the quotient is correct.

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$$\begin{array}{r}
 26. \quad \begin{array}{r} m^5 + n^5 \\ m^5 + m^4n \\ \hline -m^4n \\ -m^4n - m^3n^2 \\ \hline \quad \quad m^3n^2 \\ \quad \quad m^3n^2 + m^2n^3 \\ \quad \quad \quad - m^2n^3 \\ \quad \quad \quad - m^2n^3 - mn^4 \\ \hline \quad \quad \quad \quad mn^4 + n^5 \\ \quad \quad \quad \quad mn^4 + n^5 \end{array} \quad \left| \begin{array}{r} m + n \\ m^4 - m^3n + m^2n^2 - mn^3 + n^4 \end{array} \right.
 \end{array}$$

27.

$$\begin{array}{r|l}
 x^5 + 32 & x + 2 \\
 x^5 + 2x^4 & x^4 - 2x^3 + 4x^2 - 8x + 16 \\
 \hline
 -2x^4 & \\
 -2x^4 - 4x^3 & \\
 \hline
 4x^3 & \\
 4x^3 + 8x^2 & \\
 \hline
 -8x^2 & \\
 -8x^2 - 16x & \\
 \hline
 16x + 32 & \\
 16x + 32 & \\
 \hline
 \end{array}$$

28.

$$\begin{array}{r|l}
 x^6 + y^6 & x^2 + y^2 \\
 x^6 + x^4y^2 & x^4 - x^2y^2 + y^4 \\
 \hline
 -x^4y^2 & \\
 -x^4y^2 - x^2y^4 & \\
 \hline
 x^2y^4 + y^6 & \\
 x^2y^4 + y^6 & \\
 \hline
 \end{array}$$

29.

$$\begin{array}{r|l}
 a^6 + 5a^5 - a^3 + 2a + 3 & a - 1 \\
 a^6 - a^5 & a^5 + 6a^4 + 6a^3 + 5a^2 + 5a + 7 + \frac{10}{a-1} \\
 \hline
 6a^5 & \\
 6a^5 - 6a^4 & \\
 \hline
 6a^4 - a^3 & \\
 6a^4 - 6a^3 & \\
 \hline
 5a^3 & \\
 5a^3 - 5a^2 & \\
 \hline
 5a^2 + 2a & \\
 5a^2 - 5a & \\
 \hline
 7a + 3 & \\
 7a - 7 & \\
 \hline
 10 &
 \end{array}$$

30.

$$\begin{array}{r|l}
 2n^5 - 4n^4 - 3n^3 + 7n^2 - 3n + 2 & n - 2 \\
 2n^5 - 4n^4 & 2n^4 - 3n^2 + n - 1 \\
 \hline
 -3n^3 + 7n^2 & \\
 -3n^3 + 6n^2 & \\
 \hline
 n^2 - 3n & \\
 n^2 - 2n & \\
 \hline
 -n + 2 & \\
 -n + 2 & \\
 \hline
 \end{array}$$

31.
$$\begin{array}{r|l} x^3 - 3xyz + y^3 + z^3 & x + y + z \\ x^3 + x^2y + x^2z & x^2 - xy - xz + y^2 - yz + z^2 \\ \hline -x^2y - x^2z - 3xyz & \\ -x^2y - xy^2 - xyz & \\ \hline -x^2z + xy^2 - 2xyz & \\ -x^2z & -xyz - xz^2 \\ \hline & xy^2 - xyz + xz^2 + y^3 \\ & xy^2 & + y^2z + y^3 \\ & -xyz + xz^2 - y^2z \\ & -xyz & -y^2z - yz^2 \\ \hline & xz^2 & + yz^2 + z^3 \\ & xz^2 & + yz^2 + z^3 \end{array}$$
32.
$$\begin{array}{r|l} m^3 + 3m^2n + 3mn^2 + n^3 + x^3 & m + n + x \\ m^3 + m^2n + m^2x & m^2 + 2mn + n^2 - mx - nx + x^2 \\ \hline 2m^2n + 3mn^2 - m^2x + n^3 & \\ 2m^2n + 2mn^2 + 2mnx & \\ \hline & mn^2 + n^3 - m^2x - 2mnx \\ & mn^2 + n^3 + n^2x \\ \hline & -m^2x - 2mnx - n^2x \\ & -m^2x - mnx - mx^2 \\ \hline & -mnx - n^2x + mx^2 \\ & -mnx - n^2x - nx^2 \\ \hline & mx^2 + nx^2 + x^3 \\ & mx^2 + nx^2 + x^3 \end{array}$$
33.
$$\begin{array}{r|l} a^3 - 6a^2 + 12a - 8 - b^3 & a - 2 - b \\ a^3 - 2a^2 - a^2b & a^2 - 4a + 4 + ab - 2b + b^2 \\ \hline -4a^2 + 12a + a^2b - 8 & \\ -4a^2 + 8a + 4ab & \\ \hline & 4a - 8 + a^2b - 4ab \\ & 4a - 8 - 4b \\ \hline & a^2b - 4ab + 4b \\ & a^2b - 2ab - ab^2 \\ \hline & -2ab + 4b + ab^2 \\ & -2ab + 4b + 2b^2 \\ \hline & ab^2 - 2b^2 - b^3 \\ & ab^2 - 2b^2 - b^3 \end{array}$$
34.
$$\begin{array}{r|l} y^5 + 3y^4 + 5y^3 + 3y^2 + 3y + 5 & y + 1 \\ y^5 + y^4 & y^4 + 2y^3 + 3y^2 + 3 + \frac{2}{y+1} \\ \hline 2y^4 + 5y^3 & \\ 2y^4 + 2y^3 & \\ \hline & 3y^3 + 3y^2 \\ & 3y^3 + 3y^2 \\ \hline & 3y + 5 \\ & 3y + 3 \\ \hline & 2 \end{array}$$

$$\begin{array}{r}
 35. \quad \begin{array}{r} 2x^6 - x^7 + x^5 + 2x^4 - x^2 + 5 \\ 2x^8 + 2x^7 \\ - 3x^7 \\ - 3x^7 - 3x^6 \\ \hline 3x^6 + x^5 \\ 3x^6 + 3x^5 \\ - 2x^5 + 2x^4 \\ - 2x^5 - 2x^4 \\ \hline 4x^4 \\ 4x^4 + 4x^3 \\ - 4x^3 - x^2 \\ - 4x^3 - 4x^2 \\ \hline 3x^2 \\ 3x^2 + 3x \\ - 3x + 5 \\ - 3x - 3 \\ \hline 8 \end{array} \quad \begin{array}{r} x+1 \\ 2x^7 - 3x^6 + 3x^5 - 2x^4 + 4x^3 \\ - 4x^2 + 3x - 3 + \frac{8}{x+1} \end{array}
 \end{array}$$

$$\begin{array}{r}
 36. \quad \begin{array}{r} x^7 + 2x^6 - 2x^4 + 2x^2 - 1 \\ x^7 + x^6 \\ \hline x^6 \\ x^6 + x^5 \\ - x^5 - 2x^4 \\ - x^5 - x^4 \\ - x^4 \\ - x^4 - x^3 \\ \hline x^3 + 2x^2 \\ x^3 + x^2 \\ \hline x^2 \\ x^2 + x \\ - x - 1 \\ - x - 1 \end{array} \quad \begin{array}{r} x+1 \\ x^6 + x^5 - x^4 - x^3 + x^2 + x - 1 \end{array}
 \end{array}$$

$$\begin{array}{r}
 37. \quad \begin{array}{r} a^3 - 2a^2c + 4ac^2 - ax^2 - 4c^2x + 2cx^2 \\ a^3 - a^2x \\ \hline a^2x \\ a^2x - ax^2 \\ - 2a^2c + 4ac^2 - 4c^2x \\ - 2a^2c + 2acx \\ - 2acx + 4ac^2 - 4c^2x + 2cx^2 \\ - 2acx + 2cx^2 \\ \hline 4ac^2 - 4c^2x \\ 4ac^2 - 4c^2x \end{array} \quad \begin{array}{r} a-x \\ a^2 + ax - 2ac - 2cx + 4c^2 \end{array}
 \end{array}$$

$$\begin{array}{r}
 38. \quad \begin{array}{r} a^3 + 3abc - b^3 + c^3 \\ a^3 + a^2b - a^2c + ab^2 + abc + ac^2 \\ - a^2b + a^2c - ab^2 + 2abc - ac^2 - b^3 + c^3 \\ - a^2b - ab^2 + abc - b^3 - b^2c - bc^2 \\ \hline a^2c + abc - ac^2 + b^2c + bc^2 + c^3 \\ a^2c + abc - ac^2 + b^2c + bc^2 + c^3 \end{array} \quad \begin{array}{r} a^2 + ab - ac + b^2 + bc + c^2 \\ a - b + c \end{array}
 \end{array}$$

39.

$$\begin{array}{r}
 x^n + y^n \\
 \hline
 x^n + x^{n-1}y \\
 - x^{n-1}y \\
 \hline
 - x^{n-1}y - x^{n-2}y^2 \\
 \hline
 x^{n-2}y^2 \\
 \hline
 x^{n-2}y^2 + x^{n-3}y^3 \\
 - x^{n-3}y^3 \\
 \hline
 - x^{n-3}y^3 - x^{n-4}y^4 \\
 \hline
 x^{n-4}y^4 \\
 \hline
 x^{n-4}y^4 + x^{n-5}y^5 \\
 \hline
 - x^{n-5}y^5 + y^n
 \end{array}
 \quad \left| \begin{array}{l}
 x + y \\
 \hline
 x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4
 \end{array} \right.$$

Remainder,

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41.

$$\begin{array}{r}
 1 + 0 + 0 + 0 + 0 + 0 + 0 + 8 + 7 \\
 \hline
 1 + 2 + 1 \\
 - 2 - 1 \\
 \hline
 - 2 - 4 - 2 \\
 \hline
 3 + 2 \\
 \hline
 3 + 6 + 3 \\
 - 4 - 3 \\
 \hline
 - 4 - 8 - 4 \\
 \hline
 5 + 4 \\
 \hline
 5 + 10 + 5 \\
 - 6 - 5 + 8 \\
 \hline
 - 6 - 12 - 6 \\
 \hline
 7 + 14 + 7 \\
 \hline
 7 + 14 + 7
 \end{array}
 \quad \left| \begin{array}{l}
 1 + 2 + 1 \\
 \hline
 1 - 2 + 3 - 4 + 5 - 6 + 7 \\
 \hline
 = x^6 - 2x^5 + 3x^4 - 4x^3 + 5x^2 - 6x + 7.
 \end{array} \right.$$

42.

$$\begin{array}{r}
 1 + 0 + 0 + 0 + 0 + 0 + 38 + 12 \\
 \hline
 1 + 2 \\
 - 2 \\
 \hline
 - 2 - 4 \\
 \hline
 4 \\
 \hline
 4 + 8 \\
 - 8 \\
 \hline
 - 8 - 16 \\
 \hline
 16 + 38 \\
 \hline
 16 + 32 \\
 \hline
 6 + 12 \\
 \hline
 6 + 12
 \end{array}
 \quad \left| \begin{array}{l}
 1 + 2 \\
 \hline
 1 - 2 + 4 - 8 + 16 + 6 \\
 \hline
 = a^5 - 2a^4 + 4a^3 - 8a^2 + 16a + 6.
 \end{array} \right.$$

43.

$$\begin{array}{r}
 1 + 0 + 0 + 0 - 19 - 6 \\
 \hline
 1 + 2 \\
 - 2 \\
 \hline
 - 2 - 4 \\
 \hline
 4 \\
 \hline
 4 + 8 \\
 - 8 - 19 \\
 \hline
 - 8 - 16 \\
 \hline
 - 3 - 6 \\
 \hline
 - 3 - 6
 \end{array}
 \quad \left| \begin{array}{l}
 1 + 2 \\
 \hline
 1 - 2 + 4 - 8 - 3 \\
 \hline
 = m^4 - 2m^3 + 4m^2 - 8m - 3.
 \end{array} \right.$$

$$\begin{array}{r}
 44. \quad 1 + 0 + 0 + 0 + 0 - 32 - 4 + 8 \quad | \quad 1 - 2 \\
 \underline{1 - 2} \\
 2 \\
 \underline{2 - 4} \\
 4 \\
 \underline{4 - 8} \\
 8 \\
 \underline{8 - 16} \\
 16 - 32 \\
 \underline{16 - 32} \\
 -4 + 8 \\
 \underline{-4 + 8}
 \end{array}
 \quad
 \begin{array}{r}
 \underline{1 + 2 + 4 + 8 + 16 + 0 - 4} \\
 = m^6 + 2m^5 + 4m^4 + 8m^3 + 16m^2 - 4.
 \end{array}$$

$$\begin{array}{r}
 45. \quad 1 + 0 + 0 + 0 + 27 - 9 - 10 \quad | \quad 1 - 3 + 5 \\
 \underline{1 - 3 + 5} \\
 3 - 5 \\
 \underline{3 - 9 + 15} \\
 4 - 15 + 27 \\
 \underline{4 - 12 + 20} \\
 -3 + 7 - 9 \\
 \underline{-3 + 9 - 15} \\
 -2 + 6 - 10 \\
 \underline{-2 + 6 - 10}
 \end{array}
 \quad
 \begin{array}{r}
 \underline{1 + 3 + 4 - 3 - 2} \\
 = a^4 + 3a^3 + 4a^2 - 3a - 2.
 \end{array}$$

$$\begin{array}{r}
 46. \quad 21 - 29 - 8 + 6 + 4 \quad | \quad 3 - 2 \\
 \underline{21 - 14} \\
 -15 - 8 \\
 \underline{-15 + 10} \\
 -18 + 6 \\
 \underline{-18 + 12} \\
 -6 + 4 \\
 \underline{-6 + 4}
 \end{array}
 \quad
 \begin{array}{r}
 \underline{7 - 5 - 6 - 2} \\
 = 7x^3 - 5x^2 - 6x - 2.
 \end{array}$$

$$\begin{array}{r}
 47. \quad 2 - 11 + 16 - 12 + 9 \quad | \quad 2 - 3 \\
 \underline{2 - 3} \\
 -8 + 16 \\
 \underline{-8 + 12} \\
 4 - 12 \\
 \underline{4 - 6} \\
 -6 + 9 \\
 \underline{-6 + 9}
 \end{array}
 \quad
 \begin{array}{r}
 \underline{1 - 4 + 2 - 3} \\
 = x^3 - 4x^2 + 2x - 3.
 \end{array}$$

$$\begin{array}{r}
 48. \quad 30 - 62 + 60 - 36 + 8 \quad | \quad 5 - 2 \\
 \underline{30 - 12} \\
 -50 + 60 \\
 \underline{-50 + 20} \\
 40 - 36 \\
 \underline{40 - 16} \\
 -20 + 8 \\
 \underline{-20 + 8}
 \end{array}
 \quad
 \begin{array}{r}
 \underline{6 - 10 + 8 - 4} \\
 = 6x^3 - 10x^2 + 8x - 4.
 \end{array}$$

$$\begin{array}{r}
 49. \quad 27 - 33 + 46 - 119 + 55 \\
 \underline{27 - 15} \\
 \quad -18 + 46 \\
 \underline{-18 + 10} \\
 \quad 36 - 119 \\
 \underline{36 - 20} \\
 \quad -99 + 55 \\
 \underline{-99 + 55}
 \end{array}
 \quad
 \begin{array}{r}
 9 - 5 \\
 \underline{3 - 2 + 4 - 11} \\
 = 3x^3 - 2x^2 + 4x - 11.
 \end{array}$$

$$\begin{array}{r}
 50. \quad 1 + 0 - 2 + 0 - 1 + 0 - 10 - 36 \\
 \underline{1 - 2} \\
 \quad 2 - 2 \\
 \underline{2 - 4} \\
 \quad 2 \\
 \underline{2 - 4} \\
 \quad 4 - 1 \\
 \underline{4 - 8} \\
 \quad 7 \\
 \underline{7 - 14} \\
 \quad 14 - 10 \\
 \underline{14 - 28} \\
 \quad 18 - 36 \\
 \underline{18 - 36}
 \end{array}
 \quad
 \begin{array}{r}
 1 - 2 \\
 \underline{1 + 2 + 2 + 4 + 7 + 14 + 18} \\
 = x^6 + 2x^5 + 2x^4 + 4x^3 + 7x^2 + 14x + 18.
 \end{array}$$

$$\begin{array}{r}
 51. \quad 1 - 4 + 5 - 4 + 1 \\
 \underline{1 - 1 + 1} \\
 \quad -3 + 4 - 4 \\
 \underline{-3 + 3 - 3} \\
 \quad 1 - 1 + 1 \\
 \underline{1 - 1 + 1}
 \end{array}
 \quad
 \begin{array}{r}
 1 - 1 + 1 \\
 \underline{1 - 3 + 1} \\
 = x^2 - 3x + 1.
 \end{array}$$

$$\begin{array}{r}
 52. \quad 1 - 1 - 10 + 7 + 15 \\
 \underline{1 - 2 - 3} \\
 \quad 1 - 7 + 7 \\
 \underline{1 - 2 - 3} \\
 \quad -5 + 10 + 15 \\
 \underline{-5 + 10 + 15}
 \end{array}
 \quad
 \begin{array}{r}
 1 - 2 - 3 \\
 \underline{1 + 1 - 5} \\
 = x^2 + x - 5.
 \end{array}$$

$$\begin{array}{r}
 53. \quad 2 + 7 - 27 - 8 + 16 \\
 \underline{2 + 10 - 8} \\
 \quad -3 - 19 - 8 \\
 \underline{-3 - 15 + 12} \\
 \quad -4 - 20 + 16 \\
 \underline{-4 - 20 + 16}
 \end{array}
 \quad
 \begin{array}{r}
 1 + 5 - 4 \\
 \underline{2 - 3 - 4} \\
 = 2x^2 - 3x - 4.
 \end{array}$$

$$\begin{array}{r}
 54. \quad 28 + 6 + 6 - 6 - 2 \\
 \underline{28 + 14 + 14} \\
 \quad -8 - 8 - 6 \\
 \underline{-8 - 4 - 4} \\
 \quad -4 - 2 - 2 \\
 \underline{-4 - 2 - 2}
 \end{array}
 \quad
 \begin{array}{r}
 4 + 2 + 2 \\
 \underline{7 - 2 - 1} \\
 = 7x^2 - 2x - 1.
 \end{array}$$

$$\begin{array}{r}
 55. \quad \begin{array}{r} -6 + 7 + 2 - 1 - 2 \\ -6 - 5 - 2 \\ \hline 12 + 4 - 1 \\ 12 + 10 + 4 \\ \hline -6 - 5 - 2 \\ -6 - 5 - 2 \\ \hline \end{array} \quad \begin{array}{r} 6 + 5 + 2 \\ -1 + 2 - 1 \\ \hline \end{array} \\
 \hline
 \end{array}
 = x^2 + 2x - 1.$$

$$\begin{array}{r}
 56. \quad \begin{array}{r} 3 - 20 + 25 + 16 - 6 \\ 3 - 8 + 2 \\ \hline -12 + 23 + 16 \\ -12 + 32 - 8 \\ \hline -9 + 24 - 6 \\ -9 + 24 - 6 \\ \hline \end{array} \quad \begin{array}{r} 3 - 8 + 2 \\ 1 - 4 - 3 \\ \hline \end{array} \\
 \hline
 \end{array}
 = x^2 - 4x - 3.$$

$$\begin{array}{r}
 57. \quad \begin{array}{r} 3 + 7 + 6 + 3 - 1 \\ 3 + 3 + 3 \\ \hline 4 + 3 + 3 \\ 4 + 4 + 4 \\ \hline -1 - 1 - 1 \\ -1 - 1 - 1 \\ \hline \end{array} \quad \begin{array}{r} 1 + 1 + 1 \\ 3 + 4 - 1 \\ \hline \end{array} \\
 \hline
 \end{array}
 = 3x^2 + 4x - 1.$$

$$\begin{array}{r}
 58. \quad \begin{array}{r} 6 - 23 + 30 - 18 + 4 \\ 6 - 15 + 6 \\ \hline -8 + 24 - 18 \\ -8 + 20 - 8 \\ \hline 4 - 10 + 4 \\ 4 - 10 + 4 \\ \hline \end{array} \quad \begin{array}{r} 2 - 5 + 2 \\ 3 - 4 + 2 \\ \hline \end{array} \\
 \hline
 \end{array}
 = 3x^2 - 4x + 2.$$

$$\begin{array}{r}
 59. \quad \begin{array}{r} 24 + 32 - 16 - 25 - 4 \\ 24 - 4 - 16 \\ \hline 36 + 0 - 25 \\ 36 - 6 - 24 \\ \hline 6 - 1 - 4 \\ 6 - 1 - 4 \\ \hline \end{array} \quad \begin{array}{r} 6 - 1 - 4 \\ 4 + 6 + 1 \\ \hline \end{array} \\
 \hline
 \end{array}
 = 4x^2 + 6x + 1.$$

$$\begin{array}{r}
 60. \quad \begin{array}{r} 1 - 2 + \frac{1}{2} + \frac{2}{5} + \frac{1}{15} + \frac{5}{4} \\ 1 - \frac{3}{2} \\ \hline -\frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} + \frac{3}{4} \\ \hline -\frac{2}{3} + \frac{2}{5} \\ -\frac{2}{3} + 1 \\ \hline -\frac{2}{3} + \frac{1}{5} \\ -\frac{2}{3} + \frac{9}{10} \\ \hline -\frac{5}{6} + \frac{5}{4} \\ -\frac{5}{6} + \frac{1}{4} \\ \hline \end{array} \quad \begin{array}{r} 1 - \frac{3}{2} \\ 1 - \frac{1}{2} - \frac{2}{3} - \frac{3}{5} - \frac{5}{6} \\ \hline \end{array} \\
 \hline
 \end{array}
 = x^4 - \frac{1}{2}x^3 - \frac{2}{3}x^2 - \frac{3}{5}x - \frac{5}{6}.$$

61.
$$\begin{array}{r} 1 - \frac{5}{4} + \frac{29}{24} - \frac{31}{24} + \frac{5}{6} - \frac{1}{4} \\ \hline 1 - \frac{5}{4} \\ - \frac{1}{2} + \frac{29}{24} \\ - \frac{1}{2} + \frac{3}{8} \\ \hline \frac{5}{6} - \frac{31}{24} \\ \hline \frac{5}{6} - \frac{1}{2} \\ - \frac{2}{3} + \frac{5}{6} \\ - \frac{2}{3} + \frac{1}{2} \\ \hline \frac{1}{3} - \frac{1}{4} \\ \hline \frac{1}{3} - \frac{1}{4} \end{array} \quad \left| \begin{array}{l} 1 - \frac{5}{4} \\ \hline 1 - \frac{1}{2} + \frac{5}{6} - \frac{2}{3} + \frac{1}{4} \\ \hline \end{array} \right. = x^4 - \frac{1}{2}x^3 + \frac{5}{6}x^2 - \frac{2}{3}x + \frac{1}{4}.$$

62.
$$\begin{array}{r} 1 - \frac{1}{6} + \frac{5}{6} - \frac{13}{36} + \frac{37}{36} - \frac{17}{36} + \frac{7}{24} \\ \hline 1 - \frac{1}{6} + \frac{5}{6} \\ \hline \frac{1}{2} + \frac{1}{3} - \frac{13}{36} \\ \hline \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \\ \hline \frac{2}{3} - \frac{11}{36} + \frac{37}{36} \\ \hline \frac{2}{3} - \frac{4}{9} + \frac{1}{3} \\ \hline - \frac{1}{6} + \frac{25}{36} - \frac{17}{36} \\ \hline - \frac{1}{6} + \frac{1}{9} - \frac{1}{12} \\ \hline \frac{7}{12} - \frac{14}{36} + \frac{7}{24} \\ \hline \frac{7}{12} - \frac{1}{3} + \frac{7}{24} \end{array} \quad \left| \begin{array}{l} 1 - \frac{2}{3} + \frac{1}{2} \\ \hline 1 + \frac{1}{2} + \frac{2}{3} - \frac{1}{6} + \frac{7}{12} \\ \hline \end{array} \right. = x^4 + \frac{1}{2}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x + \frac{7}{12}.$$

63.
$$\begin{array}{r} \frac{1}{8}x^3 - \frac{1}{2}xyz + \frac{1}{2}y^3 + z^3 \\ \hline \frac{1}{8}x^3 + \frac{1}{12}x^2y + \frac{1}{4}x^2z \\ \hline - \frac{1}{12}x^2y - \frac{1}{4}x^2z - \frac{1}{2}xyz \\ \hline - \frac{1}{12}x^2y - \frac{1}{12}xy^2 - \frac{1}{6}xyz \\ \hline - \frac{1}{4}x^2z + \frac{1}{8}xy^2 - \frac{1}{3}xyz \\ \hline - \frac{1}{4}x^2z \\ \hline \frac{1}{8}xy^2 - \frac{1}{6}xyz + \frac{1}{2}xz^2 + \frac{1}{7}y^3 \\ \hline \frac{1}{8}xy^2 + \frac{1}{9}y^2z + \frac{1}{7}y^3 \\ \hline - \frac{1}{6}xyz + \frac{1}{2}xz^2 - \frac{1}{9}y^2z \\ \hline - \frac{1}{8}xyz - \frac{1}{9}y^2z - \frac{1}{8}yz^2 \\ \hline \frac{1}{8}xz^2 + \frac{1}{9}yz^2 + z^3 \\ \hline \frac{1}{8}xz^2 + \frac{1}{9}yz^2 + z^3 \end{array} \quad \left| \begin{array}{l} \frac{1}{2}x + \frac{1}{8}y + z \\ \hline \frac{1}{4}x^2 - \frac{1}{6}xy - \frac{1}{2}xz + \frac{1}{9}y^2 - \frac{1}{3}yz + z^2 \\ \hline \end{array} \right.$$

64.
$$\begin{array}{r} 1 \\ \hline 1 + x \\ - x \\ \hline - x - x^2 \\ \hline x^2 \\ \hline x^2 + x^3 \\ - x^3 \\ \hline - x^3 - x^4 \\ \hline x^4 \\ \hline x^4 + x^5 \\ - x^5 \end{array} \quad \left| \begin{array}{l} 1 + x \\ \hline 1 - x + x^2 - x^3 + x^4 \\ \hline \end{array} \right.$$

Remainder,

65.

$$\begin{array}{r}
 \frac{1}{1-x} \quad \left| \frac{1-x}{1+x+x^2+x^3+x^4} \right. \\
 \underline{x} \\
 x-x^2 \\
 \underline{x^2} \\
 x^2-x^3 \\
 \underline{x^3} \\
 x^3-x^4 \\
 \underline{x^4} \\
 x^4-x^5 \\
 \underline{x^5}
 \end{array}$$

Remainder,

66.

$$\begin{array}{r}
 x^{3n-3} + y^{3n+3} \\
 \underline{x^{3n-3} + x^{2n-2}y^{n+1}} \\
 -x^{2n-2}y^{n+1} \\
 \underline{-x^{2n-2}y^{n+1} - x^{n-1}y^{2n+2}} \\
 x^{n-1}y^{2n+2} + y^{3n+3} \\
 \underline{x^{n-1}y^{2n+2} + y^{3n+3}}
 \end{array}
 \quad \left| \frac{x^{n-1} + y^{n+1}}{x^{2n-2} - x^{n-1}y^{n+1} + y^{2n+2}} \right.$$

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PROOF OF PRINCIPLE 3

Performing the division of $x^n + y^n$ by $x - y$ to any number of terms of the quotient, the remainders obtained are:

$$\begin{array}{ll}
 \text{1st remainder,} & x^{n-1}y + y^n; \\
 \text{2d remainder,} & x^{n-2}y^2 + y^n; \\
 \text{3d remainder,} & x^{n-3}y^3 + y^n; \text{ etc.}
 \end{array}$$

Hence, it may be inferred that the n^{th} remainder is

$$\begin{aligned}
 & x^{n-n}y^n + y^n \\
 &= x^0y^n + y^n \\
 &= y^n + y^n = 2y^n.
 \end{aligned}$$

§ 103,

Hence, $x - y$ is not an exact divisor of $x^n + y^n$.

PROOF OF PRINCIPLE 4

Performing the division of $x^n + y^n$ by $x + y$ to any number of terms of the quotient, the remainders obtained are:

$$\begin{array}{ll}
 \text{1st remainder,} & -x^{n-1}y + y^n; \\
 \text{2d remainder,} & x^{n-2}y^2 + y^n; \\
 \text{3d remainder,} & -x^{n-3}y^3 + y^n; \\
 \text{4th remainder,} & x^{n-4}y^4 + y^n; \text{ etc.}
 \end{array}$$

The remainder that reduces to 0, if there be such a remainder, must reduce to the form $-y^n + y^n$. From the above examples, and from as many more as can be written, it may be inferred that no remainder occurring after an even number of terms of the quotient can reduce to $-y^n + y^n$, or 0, since the sign of the first term is +. Hence, if n is even, $x^n + y^n$ is not divisible by $x + y$.

If n is odd, the n^{th} remainder evidently becomes

$$\begin{aligned}
 & -x^{n-n}y^n + y^n \\
 &= -x^0y^n + y^n \\
 &= -y^n + y^n = 0.
 \end{aligned}$$

§ 103,

Hence, if n is odd, $x^n + y^n$ is divisible by $x + y$.

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23. Since $a^6 + b^6 = (a^2)^3 + (b^2)^3$, it is divisible by $a^2 + b^2$ (Prin. 4).
 24. $x^7 + a^7$ is divisible by $x + a$ (Prin. 4).
 25. Since $a^{10} + b^{10} = (a^2)^5 + (b^2)^5$, it is divisible by $a^2 + b^2$ (Prin. 4).
 26. Since $a^{10} + b^5 = (a^2)^5 + b^5$, it is divisible by $a^2 + b$ (Prin. 4).
 27. Since $a^{12} + b^{12} = (a^4)^3 + (b^4)^3$, it is divisible by $a^4 + b^4$ (Prin. 4).
 28. Since $a^8 - 27 = a^8 - 3^3$, it is divisible by $a - 3$ (Prin. 1).
 29. Since $a^6 - 27 = (a^2)^3 - 3^3$, it is divisible by $a^2 - 3$ (Prin. 1).
 30. As the difference of the same powers of a and b , and also of a^2 and b^2 , $a^4 - b^4$ is divisible by $a - b$ and by $a^2 - b^2$ (Prin. 1).
 As the difference of the same even powers of a and b , and also of a^2 and b^2 , $a^4 - b^4$ is divisible by $a + b$ and by $a^2 + b^2$ (Prin. 2).
 31. $a^6 - 1$ is divisible by $a - 1$ (Prin. 1) and by $a + 1$ (Prin. 2).
 Since $a^6 - 1 = (a^2)^3 - 1^3$, $a^6 - 1$ is divisible by $a^2 - 1$ (Prin. 1).
 Since $a^6 - 1 = (a^3)^2 - 1^2$, $a^6 - 1$ is divisible by $a^3 - 1$ (Prin. 1).
 Since $a^6 - 1 = (a^3)^2 - 1^2$, $a^6 - 1$ is divisible by $a^3 + 1$ (Prin. 2).
 32. $a^8 - b^8$ is divisible by $a - b$ (Prin. 1) and by $a + b$ (Prin. 2).
 Since $a^8 - b^8 = (a^2)^4 - (b^2)^4$, $a^8 - b^8$ is divisible by $a^2 - b^2$ (Prin. 1) and by $a^2 + b^2$ (Prin. 2).
 Since $a^8 - b^8 = (a^4)^2 - (b^4)^2$, $a^8 - b^8$ is divisible by $a^4 - b^4$ (Prin. 1) and by $a^4 + b^4$ (Prin. 2).
 33. $a^{10} - b^{10}$ is divisible by $a - b$ (Prin. 1) and by $a + b$ (Prin. 2).
 Since $a^{10} - b^{10} = (a^2)^5 - (b^2)^5$, $a^{10} - b^{10}$ is divisible by $a^2 - b^2$ (Prin. 1).
 Since $a^{10} - b^{10} = (a^5)^2 - (b^5)^2$, $a^{10} - b^{10}$ is divisible by $a^5 - b^5$ (Prin. 1) and by $a^5 + b^5$ (Prin. 2).
 34. $a^{16} - b^{16}$ is divisible by $a - b$ (Prin. 1) and by $a + b$ (Prin. 2).
 Since $a^{16} - b^{16} = (a^2)^8 - (b^2)^8$, $a^{16} - b^{16}$ is divisible by $a^2 - b^2$ (Prin. 1) and by $a^2 + b^2$ (Prin. 2).
 Since $a^{16} - b^{16} = (a^4)^4 - (b^4)^4$, $a^{16} - b^{16}$ is divisible by $a^4 - b^4$ (Prin. 1) and by $a^4 + b^4$ (Prin. 2).
 Since $a^{16} - b^{16} = (a^8)^2 - (b^8)^2$, $a^{16} - b^{16}$ is divisible by $a^8 - b^8$ (Prin. 1) and by $a^8 + b^8$ (Prin. 2).
 35. $a^{12} - b^{12}$ is divisible by $a - b$ (Prin. 1) and by $a + b$ (Prin. 2).
 Since $a^{12} - b^{12} = (a^2)^6 - (b^2)^6$, $a^{12} - b^{12}$ is divisible by $a^2 - b^2$ (Prin. 1) and by $a^2 + b^2$ (Prin. 2).
 Since $a^{12} - b^{12} = (a^3)^4 - (b^3)^4$, $a^{12} - b^{12}$ is divisible by $a^3 - b^3$ (Prin. 1) and by $a^3 + b^3$ (Prin. 2).
 Since $a^{12} - b^{12} = (a^4)^3 - (b^4)^3$, $a^{12} - b^{12}$ is divisible by $a^4 - b^4$ (Prin. 1).
 Since $a^{12} - b^{12} = (a^6)^2 - (b^6)^2$, $a^{12} - b^{12}$ is divisible by $a^6 - b^6$ (Prin. 1) and by $a^6 + b^6$ (Prin. 2).

REVIEW

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$$\begin{aligned}
 8. \quad & (x^{2n} + 2x^ny^n + y^{2n})(x^{2n} + 2x^ny^n + y^{2n}) \\
 &= (x^{2n} + y^{2n} + 2x^ny^n)(x^{2n} + y^{2n} + 2x^ny^n) \\
 &= (x^{2n} + y^{2n})^2 - (2x^ny^n)^2 \\
 &= x^{4n} + 2x^{2n}y^{2n} + y^{4n} - 4x^{2n}y^{2n} \\
 &= x^{4n} - 2x^{2n}y^{2n} + y^{4n}.
 \end{aligned}$$

9.
$$\begin{aligned} & \left(\frac{1}{4}x^2 + \frac{1}{9}xy + \frac{1}{9}y^2\right) \left(\frac{1}{4}x^2 - \frac{1}{9}xy + \frac{1}{9}y^2\right) \\ &= \left(\frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{9}xy\right) \left(\frac{1}{4}x^2 + \frac{1}{9}y^2 - \frac{1}{9}xy\right) \\ &= \left(\frac{1}{4}x^2 + \frac{1}{9}y^2\right)^2 - \left(\frac{1}{9}xy\right)^2 \\ &= \frac{1}{16}x^4 + \frac{1}{8}x^2y^2 + \frac{1}{81}y^4 - \frac{1}{9}x^2y^2 \\ &= \frac{1}{16}x^4 - \frac{1}{8}x^2y^2 + \frac{1}{81}y^4. \end{aligned}$$
10.
$$\begin{aligned} & \frac{.2 - .8 + 1.6}{.1 + .4 + .8} \\ & \frac{.02 - .08 + .16}{.08 - .32 + .64} \\ & \frac{.16 - .64 + 1.28}{.02 \qquad \qquad \qquad 1.28} \\ &= .02a^4 + 1.28. \end{aligned}$$
31.
$$\begin{aligned} & (a-b)(a+b)(a^2+b^2) \\ &= (a^2-b^2)(a^2+b^2) \\ &= a^4-b^4. \end{aligned}$$
33.
$$\begin{aligned} & (1-x)(1+x)(1-x)(1+x) \\ &= (1-x^2)(1-x^2) \\ &= 1-2x^2+x^4. \end{aligned}$$
32.
$$\begin{aligned} & (1-x)(1+x)(1+x^2)(1+x^4) \\ &= (1-x^2)(1+x^2)(1+x^4) \\ &= (1-x^4)(1+x^4) \\ &= 1-x^8. \end{aligned}$$
34.
$$\begin{aligned} & (m+n)(m+n)(m-n)(m-n) \\ &= (m+n)(m-n)(m+n)(m-n) \\ &= (m^2-n^2)(m^2-n^2) \\ &= m^4-2m^2n^2+n^4. \end{aligned}$$
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35. By observing 1, § 110, it will be seen that

$$(a^4 + a^3 + a^2 + a + 1)(a-1) = a^5 - 1.$$
36. By observing 1, § 110, it will be seen that

$$(x^3 + x^2y + xy^2 + y^3)(x-y) = x^4 - y^4.$$
37. By observing 2, § 110, it will be seen that

$$(x^5 - x^4 + x^3 - x^2 + x - 1)(x+1) = x^6 - 1.$$
38. By observing 1, § 110, it will be seen that

$$\begin{aligned} & (a^5 + 2a^4 + 4a^3 + 8a^2 + 16a + 32)(a-2), \\ & \text{or } (a^5 + a^4 \cdot 2 + a^3 \cdot 2^2 + a^2 \cdot 2^3 + a \cdot 2^4 + 2^5)(a-2) \\ &= a^6 - 2^6 = a^6 - 64. \end{aligned}$$
54.
$$\begin{aligned} & (x+y)(x-y)(x^2+y^2)(x^4+y^4)(x^8+y^8) \\ &= (x^2-y^2)(x^2+y^2)(x^4+y^4)(x^8+y^8) \\ &= (x^4-y^4)(x^4+y^4)(x^8+y^8) \\ &= (x^8-y^8)(x^8+y^8) \\ &= x^{16} - y^{16}. \end{aligned}$$
55.
$$\begin{aligned} & (m^8 + 1)(m^4 + 1)(m^2 + 1)(m + 1)(m - 1) \\ &= (m^8 + 1)(m^4 + 1)(m^2 + 1)(m^2 - 1) \\ &= (m^8 + 1)(m^4 + 1)(m^4 - 1) \\ &= (m^8 + 1)(m^8 - 1) \\ &= m^{16} - 1. \end{aligned}$$
56.
$$\begin{aligned} & (16x^4 + 1)(4x^2 + 1)(2x + 1)(2x - 1) \\ &= (16x^4 + 1)(4x^2 + 1)(4x^2 - 1) \\ &= (16x^4 + 1)(16x^4 - 1) \\ &= 256x^8 - 1. \end{aligned}$$

$$\begin{array}{r|l}
 59. & x^8 - 45x^5 + 45x^4 - 18x^2 + 9x - 1 \quad | \quad x^3 - 4x^2 + 3x - 1 \\
 & x^8 - 4x^7 + 3x^6 - x^5 \quad | \quad x^5 + 4x^4 + 13x^3 - 4x^2 - 6x + 1 \\
 \hline
 & 4x^7 - 3x^6 - 44x^5 + 45x^4 \\
 & 4x^7 - 16x^6 + 12x^5 - 4x^4 \\
 \hline
 & 13x^6 - 56x^5 + 49x^4 \\
 & 13x^6 - 52x^5 + 39x^4 - 13x^3 \\
 \hline
 & -4x^5 + 10x^4 + 13x^3 - 18x^2 \\
 & -4x^5 + 16x^4 - 12x^3 + 4x^2 \\
 \hline
 & -6x^4 + 25x^3 - 22x^2 + 9x \\
 & -6x^4 + 24x^3 - 18x^2 + 6x \\
 \hline
 & x^3 - 4x^2 + 3x - 1 \\
 & x^3 - 4x^2 + 3x - 1 \\
 \hline
 \end{array}$$

$$\begin{array}{r|l}
 60. & a^7 - 12a^2 - a + 12 \quad | \quad a^8 - 2a^2 + 4a - 3 \\
 & a^7 - 2a^6 + 4a^5 - 3a^4 \quad | \quad a^4 + 2a^3 - 5a - 4 \\
 \hline
 & 2a^6 - 4a^5 + 3a^4 \\
 & 2a^6 - 4a^5 + 8a^4 - 6a^3 \\
 \hline
 & -5a^4 + 6a^3 - 12a^2 - a \\
 & -5a^4 + 10a^3 - 20a^2 + 15a \\
 \hline
 & -4a^3 + 8a^2 - 16a + 12 \\
 & -4a^3 + 8a^2 - 16a + 12 \\
 \hline
 \end{array}$$

$$\begin{array}{r|l}
 61. & b^8 - 10b^2 - 5b + 4 \quad | \quad b^3 - 2b^2 + 3b - 1 \\
 & b^8 - 2b^7 + 3b^6 - b^5 \quad | \quad b^5 + 2b^4 + b^3 - 3b^2 - 7b - 4 \\
 \hline
 & 2b^7 - 3b^6 + b^5 \\
 & 2b^7 - 4b^6 + 6b^5 - 2b^4 \\
 \hline
 & b^6 - 5b^5 + 2b^4 \\
 & b^6 - 2b^5 + 3b^4 - b^3 \\
 \hline
 & -3b^5 - b^4 + b^3 - 10b^2 \\
 & -3b^5 + 6b^4 - 9b^3 + 3b^2 \\
 \hline
 & -7b^4 + 10b^3 - 13b^2 - 5b \\
 & -7b^4 + 14b^3 - 21b^2 + 7b \\
 \hline
 & -4b^3 + 8b^2 - 12b + 4 \\
 & -4b^3 + 8b^2 - 12b + 4 \\
 \hline
 \end{array}$$

$$\begin{array}{r|l}
 62. & m^{10} - 6m^8 + 5m - 2 \quad | \quad m^4 + 2m^3 - 3m - 2 \\
 & m^{10} + 2m^9 - 3m^7 - 2m^6 \quad | \quad m^8 - 2m^5 + 4m^4 - 5m^3 + 6m^2 - 4m + 1 \\
 \hline
 & -2m^9 + 3m^7 + 2m^6 \\
 & -2m^9 - 4m^8 + 6m^6 + 4m^5 \\
 \hline
 & 4m^8 + 3m^7 - 4m^6 - 4m^5 \\
 & 4m^8 + 8m^7 - 12m^5 - 8m^4 \\
 \hline
 & -5m^7 - 4m^6 + 8m^5 + 8m^4 - 6m^3 \\
 & -5m^7 - 10m^6 + 15m^4 + 10m^3 \\
 \hline
 & 6m^6 + 8m^5 - 7m^4 - 16m^3 \\
 & 6m^6 + 12m^5 - 18m^3 - 12m^2 \\
 \hline
 & -4m^5 - 7m^4 + 2m^3 + 12m^2 + 5m \\
 & -4m^5 - 8m^4 + 12m^2 + 8m \\
 \hline
 & m^4 + 2m^3 - 3m - 2 \\
 & m^4 + 2m^3 - 3m - 2 \\
 \hline
 \end{array}$$

63.
$$\begin{array}{r|l} a^7 - 160a^4 + 127a^3 - 100a^2 - 20a + 16 & a^3 - 6a^2 + 5a - 4 \\ a^7 - 6a^6 + 5a^5 - 4a^4 & a^4 + 6a^3 + 31a^2 - 4 \end{array}$$
64.
$$\begin{array}{r|l} b^{10} + 29b^4 - 170b^3 - 61b^2 + 210b - 22 & b^4 + 2b^2 - 5b - 11 \\ b^{10} + 2b^8 - 5b^7 - 11b^5 & b^6 - 2b^4 + 5b^3 + 15b^2 - 20b + 2 \end{array}$$
65.
$$\begin{array}{r|l} 6a^7 + \frac{3}{2}a^2y^5 - \frac{3}{2}ay^6 + \frac{2}{3}y^7 & a^3 + \frac{1}{2}a^2y - \frac{1}{2}ay^2 + \frac{1}{2}y^3 \\ 6a^7 + 3a^6y - \frac{3}{2}a^5y^2 + \frac{3}{4}a^4y^3 & 6a^4 - 3a^3y + 3a^2y^2 - 3ay^3 + \frac{2}{3}y^4 \end{array}$$
66. Divisor, $ac - b^2 + ed - d^2$ Quotient, $a - b - c$
 $a^2c - ab^2 + acd - ad^2 - abc + b^3 - bcd + bd^2 - ac^2 + b^2c - c^2d + cd^2$
 $a^2c - ab^2 + acd - ad^2$
 $- abc + b^3 - bcd + bd^2$
 $- ac^2 + b^2c - c^2d + cd^2$

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68.
$$4a^2 + 3b^2 - (2a^2 - 3b^2) - (7b^2 - 2a^2) - (-a^2 - b^2)$$

$$= 4a^2 + 3b^2 - 2a^2 + 3b^2 - 7b^2 + 2a^2 + a^2 + b^2 = 5a^2$$
69.
$$a^2 - (b^2 - c^2) - (b^2 + c^2 - a^2) + (c^2 - b^2 - a^2) - (a^2 - b^2 - c^2)$$

$$= a^2 - b^2 + c^2 - b^2 - c^2 + a^2 + c^2 - b^2 + a^2 - a^2 + b^2 + c^2$$

$$= 2a^2 - 2b^2 + 2c^2$$
70.
$$x^2 - (2xy - y^2) - (x^2 + xy - y^2) - x^2 - 2xy - y^2 + 5y^2$$

$$= x^2 - 2xy + y^2 - x^2 - xy + y^2 - x^2 - 2xy + y^2 + 5y^2$$

$$= -x^2 - 5xy + 8y^2$$
71.
$$m + \{2m - [n + 3p - (4p - 3n) - 5n + 2m] - 7p\}$$

$$= m + \{2m - [n + 3p - 4p + 3n - 5n + 2m] - 7p\}$$

$$= m + 2m - [-n - p + 2m] - 7p$$

$$= m + 2m + n + p - 2m - 7p = m + n - 6p$$

72. $x - 3y + \{2z - (5y - 4x - 7z) - (x - y) - z\} - x$
 $= x - 3y + \{2z - 5y + 4x - 7z - x + y - z\} - x$
 $= 3x - 7y - 6z.$
73. $1 - \{1 - [x^2 - 3 - (2x^2 - 4) + 3x^2 + 1] - (x^2 - 4)\} - 1$
 $= -\{1 - [4x^2 - 2 - (2x^2 - 4)] - (x^2 - 4)\}$
 $= -\{1 - [4x^2 - 2 - 2x^2 + 4] - x^2 + 4\}$
 $= -\{5 - [2x^2 + 2] - x^2\}$
 $= -\{5 - 2x^2 - 2 - x^2\} = 3x^2 - 3.$
74. $a - (2b + 5a) - 2b - [3a - (6b - 5a) - a] + 12a$
 $= a - 2b - 5a - 2b - [3a - 6b + 5a - a] + 12a$
 $= 8a - 4b - (7a - 6b)$
 $= 8a - 4b - 7a + 6b = a + 2b.$
75. $x - \{m - [x + (3m - 2x) + 5x - (2x + m)] - 2x + m\}$
 $= x - \{2m - 2x - [x + 3m - 2x + 5x - 2x - m]\}$
 $= x - \{2m - 2x - [2x + 2m]\}$
 $= x - \{2m - 2x - 2x - 2m\}$
 $= x - \{-4x\} = 5x.$
76. $x - \{5x - [6x - (7x - 8x - 9x) - 10x] + 11x\} + 9x$
 $= x - \{5x - [6x - (7x + x) - 10x] + 11x\} + 9x$
 $= 10x - \{16x - [6x - 8x - 10x]\}$
 $= 10x - \{16x + 12x\}$
 $= 10x - 28x = -18x.$
77. $3 - \{c - [5 - (2c - 7 - 3c - 11) + 5c] - 6c - 20\}$
 $= 3 - \{-5c - 20 - [5 - (2c - 7 + 3c - 11) + 5c]\}$
 $= 3 - \{-5c - 20 - [5 - 5c + 18 + 5c]\}$
 $= 3 - \{-5c - 20 - [23]\}$
 $= 3 + 5c + 20 + 23 = 46 + 5c.$
78. $x + \{3y + [4x - (2y - 7x) - 3y] - (10y + 4x) + 8y\}$
 $= x + 3y + 4x - (2y - 7x) - 3y - (10y + 4x) + 8y$
 $= 5x + 8y - (2y - 7x) - (10y + 4x)$
 $= 5x + 8y - 2y + 7x - 10y - 4x = 8x - 4y.$
79. $1 - \{-[-(1 - x) - 1] - 1\} - \{x - (5 - 3x) - 7 + x\}$
 $= 1 - \{-[-1 + x - 1] - 1\} - \{2x - 7 - 5 + 3x\}$
 $= 1 - \{1 - x + 1 - 1\} - 5x + 12$
 $= 13 - 1 + x - 5x$
 $= 12 - 4x.$

FACTORING

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- | | |
|--|---|
| 4. $am - an + mx - nx$ $= a(m - n) + x(m - n)$ $= (a + x)(m - n).$ | 7. $ay - by - ab + b^2$ $= y(a - b) - b(a - b)$ $= (y - b)(a - b).$ |
| 5. $bc - bd + cx - dx$ $= b(c - d) + x(c - d)$ $= (b + x)(c - d).$ | 8. $x^2 - xy - 5x + 5y$ $= x(x - y) - 5(x - y)$ $= (x - 5)(x - y).$ |
| 6. $pq - px - rq + rx$ $= p(q - x) - r(q - x)$ $= (p - r)(q - x).$ | 9. $b^2 - bc + ab - ac$ $= b(b - c) + a(b - c)$ $= (b + a)(b - c).$ |

10. $x^2 + xy - ax - ay$
 $= x(x + y) - a(x + y)$
 $= (x - a)(x + y).$
11. $c^2 - 4c + ac - 4a$
 $= c(c - 4) + a(c - 4)$
 $= (c + a)(c - 4).$
12. $2x - y + 4x^2 - 2xy$
 $= (2x - y) + 2x(2x - y)$
 $= (1 + 2x)(2x - y).$
13. $1 - m + n - mn$
 $= (1 - m) + n(1 - m)$
 $= (1 + n)(1 - m).$
14. $2p + q + 6p^2 + 3pq$
 $= (2p + q) + 3p(2p + q)$
 $= (1 + 3p)(2p + q).$
15. $ar - rs - ab + bs$
 $= r(a - s) - b(a - s)$
 $= (r - b)(a - s).$
22. $12a^3 - 8ab - 3a^4 + 2a^2b = 4a(3a^2 - 2b) - a^2(3a^2 - 2b)$
 $= (4a - a^2)(3a^2 - 2b)$
 $= a(4 - a)(3a^2 - 2b).$
23. $3m^2n - 9mn^2 + am - 3an = 3mn(m - 3n) + a(m - 3n)$
 $= (3mn + a)(m - 3n).$
24. $15ab^2 - 9b^2c - 35ab + 21bc = 3b^2(5a - 3c) - 7b(5a - 3c)$
 $= (3b^2 - 7b)(5a - 3c)$
 $= b(3b - 7)(5a - 3c).$
25. $16ax + 12ay - 8bx - 6by = 4a(4x + 3y) - 2b(4x + 3y)$
 $= (4a - 2b)(4x + 3y)$
 $= 2(2a - b)(4x + 3y).$
26. $ax^2 - ax - axy + ay + x - 1 = ax(x - 1) - ay(x - 1) + (x - 1)$
 $= (ax - ay + 1)(x - 1).$
27. $xy + x - 3y^2 - 3y - 4y - 4 = x(y + 1) - 3y(y + 1) - 4(y + 1)$
 $= (x - 3y - 4)(y + 1).$
28. $ax - a - bx + b - cx + c = a(x - 1) - b(x - 1) - c(x - 1)$
 $= (a - b - c)(x - 1).$
29. $mx - nx - x - my + ny + y = x(m - n - 1) - y(m - n - 1)$
 $= (x - y)(m - n - 1).$
30. $a^2 - a - ab + b - 2ac + 2c = a(a - 1) - b(a - 1) - 2c(a - 1)$
 $= (a - b - 2c)(a - 1).$
16. $x^3 + x^2 + x + 1$
 $= x^2(x + 1) + (x + 1)$
 $= (x^2 + 1)(x + 1).$
17. $y^3 + y^2 - 3y - 3$
 $= y^2(y + 1) - 3(y + 1)$
 $= (y^2 - 3)(y + 1).$
18. $x^5 + x^3 + x^2y + y$
 $= x^3(x^2 + 1) + y(x^2 + 1)$
 $= (x^3 + y)(x^2 + 1).$
19. $2 - 2n - n^2 + n^3$
 $= 2(1 - n) - n^2(1 - n)$
 $= (2 - n^2)(1 - n).$
20. $x^2 - x - a + ax$
 $= x^2 - x + ax - a$
 $= x(x - 1) + a(x - 1)$
 $= (x + a)(x - 1).$
21. $3x^3 - 15x + 10y - 2x^2y$
 $= 3x^3 - 15x - 2x^2y + 10y$
 $= 3x(x^2 - 5) - 2y(x^2 - 5)$
 $= (3x - 2y)(x^2 - 5).$

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31. $mp^2 - np^2 + mq - nq + m - n = p^2(m - n) + q(m - n) + (m - n)$
 $= (p^2 + q + 1)(m - n).$
32. $ax^2 - bx^2 - ax + bx + a - b = x^2(a - b) - x(a - b) + (a - b)$
 $= (x^2 - x + 1)(a - b).$

33. $2mx^2 - nx^2 + n + 2nx - 4mx - 2m$
 $= 2mx^2 - 4mx - 2m - nx^2 + 2nx + n$
 $= 2m(x^2 - 2x - 1) - n(x^2 - 2x - 1)$
 $= (2m - n)(x^2 - 2x - 1).$
34. $bx^2 - b - xy - y + yx^2 - bx = bx^2 + yx^2 - bx - xy - b - y$
 $= x^2(b + y) - x(b + y) - (b + y)$
 $= (x^2 - x - 1)(b + y).$
35. $a^2x - a^2y - ay - y + x + ax = a^2x - a^2y + ax - ay + x - y$
 $= a^2(x - y) + a(x - y) + (x - y)$
 $= (a^2 + a + 1)(x - y).$
36. $2 - 3b + 3ab - 2a + 4a^2 - 6a^2b = 2 - 3b - (2a - 3ab) + (4a^2 - 6a^2b)$
 $= (2 - 3b) - a(2 - 3b) + 2a^2(2 - 3b)$
 $= (1 - a + 2a^2)(2 - 3b).$
37. $m^2 + mn + mn + n^2 + m + n = m(m + n) + n(m + n) + (m + n)$
 $= (m + n + 1)(m + n).$

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33. $x^2 + 2x(x - y) + (x - y)^2$
 $= (x + x - y)(x + x - y)$
 $= (2x - y)(2x - y).$
34. $a^2 - 4a(a - 1) + 4(a - 1)^2$
 $= [a - 2(a - 1)][a - 2(a - 1)]$
 $= (a - 2a + 2)(a - 2a + 2)$
 $= (2 - a)(2 - a).$
35. $c^2 - 6c(a - c) + 9(a - c)^2$
 $= [c - 3(a - c)][c - 3(a - c)]$
 $= [c - 3a + 3c][c - 3a + 3c]$
 $= (4c - 3a)(4c - 3a).$
36. $m^2 + 2m(m - n) + (m - n)^2$
 $= (m + m - n)(m + m - n)$
 $= (2m - n)(2m - n).$
37. $16 - 24(a - b) + 9(a - b)^2$
 $= [4 - 3(a - b)][4 - 3(a - b)]$
 $= (4 - 3a + 3b)(4 - 3a + 3b).$
38. $x^2 + 25(y^3 - x)^2 + 10x(y^3 - x)$
 $= x^2 + 10x(y^3 - x) + 25(y^3 - x)^2$
 $= [x + 5(y^3 - x)][x + 5(y^3 - x)]$
 $= (5y^3 - 4x)(5y^3 - 4x).$
39. $14a(x - y) + (x - y)^2 + 49a^2$
 $= (x - y)^2 + 14a(x - y) + 49a^2$
 $= (x - y + 7a)(x - y + 7a).$
40. $10m(m - 4) + 25m^2 + (m - 4)^2$
 $= 25m^2 + 10m(m - 4) + (m - 4)^2$
 $= [5m + (m - 4)][5m + (m - 4)]$
 $= (6m - 4)(6m - 4)$
 $= 4(3m - 2)(3m - 2).$

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41. $(a + b)^2 - 2(a + b)(b + c) + (b + c)^2$
 $= [(a + b) - (b + c)][(a + b) - (b + c)]$
 $= (a + b - b - c)(a + b - b - c) = (a - c)(a - c).$
42. $(a - 2x)^2 + 4(a - 2x)(2x - b) + 4(2x - b)^2$
 $= [a - 2x + 2(2x - b)][a - 2x + 2(2x - b)]$
 $= (a - 2x + 4x - 2b)(a - 2x + 4x - 2b)$
 $= (a + 2x - 2b)(a + 2x - 2b).$
43. $16(a - x)^2 + 32(a - x)(x + b) + 16(x + b)^2$
 $= 16[(a - x)^2 + 2(a - x)(x + b) + (x + b)^2]$
 $= 16(a - x + x + b)(a - x + x + b)$
 $= 16(a + b)(a + b).$
44. $(a + 3b)^2 - 4(a + 3b)(3b - 2c) + 4(3b - 2c)^2$
 $= [(a + 3b) - 2(3b - 2c)][(a + 3b) - 2(3b - 2c)]$
 $= (a + 3b - 6b + 4c)(a + 3b - 6b + 4c)$
 $= (a - 3b + 4c)(a - 3b + 4c).$

45. $(x^2 + x + 1)^2 + 2(x + 1)(x^2 + x + 1) + (x + 1)^2$
 $= (x^2 + x + 1 + x + 1)(x^2 + x + 1 + x + 1)$
 $= (x^2 + 2x + 2)(x^2 + 2x + 2).$
46. $(a + b + c)^2 + 2(a + b - c)(a + b + c) + (a + b - c)^2$
 $= (a + b + c + a + b - c)(a + b + c + a + b - c)$
 $= 2(a + b) \cdot 2(a + b)$
 $= 4(a + b)(a + b).$
47. $(x^3 - x^2)^2 + 2(x^3 - x^2)(x + 1) + (x^2 + 2x + 1)$
 $= (x^3 - x^2)^2 + 2(x^3 - x^2)(x + 1) + (x + 1)^2$
 $= (x^3 - x^2 + x + 1)(x^3 - x^2 + x + 1).$
11. $x^4 - 81$
 $= (x^2 + 9)(x^2 - 9)$
 $= (x^2 + 9)(x + 3)(x - 3).$
12. $a^4 - 16$
 $= (a^2 + 4)(a^2 - 4)$
 $= (a^2 + 4)(a + 2)(a - 2).$
13. $a^4 - b^4$
 $= (a^2 + b^2)(a^2 - b^2)$
 $= (a^2 + b^2)(a + b)(a - b).$
14. $a^{16} - b^8$
 $= (a^8 + b^4)(a^8 - b^4)$
 $= (a^8 + b^4)(a^4 + b^2)(a^4 - b^2)$
 $= (a^8 + b^4)(a^4 + b^2)(a^2 + b)(a^2 - b)$
18. $m^4 - 16n^4$
 $= (m^2 + 4n^2)(m^2 - 4n^2)$
 $= (m^2 + 4n^2)(m + 2n)(m - 2n).$

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20. $81m^4 - 1$
 $= (9m^2 + 1)(9m^2 - 1)$
 $= (9m^2 + 1)(3m + 1)(3m - 1).$
25. $400x^2 - 100y^2$
 $= 100(4x^2 - y^2)$
 $= 100(2x + y)(2x - y).$
26. $2a^8 - 2b^8$
 $= 2(a^8 - b^8)$
 $= 2(a^4 + b^4)(a^4 - b^4)$
 $= 2(a^4 + b^4)(a^2 + b^2)(a^2 - b^2)$
 $= 2(a^4 + b^4)(a^2 + b^2)(a + b)(a - b).$
27. $4m^4 - 4b^4$
 $= 4(m^4 - b^4)$
 $= 4(m^2 + b^2)(m^2 - b^2)$
 $= 4(m^2 + b^2)(m + b)(m - b).$
28. $3x^4 - 3y^6$
 $= 3(x^4 - y^6)$
 $= 3(x^2 + y^3)(x^2 - y^3).$
29. $5x^4 - 5y^{10}$
 $= 5(x^4 - y^{10})$
 $= 5(x^2 + y^5)(x^2 - y^5).$
30. $8x^{10} - 8y^8$
 $= 8(x^{10} - y^8)$
 $= 8(x^5 + y^4)(x^5 - y^4).$
31. $5x^8 - 5$
 $= 5(x^8 - 1)$
 $= 5(x^4 + 1)(x^4 - 1)$
 $= 5(x^4 + 1)(x^2 + 1)(x^2 - 1)$
 $= 5(x^4 + 1)(x^2 + 1)(x + 1)(x - 1).$
32. $3a^5 - 3a$
 $= 3a(a^4 - 1)$
 $= 3a(a^2 + 1)(a^2 - 1)$
 $= 3a(a^2 + 1)(a + 1)(a - 1).$
33. $x^3 - xy^2$
 $= x(x^2 - y^2)$
 $= x(x + y)(x - y).$
34. $5a^5y - 5ay$
 $= 5ay(a^4 - 1)$
 $= 5ay(a^2 + 1)(a^2 - 1)$
 $= 5ay(a^2 + 1)(a + 1)(a - 1).$
36. $x^{2n+1} - xy^{2n}$
 $= x(x^{2n} - y^{2n})$
 $= x(x^n + y^n)(x^n - y^n).$
38. $a^2 - (a + b)^2$
 $= (a + a + b)(a - a - b)$
 $= (2a + b)(-b)$
 $= -b(2a + b).$
39. $b^2 - (2a + b)^2$
 $= (b + 2a + b)(b - 2a - b)$
 $= (2a + 2b)(-2a)$
 $= -4a(a + b).$
40. $a^2 - (b + c)^2$
 $= (a + b + c)(a - b - c).$
41. $4c^2 - (b + c)^2$
 $= (2c + b + c)(2c - b - c)$
 $= (b + 3c)(c - b).$

43. $9a^2 - (2a - 5)^2$
 $= (3a + 2a - 5)(3a - 2a + 5)$
 $= (5a - 5)(a + 5)$
 $= 5(a - 1)(a + 5).$
44. $x^4 - (3x^2 - 2y)^2$
 $= (x^2 + 3x^2 - 2y)(x^2 - 3x^2 + 2y)$
 $= 2(2x^2 - y) \cdot 2(y - x^2)$
 $= 4(2x^2 - y)(y - x^2).$
45. $49a^2 - (5a - 4b)^2$
 $= (7a + 5a - 4b)(7a - 5a + 4b)$
 $= 4(3a - b) \cdot 2(a + 2b)$
 $= 8(3a - b)(a + 2b).$
47. $(2a + 3b)^2 - (a + b)^2$
 $= (2a + 3b + a + b)(2a + 3b - a - b)$
 $= (3a + 4b)(a + 2b).$
48. $(5a - 3b)^2 - (a - b)^2$
 $= (5a - 3b + a - b)(5a - 3b - a + b)$
 $= 2(3a - 2b) \cdot 2(2a - b)$
 $= 4(3a - 2b)(2a - b).$
49. $(2x + 5)^2 - (5 - 3x)^2$
 $= (2x + 5 + 5 - 3x)(2x + 5 - 5 + 3x)$
 $= (10 - x)(5x) = 5x(10 - x).$
50. $(a - 2b)^2 - (a - 5)^2$
 $= (a - 2b + a - 5)(a - 2b - a + 5)$
 $= (2a - 2b - 5)(5 - 2b).$

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51. $(2x - 3y)^2 - (3y + z)^2$
 $= (2x - 3y + 3y + z)(2x - 3y - 3y - z)$
 $= (2x + z)(2x - 6y - z).$
52. $(5b - 4c)^2 - (3a - 2c)^2$
 $= (5b - 4c + 3a - 2c)(5b - 4c - 3a + 2c)$
 $= (5b - 6c + 3a)(5b - 2c - 3a).$
53. $(4x - 3y)^2 - (2x - 3a)^2$
 $= (4x - 3y + 2x - 3a)(4x - 3y - 2x + 3a)$
 $= 3(2x - y - a)(2x - 3y + 3a).$
54. $(9x + 6y)^2 - (4x - 3y)^2$
 $= (9x + 6y + 4x - 3y)(9x + 6y - 4x + 3y)$
 $= (13x + 3y)(5x + 9y).$
55. $(x^3 + x^2)^2 - (2x + 2)^2$
 $= (x^3 + x^2 + 2x + 2)(x^3 + x^2 - 2x - 2)$
 $= [x^2(x + 1) + 2(x + 1)][x^2(x + 1) - 2(x + 1)]$
 $= (x^2 + 2)(x + 1)(x^2 - 2)(x + 1).$
56. $(a + b + c)^2 - (a - b - c)^2$
 $= (a + b + c + a - b - c)(a + b + c - a + b + c)$
 $= 2a \cdot 2(b + c) = 4a(b + c).$
59. $a^2 - 2ax + x^2 - n^2$
 $= (a - x)^2 - n^2$
 $= (a - x + n)(a - x - n).$
60. $b^2 + 2by + y^2 - n^2$
 $= (b + y)^2 - n^2$
 $= (b + y + n)(b + y - n).$
61. $1 - 4q + 4q^2 - a^2$
 $= (1 - 2q)^2 - a^2$
 $= (1 - 2q + a)(1 - 2q - a).$
62. $r^2 - 2rx + x^2 - 16t^2$
 $= (r - x)^2 - (4t)^2$
 $= (r - x + 4t)(r - x - 4t).$
63. $9a^2b - 6ab^2 + b^3 - 4bc^2$
 $= b(9a^2 - 6ab + b^2 - 4c^2)$
 $= b[(3a - b)^2 - (2c)^2]$
 $= b(3a - b + 2c)(3a - b - 2c).$
64. $4a^2c + 12abc + 9b^2c - 4c^3$
 $= c(4a^2 + 12ab + 9b^2 - 4c^2)$
 $= c[(2a + 3b)^2 - (2c)^2]$
 $= c(2a + 3b + 2c)(2a + 3b - 2c).$
65. $3x^2y - 12xy^2 + 12y^3 - 3x^6y$
 $= 3y(x^2 - 4xy + 4y^2 - x^6)$
 $= 3y[(x - 2y)^2 - (x^3)^2]$
 $= 3y(x - 2y + x^3)(x - 2y - x^3).$

$$\begin{aligned}
 66. \quad & 4an^4 - 16a^2n^2 + 16a^3 - 4an^6 \\
 &= 4a(n^4 - 4an^2 + 4a^2 - n^6) \\
 &= 4a[(n^2 - 2a)^2 - (n^3)^2] \\
 &= 4a(n^2 - 2a + n^3)(n^2 - 2a - n^3).
 \end{aligned}$$

$$\begin{aligned}
 67. \quad & c^2 - a^2 - b^2 - 2ab \\
 &= c^2 - (a^2 + 2ab + b^2) \\
 &= c^2 - (a + b)^2 \\
 &= (c + a + b)(c - a - b).
 \end{aligned}$$

$$\begin{aligned}
 68. \quad & b^2 - x^2 - y^2 + 2xy \\
 &= b^2 - (x^2 - 2xy + y^2) \\
 &= b^2 - (x - y)^2 \\
 &= (b + x - y)(b - x + y).
 \end{aligned}$$

$$\begin{aligned}
 69. \quad & 4c^2 - x^2 - y^2 - 2xy \\
 &= 4c^2 - (x^2 + 2xy + y^2) \\
 &= (2c)^2 - (x + y)^2 \\
 &= (2c + x + y)(2c - x - y).
 \end{aligned}$$

$$\begin{aligned}
 70. \quad & 9c^2 - x^2 - y^2 + 2xy \\
 &= 9c^2 - (x^2 - 2xy + y^2) \\
 &= (3c)^2 - (x - y)^2 \\
 &= (3c + x - y)(3c - x + y).
 \end{aligned}$$

77.

$$\begin{aligned}
 & 4x^2 + 9 - 12x + 10mn - m^2 - 25n^2 \\
 &= 4x^2 - 12x + 9 - m^2 + 10mn - 25n^2 \\
 &= (2x - 3)^2 - (m - 5n)^2 \\
 &= (2x - 3 + m - 5n)(2x - 3 - m + 5n).
 \end{aligned}$$

$$\begin{aligned}
 71. \quad & x^3 - a^2x - 4b^2x - 4abx \\
 &= x(x^2 - a^2 - 4ab - 4b^2) \\
 &= x[x^2 - (a + 2b)^2] \\
 &= x(x + a + 2b)(x - a - 2b)
 \end{aligned}$$

$$\begin{aligned}
 72. \quad & bc^2 - 9a^2b - b^3 - 6ab^2 \\
 &= b(c^2 - 9a^2 - 6ab - b^2) \\
 &= b[c^2 - (3a + b)^2] \\
 &= b(c + 3a + b)(c - 3a - b).
 \end{aligned}$$

$$\begin{aligned}
 73. \quad & ab^2 - 4a^3 - 12a^2c - 9ac^2 \\
 &= a(b^2 - 4a^2 - 12ac - 9c^2) \\
 &= a[b^2 - (2a + 3c)^2] \\
 &= a(b + 2a + 3c)(b - 2a - 3c).
 \end{aligned}$$

$$\begin{aligned}
 74. \quad & 27c^2 - 12a^2 + 36ab - 27b^2 \\
 &= 3(9c^2 - 4a^2 + 12ab - 9b^2) \\
 &= 3[(3c)^2 - (2a - 3b)^2] \\
 &= 3(3c + 2a - 3b)(3c - 2a + 3b).
 \end{aligned}$$

$$\begin{aligned}
 75. \quad & a^2 - 2ab + b^2 - c^2 + 2cd - d^2 \\
 &= (a - b)^2 - (c - d)^2 \\
 &= (a - b + c - d)(a - b - c + d).
 \end{aligned}$$

$$\begin{aligned}
 76. \quad & x^2 - 2xy + y^2 - m^2 + 10m - 25 \\
 &= (x - y)^2 - (m - 5)^2 \\
 &= (x - y + m - 5)(x - y - m + 5)
 \end{aligned}$$

78.

$$\begin{aligned}
 & x^2 - a^2 + y^2 - b^2 + 2xy - 2ab \\
 &= x^2 + 2xy + y^2 - a^2 - 2ab - b^2 \\
 &= (x + y)^2 - (a + b)^2 \\
 &= (x + y + a + b)(x + y - a - b).
 \end{aligned}$$

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$$\begin{aligned}
 18. \quad & +6a^2 = (+2a) \times (+3a) \text{ and } +5a = (+2a) + (+3a); \\
 & \therefore x^2 + 5ax + 6a^2 = (x + 2a)(x + 3a).
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & +5a^2 = (-a) \times (-5a) \text{ and } (-a) + (-5a) = -6a; \\
 & \therefore x^2 - 6ax + 5a^2 = (x - a)(x - 5a).
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & -12b^2 = (+2b) \times (-6b) \text{ and } -4b = (+2b) + (-6b); \\
 & \therefore y^2 - 4by - 12b^2 = (y + 2b)(y - 6b).
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & -28n^2 = (+4n) \times (-7n) \text{ and } -3n = (+4n) + (-7n); \\
 & \therefore y^2 - 3ny - 28n^2 = (y + 4n)(y - 7n).
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & -2a^2n^2 = (+an) \times (-2an) \text{ and } -an = (+an) + (-2an); \\
 & \therefore z^2 - anz - 2a^2n^2 = (z + an)(z - 2an).
 \end{aligned}$$

$$\begin{aligned}
 23. \quad & +90c^2 = (+9c) \times (+10c) \text{ and } +19c = (+9c) + (+10c); \\
 & \therefore x^4 + 19cx^2 + 90c^2 = (x^2 + 9c)(x^2 + 10c).
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & +20a^2 = (+2a) \times (+10a) \text{ and } +12a = (+2a) + (+10a); \\
 & \therefore x^6 + 12ax^3 + 20a^2 = (x^3 + 2a)(x^3 + 10a).
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & +24b^4 = (-3b^2) \times (-8b^2) \text{ and } -11b^2 = (-3b^2) + (-8b^2); \\
 & \therefore x^{10} - 11b^2x^5 + 24b^4 = (x^5 - 3b^2)(x^5 - 8b^2).
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & 5nx^2 - 55nx + 150n = 5n(x^2 - 11x + 30) \\
 & \quad \quad \quad = 5n(x - 5)(x - 6).
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & 3a^2bx^2 - 3a^2bx - 6a^2b = 3a^2b(x^2 - x - 2) \\
 & \quad \quad \quad = 3a^2b(x + 1)(x - 2).
 \end{aligned}$$

$$28. \quad 12 m^2 x^2 - 60 m^2 b x + 72 m^2 b^2 = 12 m^2 (x^2 - 5 b x + 6 b^2) \\ = 12 m^2 (x - 2 b) (x - 3 b).$$

$$29. \quad 4 a x + 2 a x^2 - 48 a = 2 a (x^2 + 2 x - 24) \\ = 2 a (x - 4) (x + 6).$$

$$30. \quad 11 a^2 x - 55 a x + 66 x = 11 x (a^2 - 5 a + 6) \\ = 11 x (a - 2) (a - 3).$$

$$31. \quad 20 b x + 10 b^2 - 630 x^2 = 10 (b^2 + 2 b x - 63 x^2) \\ = 10 (b - 7 x) (b + 9 x).$$

$$32. \quad -ab = (+b) \times (-a) \text{ and } b - a = (+b) + (-a); \\ \therefore x^2 + (b - a)x - ab = (x + b)(x - a).$$

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$$5. \quad \text{First factor, try } 2x + 1, 2x - 1, 2x + 3, 2x - 3, 2x + 5, 2x - 5. \\ \text{Second factor, try } \frac{x - 15}{x + 15}, \frac{x - 5}{x + 5}, \frac{x - 3}{x + 3}, \frac{x - 1}{x + 1}. \\ \text{Products, 2d terms, } -29x, +29x, -7x, +7x, -x, +x. \\ \therefore 2x^2 + x - 15 = (2x - 5)(x + 3).$$

$$6. \quad 9x^2 - 42x + 40 = (3x)^2 - 14(3x) + 40 \\ \text{Put } m \text{ for } 3x, \quad = m^2 - 14m + 40 \\ = (m - 4)(m - 10) \\ \text{Put } 3x \text{ for } m, \quad = (3x - 4)(3x - 10).$$

$$7. \quad \text{First factor, try } \frac{5x + 1}{x + 6}, \frac{5x + 2}{x + 3}, \frac{5x + 3}{x + 2}, \dots \\ \text{Second factor, try } \dots \\ \text{Products, 2d terms, } +31x, +17x, +13x, \dots \\ \therefore 5x^2 + 13x + 6 = (5x + 3)(x + 2).$$

$$8. \quad \text{First factor, try } 3x - 1, 3x - 2, \dots \\ \text{Second factor, try } \frac{x - 10}{x - 5}, \dots \\ \text{Products, 2d terms, } -31x, -17x, \dots \\ \therefore 3x^2 - 17x + 10 = (3x - 2)(x - 5).$$

$$9. \quad 25x^2 + 15x + 2 = (5x)^2 + 3(5x) + 2 \\ \text{Put } m \text{ for } 5x, \quad = m^2 + 3m + 2 \\ = (m + 1)(m + 2) \\ \text{Put } 5x \text{ for } m, \quad = (5x + 1)(5x + 2).$$

$$10. \quad 16x^2 + 20x - 66 = (4x)^2 + 5(4x) - 66 \\ \text{Put } m \text{ for } 4x, \quad = m^2 + 5m - 66 \\ = (m - 6)(m + 11) \\ \text{Put } 4x \text{ for } m, \quad = (4x - 6)(4x + 11) \\ = 2(2x - 3)(4x + 11).$$

$$11. \quad 36x^2 - 48x - 20 = 4(9x^2 - 12x - 5) \\ \text{Put } m \text{ for } 3x, \quad = 4(m^2 - 4m - 5) \\ = 4(m + 1)(m - 5) \\ \text{Put } 3x \text{ for } m, \quad = 4(3x + 1)(3x - 5).$$

12. Since $\sqrt{9x^2}$, or $3x$, is not exactly contained in $43x$, the factors of $9x^2$ are not $3x$ and $3x$, and hence are $9x$ and x .

$$\text{First factor, try } 9x - 1, 9x - 2, \dots \\ \text{Second factor, try } \frac{x + 10}{x + 5}, \dots \\ \text{Products, 2d terms, } +89x, +43x, \dots \\ \therefore 9x^2 + 43x - 10 = (9x - 2)(x + 5).$$

$$\begin{aligned}
24. \quad 6x^2 - 13x + 6 &= \frac{36x^2 - 78x + 36}{6} = \frac{(6x)^2 - 13(6x) + 36}{6} \\
&= \frac{(6x - 4)(3x - 9)}{6} = \frac{2(3x - 2) \cdot 3(2x - 3)}{6} \\
&= (3x - 2)(2x - 3). \\
25. \quad 15x^2 - 14x - 8 &= \frac{225x^2 - 210x - 120}{15} = \frac{(15x)^2 - 14(15x) - 120}{15} \\
&= \frac{(15x + 6)(15x - 20)}{15} = \frac{3(5x + 2) \cdot 5(3x - 4)}{15} \\
&= (5x + 2)(3x - 4). \\
26. \quad 15x^2 + 17x - 4 &= \frac{225x^2 + 255x - 60}{15} = \frac{(15x)^2 + 17(15x) - 60}{15} \\
&= \frac{(15x - 3)(15x + 20)}{15} = \frac{3(5x - 1) \cdot 5(3x + 4)}{15} \\
&= (5x - 1)(3x + 4). \\
27. \quad 21a^2 - a - 10 &= \frac{441a^2 - 21a - 210}{21} = \frac{(21a)^2 - 1(21a) - 210}{21} \\
&= \frac{(21a + 14)(21a - 15)}{21} = \frac{7(3a + 2) \cdot 3(7a - 5)}{21} \\
&= (3a + 2)(7a - 5). \\
28. \quad 18x^2 - 3x - 36 &= \frac{36x^2 - 6x - 72}{2} = \frac{(6x)^2 - 1(6x) - 72}{2} \\
&= \frac{(6x + 8)(6x - 9)}{2} = \frac{2(3x + 4) \cdot 3(2x - 3)}{2} \\
&= 3(3x + 4)(2x - 3).
\end{aligned}$$

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$$\begin{aligned}
2. \quad x^6 - y^6 &= (x^3)^2 - (y^3)^2 \\
&= (x^3 + y^3)(x^3 - y^3) \\
&= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2). \\
3. \quad x^6 - 1 &= (x^3)^2 - (1)^2 \\
&= (x^3 + 1)(x^3 - 1), \text{ or } (x^3 + 1^3)(x^3 - 1^3) \\
&= (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1). \\
4. \quad a^8 - b^8 &= (a^4 + b^4)(a^4 - b^4) \\
&= (a^4 + b^4)(a^2 + b^2)(a^2 - b^2) \\
&= (a^4 + b^4)(a^2 + b^2)(a + b)(a - b). \\
5. \quad x^4 - 16 &= (x^2 + 4)(x^2 - 4) \\
&= (x^2 + 4)(x + 2)(x - 2). \\
6. \quad x^4 - 81 &= (x^2 + 9)(x^2 - 9) \\
&= (x^2 + 9)(x + 3)(x - 3). \\
7. \quad a^4 - 625 &= (a^2 + 25)(a^2 - 25) \\
&= (a^2 + 25)(a + 5)(a - 5). \\
8. \quad 1 - b^8 &= (1^2 - (b^4)^2) \\
&= (1 + b^4)(1 - b^4), \text{ or } (1^4 + b^4)(1^4 - b^4) \\
&= (1 + b)(1 - b + b^2)(1 - b)(1 + b + b^2). \\
9. \quad 64 - y^6 &= (8)^2 - (y^3)^2 \\
&= (8 + y^3)(8 - y^3), \text{ or } (2^3 + y^3)(2^3 - y^3) \\
&= (2 + y)(4 - 2y + y^2)(2 - y)(4 + 2y + y^2).
\end{aligned}$$

10. $1 - x^8 = 1^2 - (x^4)^2$
 § 128, $= (1 + x^4)(1 - x^4)$
 $= (1 + x^4)(1 + x^2)(1 - x^2)$
 $= (1 + x^4)(1 + x^2)(1 + x)(1 - x).$
18. $r^6 + s^6 = (r^2)^3 + (s^2)^3$
 § 132, $= (r^2 + s^2)(r^4 - r^2s^2 + s^4).$
23. $1 + x^6 = (1)^3 + (x^2)^3$
 § 132, $= (1 + x^2)(1 - x^2 + x^4).$
24. § 123, $x + x^6 = x(1 + x^5), \text{ or } x(1^5 + x^5)$
 § 132, $= x(1 + x)(1 - x + x^2 - x^3 + x^4).$
25. § 123, $32n - n^6 = n(32 - n^5), \text{ or } n(2^5 - n^5)$
 § 133, $= n(2 - n)(16 + 8n + 4n^2 + 2n^3 + n^4).$
26. $m^3 - x^6 = (m)^3 - (x^2)^3$
 § 133, $= (m - x^2)(m^2 + mx^2 + x^4).$
27. § 123, $b^4 - a^2b^4 = b^4(1 - a^2)$
 § 128, $= b^4(1 + a)(1 - a).$
28. $125 + a^3 = 5^3 + a^3$
 § 132, $= (5 + a)(25 - 5a + a^2).$

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8. $x^3 - 9x^2 + 23x - 15$
 Substituting 1 for x , $= 1 - 9 + 23 - 15 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 9x^2 + 23x - 15 = (x - 1)(x^2 - 8x + 15)$
 § 130, $= (x - 1)(x - 3)(x - 5).$
9. $x^3 - 13x^2 + 47x - 35$
 Substituting 1 for x , $= 1 - 13 + 47 - 35 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 13x^2 + 47x - 35 = (x - 1)(x^2 - 12x + 35)$
 § 130, $= (x - 1)(x - 5)(x - 7).$
10. $x^3 - 14x^2 + 35x - 22$
 Substituting 1 for x , $= 1 - 14 + 35 - 22 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 14x^2 + 35x - 22 = (x - 1)(x^2 - 13x + 22)$
 § 130, $= (x - 1)(x - 2)(x - 11).$
11. $x^3 - 4x^2 - 7x + 10$
 Substituting 1 for x , $= 1 - 4 - 7 + 10 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 4x^2 - 7x + 10 = (x - 1)(x^2 - 3x - 10)$
 § 130, $= (x - 1)(x + 2)(x - 5).$
12. $x^3 - 6x^2 - 9x + 14$
 Substituting 1 for x , $= 1 - 6 - 9 + 14 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 6x^2 - 9x + 14 = (x - 1)(x^2 - 5x - 14)$
 § 130, $= (x - 1)(x + 2)(x - 7).$
13. $x^3 - 12x^2 + 41x - 30$
 Substituting 1 for x , $= 1 - 12 + 41 - 30 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 12x^2 + 41x - 30 = (x - 1)(x^2 - 11x + 30)$
 § 130, $= (x - 1)(x - 5)(x - 6).$
14. $x^3 - 11x^2 + 31x - 21$
 Substituting 1 for x , $= 1 - 11 + 31 - 21 = 0; \therefore x - 1$ is a factor,
 and $x^3 - 11x^2 + 31x - 21 = (x - 1)(x^2 - 10x + 21)$
 § 130, $= (x - 1)(x - 3)(x - 7).$

15. Substituting 1 for x , $x^3 - 10x^2 + 29x - 20 = 1 - 10 + 29 - 20 = 0$; $\therefore x - 1$ is a factor,
 and $x^3 - 10x^2 + 29x - 20 = (x - 1)(x^2 - 9x + 20)$
 § 130, $= (x - 1)(x - 4)(x - 5)$.

16. Substituting 1 for x , $x^3 - 16x^2 + 71x - 56 = 1 - 16 + 71 - 56 = 0$; $\therefore x - 1$ is a factor,
 and $x^3 - 16x^2 + 71x - 56 = (x - 1)(x^2 - 15x + 56)$
 § 130, $= (x - 1)(x - 7)(x - 8)$.

17. Substituting 1 for x , $x^3 - 57x + 56 = 1 - 57 + 56 = 0$; $\therefore x - 1$ is a factor,
 and $x^3 - 57x + 56 = (x - 1)(x^2 + x - 56)$
 § 130, $= (x - 1)(x - 7)(x + 8)$.

18. Substituting 1 for x , $x^3 - 21x + 20 = 1 - 21 + 20 = 0$; $\therefore x - 1$ is a factor,
 and $x^3 - 21x + 20 = (x - 1)(x^2 + x - 20)$
 § 130, $= (x - 1)(x - 4)(x + 5)$.

19. Substituting -1 for x , $x^3 - 31x - 30 = -1 + 31 - 30 = 0$; $\therefore x + 1$ is a factor,
 and $x^3 - 31x - 30 = (x + 1)(x^2 - x - 30)$
 § 130, $= (x + 1)(x + 5)(x - 6)$.

20. Substituting 1 for x , $x^3 - 13x + 12 = 1 - 13 + 12 = 0$; $\therefore x - 1$ is a factor,
 and $x^3 - 13x + 12 = (x - 1)(x^2 + x - 12)$
 § 130, $= (x - 1)(x - 3)(x + 4)$.

21. Substituting 1 for x , $x^3 - 7x + 6 = 1 - 7 + 6 = 0$; $\therefore x - 1$ is a factor,
 and $x^3 - 7x + 6 = (x - 1)(x^2 + x - 6)$
 § 130, $= (x - 1)(x - 2)(x + 3)$.

22. Substituting 2 for x , $x^3 - 19x + 30 = 8 - 38 + 30 = 0$; $\therefore x - 2$ is a factor,
 and $x^3 - 19x + 30 = (x - 2)(x^2 + 2x - 15)$
 § 130, $= (x - 2)(x - 3)(x + 5)$.

23. Substituting -2 for x , $x^3 - 67x - 126 = -8 + 134 - 126 = 0$; $\therefore x + 2$ is a factor,
 and $x^3 - 67x - 126 = (x + 2)(x^2 - 2x - 63)$
 $= (x + 2)(x + 7)(x - 9)$.

24. Substituting -2 for x , $x^3 - 39x - 70 = -8 + 78 - 70 = 0$; $\therefore x + 2$ is a factor,
 and $x^3 - 39x - 70 = (x + 2)(x^2 - 2x - 35)$
 § 130, $= (x + 2)(x + 5)(x - 7)$.

25. Substituting -2 for a , $a^3 + 4a^2 - 11a - 30 = -8 + 16 + 22 - 30 = 0$; $\therefore a + 2$ is a factor,
 and $a^3 + 4a^2 - 11a - 30 = (a + 2)(a^2 + 2a - 15)$
 § 130, $= (a + 2)(a - 3)(a + 5)$.

26. Substituting -2 for a , $a^3 + 9a^2 + 26a + 24 = -8 + 36 + 52 + 24 = 0$; $\therefore a + 2$ is a factor,
 and $a^3 + 9a^2 + 26a + 24 = (a + 2)(a^2 + 7a + 12)$
 § 130, $= (a + 2)(a + 3)(a + 4)$.

27. $m^3 - 6m^2 - m + 30$
 Substituting -2 for m , $= -8 - 24 + 2 + 30 = 0$; $\therefore m + 2$ is a factor,
 and $m^3 - 6m^2 - m + 30 = (m + 2)(m^2 - 8m + 15)$
 § 130, $= (m + 2)(m - 3)(m - 5)$.

28. $b^3 - 5b^2 - 29b + 105$
 Substituting 3 for b , $= 27 - 45 - 87 + 105 = 0$; $\therefore b - 3$ is a factor,
 and $b^3 - 5b^2 - 29b + 105 = (b - 3)(b^2 - 2b - 35)$
 § 130, $= (b - 3)(b + 5)(b - 7)$.

29. $a^3 + 10a^2 - 17a - 66$
 Substituting -2 for a , $= -8 + 40 + 34 - 66 = 0$; $\therefore a + 2$ is a factor,
 and $a^3 + 10a^2 - 17a - 66 = (a + 2)(a^2 + 8a - 33)$
 § 130, $= (a + 2)(a - 3)(a + 11)$.

30. $m^3 + 7m^2 + 2m - 40$
 Substituting 2 for m , $= 8 + 28 + 4 - 40 = 0$; $\therefore m - 2$ is a factor,
 and $m^3 + 7m^2 + 2m - 40 = (m - 2)(m^2 + 9m + 20)$
 § 130, $= (m - 2)(m + 4)(m + 5)$.

31. $b^3 + 16b^2 + 73b + 90$
 Substituting -2 for b , $= -8 + 64 - 146 + 90 = 0$; $\therefore b + 2$ is a factor,
 and $b^3 + 16b^2 + 73b + 90 = (b + 2)(b^2 + 14b + 45)$
 § 130, $= (b + 2)(b + 5)(b + 9)$.

32. $n^3 + 12n^2 + 41n + 42$
 Substituting -2 for n , $= -8 + 48 - 82 + 42 = 0$; $n + 2$ is a factor,
 and $n^3 + 12n^2 + 41n + 42 = (n + 2)(n^2 + 10n + 21)$
 § 130, $= (n + 2)(n + 3)(n + 7)$.

33. $x^4 - 15x^2 + 10x + 24$
 Substituting -1 for x , $= 1 - 15 - 10 + 24 = 0$.
 Substituting 2 for x , $= 16 - 60 + 20 + 24 = 0$.
 Hence, $x + 1$ and $x - 2$ are factors.

$\therefore x^4 - 15x^2 + 10x + 24 = (x + 1)(x - 2)(x^2 + x - 12)$
 § 130, $= (x + 1)(x - 2)(x - 3)(x + 4)$.

34. $x^4 - 25x^2 + 60x - 36$
 Substituting 1 for x , $= 1 - 25 + 60 - 36 = 0$.
 Substituting 2 for x , $= 16 - 100 + 120 - 36 = 0$.
 Hence, $x - 1$ and $x - 2$ are factors.

$\therefore x^4 - 25x^2 + 60x - 36 = (x - 1)(x - 2)(x^2 + 3x - 18)$
 § 130, $= (x - 1)(x - 2)(x - 3)(x + 6)$.

35. $x^4 + 13x^2 - 54x + 40$
 Substituting 1 for x , $= 1 + 13 - 54 + 40 = 0$.
 Substituting 2 for x , $= 16 + 52 - 108 + 40 = 0$.
 Hence, $x - 1$ and $x - 2$ are factors.

$\therefore x^4 + 13x^2 - 54x + 40 = (x - 1)(x - 2)(x^2 + 3x + 20)$.

36. $x^4 + 22x^2 + 27x - 50$
 Substituting 1 for x , $= 1 + 22 + 27 - 50 = 0$.
 Substituting -2 for x , $= 16 + 88 - 54 - 50 = 0$.
 Hence, $x - 1$ and $x + 2$ are factors.

$\therefore x^4 + 22x^2 + 27x - 50 = (x - 1)(x + 2)(x^2 - x + 25)$.

37. $x^4 - 9x^2y^2 - 4xy^3 + 12y^4$
 Substituting y for x , $= y^4 - 9y^4 - 4y^4 + 12y^4 = 0$.
 Substituting $-2y$ for x , $= 16y^4 - 36y^4 + 8y^4 + 12y^4 = 0$.
 Hence, $x - y$ and $x + 2y$ are factors.

$\therefore x^4 - 9x^2y^2 - 4xy^3 + 12y^4 = (x - y)(x + 2y)(x^2 - xy - 6y^2)$
 § 130, $= (x - y)(x + 2y)(x + 2y)(x - 3y)$.

38. $x^4 - 9x^2y^2 + 12xy^3 - 4y^4$
 Substituting y for x , $= y^4 - 9y^4 + 12y^4 - 4y^4 = 0$.
 Substituting $2y$ for x , $= 16y^4 - 36y^4 + 24y^4 - 4y^4 = 0$.
 Hence, $x - y$ and $x - 2y$ are factors.
 $\therefore x^4 - 9x^2y^2 + 12xy^3 - 4y^4 = (x - y)(x - 2y)(x^2 + 3xy - 2y^2)$.

39. $x^4 - x^3 - 7x^2 + x + 6$
 Substituting 1 for x , $= 1 - 1 - 7 + 1 + 6 = 0$.
 Substituting -1 for x , $= 1 + 1 - 7 - 1 + 6 = 0$.
 Hence, $x - 1$ and $x + 1$ are factors.
 $\therefore x^4 - x^3 - 7x^2 + x + 6 = (x - 1)(x + 1)(x^2 - x - 6)$
 § 130, $= (x - 1)(x + 1)(x + 2)(x - 3)$.

40. $x^4 - 9x^3 + 21x^2 + x - 30$
 Substituting -1 for x , $= 1 + 9 + 21 - 1 - 30 = 0$.
 Substituting 2 for x , $= 16 - 72 + 84 + 2 - 30 = 0$.
 Hence, $x + 1$ and $x - 2$ are factors.
 $\therefore x^4 - 9x^3 + 21x^2 + x - 30 = (x + 1)(x - 2)(x^2 - 8x + 15)$
 § 130, $= (x + 1)(x - 2)(x - 3)(x - 5)$.

41. $x^4 + 8x^3 + 14x^2 - 8x - 15$
 Substituting 1 for x , $= 1 + 8 + 14 - 8 - 15 = 0$.
 Substituting -1 for x , $= 1 - 8 + 14 + 8 - 15 = 0$.
 Hence, $x - 1$ and $x + 1$ are factors.
 $\therefore x^4 + 8x^3 + 14x^2 - 8x - 15 = (x - 1)(x + 1)(x^2 + 8x + 15)$
 § 130, $= (x - 1)(x + 1)(x + 3)(x + 5)$.

42. $x^5 - 4x^4 + 19x^3 - 28x^2 + 12x$
 Substituting 1 for x , $= 1 - 4 + 19 - 28 + 12 = 0$.
 Substituting 2 for x , $= 32 - 64 + 76 - 56 + 12 = 0$.
 Hence, $x - 1$ and $x - 2$ are factors. Removing these factors by division,
 the quotient is $x^3 - x^2 - 5x + 6$.
 Substituting 2 for x , $x^3 - x^2 - 5x + 6 = 8 - 4 - 10 + 6 = 0$.
 Hence, $x - 2$ is a factor of $x^3 - x^2 - 5x + 6$, and the other factor is
 $x^2 + x - 3$.
 $\therefore x^5 - 4x^4 + 19x^3 - 28x^2 + 12x = (x - 1)(x - 2)(x - 2)(x^2 + x - 3)$.

43. $x^5 - 18x^3 + 30x^2 - 19x + 30$
 Substituting 2 for x , $= 32 - 144 + 120 - 38 + 30 = 0$.
 Substituting 3 for x , $= 243 - 486 + 270 - 57 + 30 = 0$.
 Hence, $x - 2$ and $x - 3$ are factors. Removing these factors by division,
 the remaining factor is found to be $x^3 + 5x^2 + x + 5$.
 $x^3 + 5x^2 + x + 5 = x^2(x + 5) + 1(x + 5)$
 $= (x^2 + 1)(x + 5)$.
 $\therefore x^5 - 18x^3 + 30x^2 - 19x + 30 = (x - 2)(x - 3)(x^2 + 1)(x + 5)$.

44. $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
 Substituting 2 for x , $= 32 - 160 + 320 - 320 + 160 - 32 = 0$.
 Hence, $x - 2$ is a factor. Removing this factor by division, the quotient
 is $x^4 - 8x^3 + 24x^2 - 32x + 16$.
 Substituting 2 for x , $= 16 - 64 + 96 - 64 + 16 = 0$.
 Hence, $x - 2$ is a factor. Removing this factor by division, the quotient
 is $x^3 - 6x^2 + 12x - 8$.
 Substituting 2 for x , $= 8 - 24 + 24 - 8 = 0$.
 Hence, $x - 2$ is a factor, and the other factor is $x^2 - 4x + 4$, which is
 equal to $(x - 2)(x - 2)$.
 $\therefore x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32 = (x - 2)(x - 2)(x - 2)(x - 2)(x - 2)$.

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$$\begin{aligned}
 & 2. \quad 9x^2 + 4y^2 + 25z^2 - 12xy + 30xz - 20yz \\
 \text{Sol. ex. 1,} &= (3x)^2 + (-2y)^2 + (5z)^2 + 2(3x)(-2y) + 2(3x)(5z) + 2(-2y)(5z) \\
 \S 95, &= (3x - 2y + 5z)(3x - 2y + 5z). \\
 & 3. \quad 25m^2 + 36n^2 + p^2 - 60mn - 10mp + 12np \\
 \text{Sol. ex. 1,} &= (5m)^2 + (-6n)^2 + (-p)^2 + 2(5m)(-6n) + 2(5m)(-p) \\
 & \quad + 2(-6n)(-p) \\
 \S 95, &= (5m - 6n - p)(5m - 6n - p). \\
 & 4. \quad a^2 + 16x^4 + 36y^2 - 8ax^2 + 12ay - 48x^2y \\
 \text{Sol. ex. 1,} &= (a)^2 + (-4x^2)^2 + (6y)^2 + 2(a)(-4x^2) + 2(a)(6y) + 2(-4x^2)(6y) \\
 \S 95, &= (a - 4x^2 + 6y)(a - 4x^2 + 6y). \\
 & 5. \quad x^2 + 4a^2 + b^2 + y^2 + 4ax - 2bx + 2xy - 4ab + 4ay - 2by \\
 \text{Sol. ex. 1,} &= (x)^2 + (2a)^2 + (-b)^2 + (y)^2 + 2(x)(2a) + 2(x)(-b) + 2(x)(y) \\
 & \quad + 2(2a)(-b) + 2(2a)(y) + 2(-b)(y) \\
 \S 95, &= (x + 2a - b + y)(x + 2a - b + y). \\
 & 6. \quad m^2 + 4n^2 + a^2 + 9 - 4mn - 2am + 6m + 4an - 12n - 6a \\
 \text{Sol. ex. 1,} &= (m)^2 + (-2n)^2 + (-a)^2 + (3)^2 + 2(m)(-2n) + 2(m)(-a) \\
 & \quad + 2(m)(3) + 2(-2n)(-a) + 2(-2n)(3) + 2(-a)(3) \\
 \S 95, &= (m - 2n - a + 3)(m - 2n - a + 3).
 \end{aligned}$$

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$$\begin{aligned}
 & 4. \quad x^4 + x^2y^2 + y^4 \\
 & \quad = x^4 + 2x^2y^2 + y^4 - x^2y^2 \\
 & \quad = (x^2 + y^2)^2 - (xy)^2 \\
 & \quad = (x^2 + xy + y^2)(x^2 - xy + y^2). \\
 & 5. \quad a^8 + a^4b^4 + b^8 \\
 & \quad = a^8 + 2a^4b^4 + b^8 - a^4b^4 \\
 & \quad = (a^4 + b^4)^2 - (a^2b^2)^2 \\
 & \quad = (a^4 + a^2b^2 + b^4)(a^4 - a^2b^2 + b^4) \\
 \text{Ex. 1,} & \quad = (a^2 + ab + b^2)(a^2 - ab + b^2)(a^4 - a^2b^2 + b^4). \\
 & 6. \quad p^4 + p^2q^2 + q^4 \\
 & \quad = p^4 + 2p^2q^2 + q^4 - p^2q^2 \\
 & \quad = (p^2 + q^2)^2 - (pq)^2 \\
 & \quad = (p^2 + pq + q^2)(p^2 - pq + q^2). \\
 & 7. \quad 9x^4 + 20x^2y^2 + 16y^4 \\
 & \quad = 9x^4 + 24x^2y^2 + 16y^4 - 4x^2y^2 \\
 & \quad = (3x^2 + 4y^2)^2 - (2xy)^2 \\
 & \quad = (3x^2 + 2xy + 4y^2)(3x^2 - 2xy + 4y^2). \\
 & 8. \quad 4a^4 + 11a^2b^2 + 9b^4 \\
 & \quad = 4a^4 + 12a^2b^2 + 9b^4 - a^2b^2 \\
 & \quad = (2a^2 + 3b^2)^2 - (ab)^2 \\
 & \quad = (2a^2 + ab + 3b^2)(2a^2 - ab + 3b^2). \\
 & 9. \quad 16a^4 - 17a^2x^2 + x^4 \\
 & \quad = 16a^4 + 8a^2x^2 + x^4 - 25a^2x^2 \\
 & \quad = (4a^2 + 5ax + x^2)(4a^2 - 5ax + x^2) \\
 & \quad = (4a + x)(a + x)(4a - x)(a - x). \\
 & 10. \quad 25x^4 - 29x^2y^2 + 4y^4 \\
 & \quad = 25x^4 + 20x^2y^2 + 4y^4 - 49x^2y^2 \\
 & \quad = (5x^2 + 7xy + 2y^2)(5x^2 - 7xy + 2y^2) \\
 & \quad = (5x + 2y)(x + y)(5x - 2y)(x - y). \\
 & 11. \quad x^4 + x^2 + 1 \\
 & \quad = x^4 + 2x^2 + 1 - x^2 \\
 & \quad = (x^2 + 1)^2 - x^2 \\
 & \quad = (x^2 + x + 1)(x^2 - x + 1). \\
 & 12. \quad n^8 + n^4 + 1 \\
 & \quad = n^8 + 2n^4 + 1 - n^4 \\
 & \quad = (n^4 + 1)^2 - (n^2)^2 \\
 & \quad = (n^4 + n^2 + 1)(n^4 - n^2 + 1) \\
 & \quad = (n^2 + n + 1)(n^2 - n + 1)(n^4 - n^2 + 1). \\
 & 13. \quad 16x^4 + 4x^2y^2 + y^4 \\
 & \quad = 16x^4 + 8x^2y^2 + y^4 - 4x^2y^2 \\
 & \quad = (4x^2 + y^2)^2 - (2xy)^2 \\
 & \quad = (4x^2 + 2xy + y^2)(4x^2 - 2xy + y^2).
 \end{aligned}$$

14.

$$\begin{aligned}
 & a^4b^4 - 21 a^2b^2 + 36 \\
 = & a^4b^4 - 12 a^2b^2 + 36 - 9 a^2b^2 \\
 = & (a^2b^2 - 6)^2 - (3ab)^2 \\
 = & (a^2b^2 + 3ab - 6)(a^2b^2 - 3ab - 6).
 \end{aligned}$$

16.

$$\begin{aligned}
 & 25 a^4 - 14 a^2b^2 + b^8 \\
 = & 25 a^4 - 10 a^2b^4 + b^8 - 4 a^2b^4 \\
 = & (5 a^2 - b^4)^2 - (2 ab^2)^2 \\
 = & (5 a^2 + 2 ab^2 - b^4)(5 a^2 - 2 ab^2 - b^4).
 \end{aligned}$$

18.

$$\begin{aligned}
 & b^4 + 64 \\
 = & b^4 + 16 b^2 + 64 - 16 b^2 \\
 = & (b^2 + 8)^2 - (4b)^2 \\
 = & (b^2 + 4b + 8)(b^2 - 4b + 8).
 \end{aligned}$$

20.

$$\begin{aligned}
 & m^8 + 4 \\
 = & m^8 + 4 m^4 + 4 - 4 m^4 \\
 = & (m^4 + 2)^2 - (2 m^2)^2 \\
 = & (m^4 + 2 m^2 + 2)(m^4 - 2 m^2 + 2).
 \end{aligned}$$

22.

$$\begin{aligned}
 & a^8 - 16 \\
 = & (a^4 + 4)(a^4 - 4) \\
 = & (a^4 + 4 a^2 + 4 - 4 a^2)(a^4 - 4) \\
 = & [(a^2 + 2)^2 - (2 a)^2](a^2 + 2)(a^2 - 2) \\
 = & (a^2 + 2 a + 2)(a^2 - 2 a + 2)(a^2 + 2)(a^2 - 2).
 \end{aligned}$$

23.

$$\begin{aligned}
 & m^5 + 4 mn^4 \\
 = & m(m^4 + 4 n^4) \\
 = & m(m^4 + 4 m^2 n^2 + 4 n^4 - 4 m^2 n^2) \\
 = & m[(m^2 + 2 n^2)^2 - (2 mn)^2] \\
 = & m(m^2 + 2 mn + 2 n^2)(m^2 - 2 mn + 2 n^2).
 \end{aligned}$$

24.

$$\begin{aligned}
 & x^4 + 64 y^4 \\
 = & x^4 + 16 x^2 y^2 + 64 y^4 - 16 x^2 y^2 \\
 = & (x^2 + 8 y^2)^2 - (4 xy)^2 \\
 = & (x^2 + 4 xy + 8 y^2)(x^2 - 4 xy + 8 y^2).
 \end{aligned}$$

25. $4 a^4 + 81$

$$\begin{aligned}
 & = 4 a^4 + 36 a^2 + 81 - 36 a^2 \\
 & = (2 a^2 + 9)^2 - (6 a)^2 \\
 & = (2 a^2 + 6 a + 9)(2 a^2 - 6 a + 9).
 \end{aligned}$$

15.

$$\begin{aligned}
 & c^4 + c^2 d^2 x^2 + d^4 x^4 \\
 = & c^4 + 2 c^2 d^2 x^2 + d^4 x^4 - c^2 d^2 x^2 \\
 = & (c^2 + d^2 x^2)^2 - (cdx)^2 \\
 = & (c^2 + cdx + d^2 x^2)(c^2 - cdx + d^2 x^2).
 \end{aligned}$$

17.

$$\begin{aligned}
 & 9 a^4 + 26 a^2 b^2 + 25 b^4 \\
 = & 9 a^4 + 30 a^2 b^2 + 25 b^4 - 4 a^2 b^2 \\
 = & (3 a^2 + 5 b^2)^2 - (2 ab)^2 \\
 = & (3 a^2 + 2 ab + 5 b^2)(3 a^2 - 2 ab + 5 b^2).
 \end{aligned}$$

19.

$$\begin{aligned}
 & a^4 + 4 b^4 \\
 = & a^4 + 4 a^2 b^2 + 4 b^4 - 4 a^2 b^2 \\
 = & (a^2 + 2 b^2)^2 - (2 ab)^2 \\
 = & (a^2 + 2 ab + 2 b^2)(a^2 - 2 ab + 2 b^2).
 \end{aligned}$$

21.

$$\begin{aligned}
 & a^4 + 324 \\
 = & a^4 + 36 a^2 + 324 - 36 a^2 \\
 = & (a^2 + 18)^2 - (6 a)^2 \\
 = & (a^2 + 6 a + 18)(a^2 - 6 a + 18).
 \end{aligned}$$

26. $x^5 y^2 + 4 xy^2$

$$\begin{aligned}
 & = xy^2(x^4 + 4) \\
 & = xy^2(x^4 + 4x^2 + 4 - 4x^2) \\
 & = xy^2[(x^2 + 2)^2 - (2x)^2] \\
 & = xy^2(x^2 + 2x + 2)(x^2 - 2x + 2).
 \end{aligned}$$

2.

$$\begin{aligned}
 & a^2 + 2 ab + b^2 + 8 ac + 8 bc + 15 c^2 \\
 = & (a + b)^2 + 8 c(a + b) + 3 c \cdot 5 c \\
 = & (a + b + 3 c)(a + b + 5 c).
 \end{aligned}$$

3.

$$\begin{aligned}
 & x^2 - 6 xy + 9 y^2 + 6 xz - 18 yz + 5 z^2 \\
 = & (x - 3 y)^2 + 6 z(x - 3 y) + z \cdot 5 z \\
 = & (x - 3 y + z)(x - 3 y + 5 z).
 \end{aligned}$$

4.

$$\begin{aligned}
 & m^2 + n^2 - 2 mn + 7 mp - 7 np - 30 p^2 \\
 = & (m - n)^2 + 7 p(m - n) + (-3 p)(+10 p) \\
 = & (m - n - 3 p)(m - n + 10 p).
 \end{aligned}$$

5.

$$\begin{aligned}
 & 16 n^2 + 55 - 64 n - 16 m + m^2 + 8 mn \\
 = & m^2 + 8 mn + 16 n^2 - 16 m - 64 n + 55 \\
 = & (m + 4 n)^2 - 16(m + 4 n) + (-5)(-11) \\
 = & (m + 4 n - 5)(m + 4 n - 11).
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & 9m^4 + k^2 - 30 + 39m^2 + 13k + 6m^2k \\
 &= 9m^4 + 6m^2k + k^2 + 39m^2 + 13k - 30 \\
 &= (3m^2 + k)^2 + 13(3m^2 + k) + (-2)(+15) \\
 &= (3m^2 + k - 2)(3m^2 + k + 15).
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & 25a^2 + y^2 + 10x^2 + 10ay - 35ax - 7xy \\
 &= 25a^2 + 10ay + y^2 - 35ax - 7xy + 10x^2 \\
 &= (5a + y)^2 - 7x(5a + y) + (-2x)(-5x) \\
 &= (5a + y - 2x)(5a + y + 5x).
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & 4x^2 + y^2 - 6z^2 - 4xy + 2xz - yz \\
 &= 4x^2 - 4xy + y^2 + 2xz - yz - 6z^2 \\
 &= (2x - y)^2 + z(2x - y) + (-2z)(+3z) \\
 &= (2x - y - 2z)(2x - y + 3z).
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & a^2 + b^2 + c^2 + 2ab + 2ac + 2bc + 5a + 5b + 5c + 6 \\
 &= (a + b + c)^2 + 5(a + b + c) + 2 \cdot 3 \\
 &= (a + b + c + 2)(a + b + c + 3).
 \end{aligned}$$

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$$3. \quad y^4 - 1 = (y^2 + 1)(y^2 - 1) = (y^2 + 1)(y + 1)(y - 1).$$

$$4. \quad 1 - x^8 = (1 + x^4)(1 - x^4) = (1 + x^4)(1 + x^2)(1 + x)(1 - x).$$

$$\begin{aligned}
 5. \quad x^{10} - 1 &= (x^5 + 1)(x^5 - 1) \\
 &= (x + 1)(x^4 - x^3 + x^2 - x + 1)(x - 1)(x^4 + x^3 + x^2 + x + 1).
 \end{aligned}$$

$$6. \quad x^6 - 1 = (x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1).$$

$$8. \quad 1 - b^4 = (1 + b^2)(1 - b^2) = (1 + b^2)(1 + b)(1 - b).$$

$$\begin{aligned}
 9. \quad a - a^7 &= a(1 - a^6) = a(1 + a^3)(1 - a^3) \\
 &= a(1 + a)(1 - a + a^2)(1 - a)(1 + a + a^2).
 \end{aligned}$$

$$10. \quad b^7 + b = b(b^6 + 1) = b(b^2 + 1)(b^4 - b^2 + 1).$$

$$\begin{aligned}
 11. \quad p^4 + 4 &= p^4 + 4p^2 + 4 - 4p^2 = (p^2 + 2)^2 - (2p)^2 \\
 &= (p^2 + 2p + 2)(p^2 - 2p + 2).
 \end{aligned}$$

$$12. \quad 1 + x^{12} = 1^3 + (x^4)^3 = (1 + x^4)(1 - x^4 + x^8).$$

$$13. \quad y - a^4y = y(1 - a^4) = y(1 + a^2)(1 + a)(1 - a).$$

$$14. \quad x^2y - y^3 = y(x^2 - y^2) = y(x + y)(x - y).$$

$$\begin{aligned}
 15. \quad a^{13} - ab^{12} &= a(a^{12} - b^{12}) \\
 &= a(a^6 + b^6)(a^6 - b^6) \\
 &= a(a^2 + b^2)(a^4 - a^2b^2 + b^4)(a^3 + b^3)(a^3 - b^3) \\
 &= a(a^2 + b^2)(a^4 - a^2b^2 + b^4)(a + b)(a^2 - ab + b^2)(a - b) \\
 &\quad (a^2 + ab + b^2).
 \end{aligned}$$

$$16. \quad a^4 - 256 = (a^2 + 16)(a^2 - 16) = (a^2 + 16)(a + 4)(a - 4).$$

$$\begin{aligned}
 18. \quad 64 - 2y^5 &= 2(32 - y^5) = 2(2^5 - y^5) \\
 &= 2(2 - y)(16 + 8y + 4y^2 + 2y^3 + y^4).
 \end{aligned}$$

$$\begin{aligned}
 19. \quad 7n^7 - 7n &= 7n(n^6 - 1) = 7n(n^3 + 1)(n^3 - 1) \\
 &= 7n(n + 1)(n^2 - n + 1)(n - 1)(n^2 + n + 1).
 \end{aligned}$$

$$20. \quad a^7 - 9a = a(a^6 - 9) = a(a^3 + 3)(a^3 - 3).$$

$$21. \quad 4x^4 - 4x = 4x(x^3 - 1) = 4x(x - 1)(x^2 + x + 1).$$

$$22. \quad 7y^4 - 175 = 7(y^4 - 25) = 7(y^2 + 5)(y^2 - 5).$$

$$\begin{aligned}
 23. \quad 8 - 27a^3x^3 &= 2^3 - (3ax)^3 \\
 &= (2 - 3ax)[2^2 + 2 \cdot 3ax + (3ax)^2] \\
 &= (2 - 3ax)(4 + 6ax + 9a^2x^2).
 \end{aligned}$$

$$24. \quad 32x - 2x^3 = 2x(16 - x^2) = 2x(4 + x)(4 - x).$$

25. $6b^4 + 24 = 6(b^4 + 4) = 6(b^4 + 4b^2 + 4 - 4b^2)$
 $= 6[(b^2 + 2)^2 - (2b)^2]$
 $= 6(\bar{b}^2 + 2\bar{b} + 2)(b^2 - 2b + 2).$
26. $a^6 + 27a^2 = a^2(a^4 + 27) = a^2(a + 3)(a^2 - 3a + 9).$
27. $b^2 - 196 = (b + 14)(b - 14).$
28. $450 - 2a^2 = 2(225 - a^2) = 2(15 + a)(15 - a).$
29. $4m^3 + .004 = .004(1000m^3 + 1) = .004[(10m)^3 + 1^3]$
 $= .004(10m + 1)(100m^2 - 10m + 1).$
30. $125 - 8x^6 = 5^3 - (2x^2)^3 = (5 - 2x^2)[5^2 + 5 \cdot 2x^2 + (2x^2)^2]$
 $= (5 - 2x^2)(25 + 10x^2 + 4x^4).$
32. $x^2 - xy - 132y^2 = x^2 + (-11y - 12y)x + (-11y)(-12y)$
 $= (x + 11y)(x - 12y).$
33. $ax^2 - 3ax - 4a = a(x^2 - 3x - 4) = a(x + 1)(x - 4).$
34. $x^2 + 5x^2 - 6x = x(x^2 + 5x - 6) = x(x - 1)(x + 6).$
35. $3x^2 + 30x + 27 = 3(x^2 + 10x + 9) = 3(x + 1)(x + 9).$
36. $128a^2 - 250a^5 = 2a^2(64 - 125a^3) = 2a^2[4^3 - (5a)^3]$
 $= 2a^2(4 - 5a)[4^2 + 4 \cdot 5a + (5a)^2]$
 $= 2a^2(4 - 5a)(16 + 20a + 25a^2).$
37. $5x^{10} + 10x^5 - 15 = 5(x^{10} + 2x^5 - 3) = 5(x^5 - 1)(x^5 + 3)$
 $= 5(x - 1)(x^4 + x^3 + x^2 + x + 1)(x^5 + 3)$
38. $6x^2 - 19x + 15 = 6x^2 - 9x - 10x + 15$
 $= 3x(2x - 3) - 5(2x - 3)$
 $= (3x - 5)(2x - 3).$
39. $x^6 + 2x^3y^2 + y^6 = (x^3)^2 + 2 \cdot x^3 \cdot y^2 + (y^3)^2$
 $= (x^3 + y^3)(x^3 + y^3).$
40. $7x^2 - 77xy - 84y^2 = 7(x^2 - 11xy - 12y^2) = 7(x + y)(x - 12y)$
42. $9x^2 - 24xy + 16y^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$
 $= (3x - 4y)(3x - 4y).$
43. $289x^2 - 34xy + y^2 = (17x)^2 - 2(17x)y + y^2$
 $= (17x - y)(17x - y).$
44. $3bx^2 + bxy - 10by^2 = b(3x^2 + xy - 10y^2)$
 $= b(x + 2y)(3x - 5y).$
49. $10a^2c + 33ac - 7c = c(10a^2 + 33a - 7)$
 $= \frac{c(100a^2 + 330a - 70)}{10} = \frac{c[(10a)^2 + 33(10a) + (-2)(+35)]}{10}$
 $= \frac{c(10a - 2)(10a + 35)}{10} = \frac{c \cdot 2(5a - 1) \cdot 5(2a + 7)}{10} = c(5a - 1)(2a + 7)$
50. $60ny^2 - 61ny - 56n = n(60y^2 - 61y - 56)$
 $= \frac{n(3600y^2 - 61 \cdot 60y - 7 \cdot 8 \cdot 5 \cdot 12)}{60}$
 $= \frac{n[(60y)^2 - 61(60y) - 35 \cdot 96]}{60}$
 $= \frac{n(60y + 35)(60y - 96)}{60}$
 $= \frac{n \cdot 5(12y + 7) \cdot 12(5y - 8)}{60}$
 $= n(12y + 7)(5y - 8).$

$$\begin{aligned} 51. \quad 25x^2 + 60xy + 36y^2 &= (5x)^2 + 2(5x)(6y) + (6y)^2 \\ &= (5x + 6y)(5x + 6y). \end{aligned}$$

$$\begin{aligned} 52. \quad 6ax^2 + 5axy - 6ay^2 &= a(6x^2 + 5xy - 6y^2) \\ \text{By trial,} &= a(2x + 3y)(3x - 2y). \end{aligned}$$

$$\begin{aligned} 53. \quad 169x^4 - 26ax^3 + a^2x^2 &= x^2(169x^2 - 26ax + a^2) \\ &= x^2[(13x)^2 - 2(13x)a + a^2] \\ &= x^2(13x - a)(13x - a). \end{aligned}$$

$$\begin{aligned} 54. \quad a^4c^4 + a^2b^2c^2 + b^4 &= a^4c^4 + 2a^2b^2c^2 + b^4 - a^2b^2c^2 \\ &= (a^2c^2 + b^2)^2 - (abc)^2 \\ &= (a^2c^2 + abc + b^2)(a^2c^2 - abc + b^2). \end{aligned}$$

$$\begin{aligned} 55. \quad 16x^4 + 4x^2y^2 + y^4 &= 16x^4 + 8x^2y^2 + y^4 - 4x^2y^2 \\ &= (4x^2 + y^2)^2 - (2xy)^2 \\ &= (4x^2 + 2xy + y^2)(4x^2 - 2xy + y^2). \end{aligned}$$

$$\begin{aligned} 56. \quad b^4c - 13b^2c + 42c &= c(b^4 - 13b^2 + 42) \\ &= c(b^2 - 6)(b^2 - 7). \end{aligned}$$

$$57. \quad 2a^2 - 6ab - 140b^2 = 2(a^2 - 3ab - 70b^2) = 2(a + 7b)(a - 10b).$$

$$\begin{aligned} 58. \quad m^2n - 21mn^2 + 80n^3 &= n(m^2 - 21mn + 80n^2) \\ &= n(m - 5n)(m - 16n). \end{aligned}$$

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$$\begin{aligned} 59. \quad 17x^2 + 25x - 18 &= \frac{289x^2 + 25 \cdot 17x - 306}{17} = \frac{(17x)^2 + 25(17x) - 9 \cdot 34}{17} \\ &= \frac{(17x - 9)(17x + 34)}{17} = (17x - 9)(x + 2). \end{aligned}$$

$$\begin{array}{lll} 60. \quad \text{First factor, try} & 5x - y, & 5x - 5y. \\ \text{Second factor, try} & x - 5y, & x - y. \\ \text{Products, 2d terms,} & -26xy, & -10xy. \\ \therefore 5x^2 - 26xy + 5y^2 &= (5x - y)(x - 5y). \end{array}$$

$$\begin{aligned} 62. \quad 8a^2 - 21ab - 9b^2 &= -(9b^2 + 21ab - 8a^2) \\ &= -[(3b)^2 + 7a(3b) + (-a)(+8a)] \\ &= -(3b - a)(3b + 8a) \\ &= (a - 3b)(8a + 3b). \end{aligned}$$

$$\begin{aligned} 63. \quad 60a^2 + 8ax - 3x^2 &= \frac{900a^2 + 120ax - 45x^2}{15} = \frac{(30a)^2 + 4x(30a) + (-5x)(+9x)}{15} \\ &= \frac{(30a - 5x)(30a + 9x)}{15} = \frac{5(6a - x) \cdot 3(10a + 3x)}{15} \\ &= (6a - x)(10a + 3x). \end{aligned}$$

$$\begin{aligned} 64. \quad 30x^2 - 37x - 77 &= \frac{900x^2 - 37(30x) - 77 \cdot 30}{30} = \frac{(30x)^2 - 37(30x) - 33 \cdot 70}{30} \\ &= \frac{(30x + 33)(30x - 70)}{30} = \frac{3(10x + 11) \cdot 10(3x - 7)}{30} \\ &= (10x + 11)(3x - 7). \end{aligned}$$

$$\begin{aligned} 65. \quad 2x^3 + 28x^2 + 66x &= 2x(x^2 + 14x + 33) \\ &= 2x(x + 3)(x + 11). \end{aligned}$$

$$\begin{aligned} 66. \quad a^2 + b^2 - c^2 - 2ab &= a^2 - 2ab + b^2 - c^2 = (a - b)^2 - c^2 \\ &= (a - b + c)(a - b - c). \end{aligned}$$

$$\begin{aligned} 67. \quad ax^2 + 10ax - 39a &= a(x^2 + 10x - 39) \\ &= a(x - 3)(x + 13). \end{aligned}$$

$$\begin{aligned} 68. \quad n^4 + n^2a^2b^4 + a^4b^8 &= n^4 + 2n^2a^2b^4 + a^4b^8 - n^2a^2b^4 \\ &= (n^2 + a^2b^4)^2 - (nab^2)^2 \\ &= (n^2 + nab^2 + a^2b^4)(n^2 - nab^2 + a^2b^4). \end{aligned}$$

$$\begin{aligned} 69. \quad a^2z^4 + a^2z^2 + a^2 &= a^2(z^4 + z^2 + 1) \\ &= a^2(z^4 + 2z^2 + 1 - z^2) \\ &= a^2(z^2 + z + 1)(z^2 - z + 1). \end{aligned}$$

$$\begin{aligned} 72. \quad b^8 + b^4y^2 + y^4 &= b^8 + 2b^4y^2 + y^4 - b^4y^2 \\ &= (b^4 + b^2y + y^2)(b^4 - b^2y + y^2). \end{aligned}$$

$$73. \quad x^7 - 2x^6 + x = x(x^6 - 2x^5 + 1)$$

$$\text{Substituting 1 for } x, x^6 - 2x^5 + 1 = 1 - 2 + 1 = 0.$$

Hence, $x - 1$ is a factor.

$$\therefore x^7 - 2x^6 + x = x(x - 1)(x^5 - x^4 - x^3 - x^2 - x - 1).$$

$$\begin{aligned} 74. \quad x^5 + x^2y - 41xy^2 - 105y^3 &= x^5 + x^2y - 41xy^2 - 105y^3 \\ \text{Substituting } -5y \text{ for } x, &= -125y^5 + 25y^3 + 205y^3 - 105y^3 = 0. \end{aligned}$$

Hence, $x + 5y$ is a factor.

$$\begin{aligned} \therefore x^5 + x^2y - 41xy^2 - 105y^3 &= (x + 5y)(x^2 - 4xy - 21y^2) \\ \S 130, &= (x + 5y)(x + 3y)(x - 7y). \end{aligned}$$

$$\begin{aligned} 75. \quad x^2 - cx + 2dx - 2cd &= x(x - c) + 2d(x - c) \\ &= (x + 2d)(x - c). \end{aligned}$$

$$\begin{aligned} 76. \quad x^3y + 4x^2y - 31xy - 70y &= y(x^3 + 4x^2 - 31x - 70) \\ 136, &= y(x + 2)(x^2 + 2x - 35) \\ 130, &= y(x + 2)(x - 5)(x + 7). \end{aligned}$$

$$\begin{aligned} 77. \quad x^2 - 3ax + 4bx - 12ab &= x(x - 3a) + 4b(x - 3a) \\ &= (x + 4b)(x - 3a). \end{aligned}$$

$$\begin{aligned} 78. \quad ax^3 - 9ax^2 + 26ax - 24a &= a(x^3 - 9x^2 + 26x - 24) \\ 136, &= a(x - 2)(x^2 - 7x + 12) \\ 130, &= a(x - 2)(x - 3)(x - 4). \end{aligned}$$

$$\begin{aligned} 79. \quad 12ax - 8bx - 9ay + 6by &= 4x(3a - 2b) - 3y(3a - 2b) \\ &= (4x - 3y)(3a - 2b). \end{aligned}$$

$$\begin{aligned} 80. \quad 25x^2 - 9y^2 - 24yz - 16z^2 &= 25x^2 - (9y^2 + 24yz + 16z^2) \\ &= (5x)^2 - (3y + 4z)^2 \\ &= (5x + 3y + 4z)(5x - 3y - 4z). \end{aligned}$$

$$\begin{aligned} 81. \quad x^2 - z^2 + y^2 - a^2 - 2xy + 2az &= x^2 - 2xy + y^2 - (a^2 - 2az + z^2) \\ &= (x - y)^2 - (a - z)^2 \\ &= (x - y + a - z)(x - y - a + z). \end{aligned}$$

$$\begin{aligned} 82. \quad 2b^2m - 3ab^2 + 2bmx - 3abx &= b^2(2m - 3a) + bx(2m - 3a) \\ &= b(b + x)(2m - 3a). \end{aligned}$$

$$\begin{aligned} 83. \quad a^2 + b^2 + c^2 - 2ab - 2ac + 2bc &= a^2 - 2ab + b^2 - 2ac + 2bc + c^2 \\ &= (a - b)^2 - 2c(a - b) + c^2 \\ &= (a - b - c)(a - b + c). \end{aligned}$$

$$\begin{aligned} 84. \quad x^3y + 14x^2y + 43xy + 30y &= y(x^3 + 14x^2 + 43x + 30) \\ 136, &= y(x + 1)(x^2 + 13x + 30) \\ 130, &= y(x + 1)(x + 3)(x + 10). \end{aligned}$$

$$\begin{aligned} 85. \quad x^3y - 15x^2y + 38xy - 24y &= y(x^3 - 15x^2 + 38x - 24) \\ 136, &= y(x - 1)(x^2 - 14x + 24) \\ 130, &= y(x - 1)(x - 2)(x - 12). \end{aligned}$$

86. $abx^3 + 3abx^2 - abx - 3ab = ab(x^3 + 3x^2 - x - 3)$
 $= ab[x^2(x+3) - (x+3)]$
 $= ab(x^2 - 1)(x+3)$
 $= ab(x+1)(x-1)(x+3).$
87. $3bmx + 2bm - 3anx - 2an = bm(3x+2) - an(3x+2)$
 $= (bm - an)(3x+2).$
88. $20ax^3 - 28ax^2 + 5a^2x - 7a^2 = 4ax^2(5x-7) + a^2(5x-7)$
 $= a(4x^2 + a)(5x-7).$
89. $x^2 + 9y^2 + 25z^2 - 6xy - 10xz + 30yz$
 $= (x)^2 + (-3y)^2 + (-5z)^2 + 2x(-3y) + 2x(-5z) + 2(-3y)(-5z)$
 $= (x-3y-5z)(x-3y-5z).$
90. $9x^2 + y^2 + 16z^2 - 6xy - 8yz + 24zx$
 $= (3x)^2 + (-y)^2 + (4z)^2 + 2(3x)(-y) + 2(3x)(4z) + 2(-y)(4z)$
 $= (3x-y+4z)(3x-y+4z).$
91. $x^2y^2z^2 + a^2b^2 + 1 + 2abxyz + 2xyz + 2ab$
 $= (xyz)^2 + (ab)^2 + (1)^2 + 2(xyz)(ab) + 2(xyz)(1) + 2(ab)(1)$
 $= (xyz+ab+1)(xyz+ab+1).$
92. $a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c + 2abcd - 2bc^2d$
 $= (ab)^2 + (-bc)^2 + (cd)^2 + 2(ab)(-bc) + 2(ab)(cd) + 2(-bc)(cd)$
 $= (ab-bc+cd)(ab-bc+cd).$
93. $x^8 + n^4x^4 + n^8 + 2n^2x^6 + 2n^4x^4 + 2n^6x^2$
 $= (x^4)^2 + (n^2x^2)^2 + (n^4)^2 + 2(x^4)(n^2x^2) + 2(x^4)(n^4) + 2(n^2x^2)(n^4)$
 $= (x^4 + n^2x^2 + n^4)(x^4 + n^2x^2 + n^4)$
 $= (x^4 + 2n^2x^2 + n^4 - n^2x^2)(x^4 + 2n^2x^2 + n^4 - n^2x^2)$
 $= [(x^2 + n^2)^2 - (nx)^2][(x^2 + n^2)^2 - (nx)^2]$
 $= (x^2 + nx + n^2)(x^2 - nx + n^2)(x^2 + nx + n^2)(x^2 - nx + n^2).$
94. $a^2b^2x^2 - a^2b^2 - b^2x^2 + b^2 - a^2x^2 + a^2 + x^2 - 1$
 $= a^2b^2(x^2-1) - b^2(x^2-1) - a^2(x^2-1) + (x^2-1)$
 $= (a^2b^2 - b^2 - a^2 + 1)(x^2-1)$
 $= [b^2(a^2-1) - (a^2-1)](x^2-1)$
 $= (b^2-1)(a^2-1)(x^2-1)$
 $= (a+1)(a-1)(b+1)(b-1)(x+1)(x-1).$
95. $(a+b)^6 - 1$
 $= [(a+b)^3 + 1][(a+b)^3 - 1]$
 $= (a+b+1)[(a+b)^2 - (a+b) + 1][a^2 + b^2 + ab + a + b + 1]$
 $= (a+b+1)(a^2+2ab+b^2-a-b+1)(a+b-1)(a^2+2ab+b^2+a+b+1).$
96. Substituting 1 for a , $a^3 - 2a^2 + 1 = 1 - 2 + 1 = 0$;
 $\therefore a^3 - 2a^2 + 1 = (a-1)(a^2 - a - 1).$
97. Substituting 2 for b , $b^3 - 4b^2 + 8 = 8 - 16 + 8 = 0$;
 $\therefore b^3 - 4b^2 + 8 = (b-2)(b^2 - 2b - 4).$
98. Substituting 5 for x , $x^3 - 10x^2 + 125 = 125 - 250 + 125 = 0$;
 $\therefore x^3 - 10x^2 + 125 = (x-5)(x^2 - 5x - 25).$
99. $8x^4 - 6x^2 - 35 = \frac{16x^4 - 12x^2 - 70}{2} = \frac{(4x^2)^2 - 3(4x^2) - 70}{2}$
 $= \frac{(4x^2+7)(4x^2-10)}{2} = (4x^2+7)(2x^2-5).$
100. $3x^6 + 96x = 3x(x^5 + 32) = 3x(x^5 + 2^5)$
 $= 3x(x+2)(x^4 - 2x^3 + 4x^2 - 8x + 16).$

101. $(a-2)^3 + (a-1)^3 = [(a-2) + (a-1)] [(a-2)^2 - (a-2)(a-1) + (a-1)^2]$
 $= (a-2+a-1) (a^2 - 4a + 4 - a^2 + 3a - 2 + a^2 - 2a + 1)$
 $= (2a-3) (a^2 - 3a + 3).$
102. $12x^3 + 3x^2 - 8x - 2 = 3x^2(4x+1) - 2(4x+1)$
 $= (3x^2 - 2)(4x+1).$
103. $2x^2 + 10x + ax + 5a = 2x(x+5) + a(x+5)$
 $= (2x+a)(x+5).$
104. $x^3 + 5x^2 - 29x - 105$
 136, $= (x-5)(x^2 + 10x + 21)$
 130, $= (x-5)(x+3)(x+7).$
105. $m^2n^2 + a^2b^2 + b^2n^2 + 2bmn^2 + 2ab^2n + 2abmn$
 $= (mn)^2 + (ab)^2 + (bn)^2 + 2(mn)(ab) + 2(mn)(bn) + 2(ab)(bn)$
 § 95, $= (mn + ab + bn)(mn + ab + bn).$

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106. $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$
 $= a^4(a+b) + a^3b^2(a+b) + b^4(a+b)$
 $= (a^4 + a^3b^2 + b^4)(a+b)$
 $= (a^4 + 2a^2b^2 + b^4 - a^2b^2)(a+b)$
 $= (a^2 + ab + b^2)(a^2 - ab + b^2)(a+b).$
107. $a^2b^2 - 4abx - 4x + 2ab + 4x^2$
 $= a^2b^2 - 4abx + 4x^2 + 2ab - 4x$
 $= (ab - 2x)^2 + 2(ab - 2x)$
 $= (ab - 2x + 2)(ab - 2x).$
108. $(a+b)^2(x-y) - (a+b)(x^2 - y^2).$
 $(a+b)(x-y)$ is a monomial factor. Hence, § 123, finding its coefficient by dividing the given polynomial by $(a+b)(x-y)$,
 $(a+b)^2(x-y) - (a+b)(x^2 - y^2) = [(a+b) - (x+y)](a+b)(x-y)$
 $= (a+b-x-y)(a+b)(x-y).$
109. $1 - x^2 + abx^2 + bx^3 - bx - ab$
 $= 1 - ab - bx - x^2 + abx^2 + bx^3$
 $= (1 - ab - bx) - x^2(1 - ab - bx)$
 $= (1 - x^2)(1 - ab - bx)$
 $= (1+x)(1-x)(1-ab-bx).$
110. $x^2 - x^3 + x^2y - xy + x^2y - xy^2$
 $= x(x-y-x^2+xy+x^2y-y^2)$
 $= x[(x-y) - x(x-y) + y(x-y)]$
 $= x(1-x+y)(x-y).$
112. $x^3 + 15x^2 + 75x + 125 = x^3 + 5^3 + 15x(x+5)$
 $= (x+5)(x^2 - 5x + 25) + 15x(x+5)$
 $= (x^2 - 5x + 25 + 15x)(x+5)$
 $= (x^2 + 10x + 25)(x+5)$
 $= (x+5)(x+5)(x+5).$
113. $4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2$
 $= (2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2$
 $= (2ab+2cd+a^2+b^2-c^2-d^2)(2ab+2cd-a^2-b^2+c^2+d^2)$
 $= [a^2+2ab+b^2-(c^2-2cd+d^2)][c^2+2cd+d^2-(a^2-2ab+b^2)]$
 $= (a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$
 $= (a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d).$
114. $x^{3n} - a^{3n} = (x^n)^3 - (a^n)^3 = (x^n - a^n)(x^{2n} + x^na^n + a^{2n}).$
115. $(a^2+b^2-c^2)^2 - 4a^2b^2 = (a^2+b^2-c^2+2ab)(a^2+b^2-c^2-2ab)$
 $= (a^2+2ab+b^2-c^2)(a^2-2ab+b^2-c^2)$
 $= (a+b+c)(a+b-c)(a-b+c)(a-b-c).$
117. $x^3 - xy - x^2y + y^2 = x(x^2-y) - y(x^2-y)$
 $= (x-y)(x^2-y).$
118. $x^4 - 4x^2y^2 + 2x^3 - 16y^3$
 $= x^2(x^2 - 4y^2) + 2(x^3 - 8y^3)$
 $= x^2(x+2y)(x-2y) + 2(x-2y)(x^2+2xy+4y^2)$
 $= [x^2(x+2y) + 2(x^2+2xy+4y^2)](x-2y)$
 $= (x^3+2x^2y+2x^2+4xy+8y^2)(x-2y).$

$$\begin{aligned} 119. \quad a^4 - b^4 - (a+b)(a-b) &= (a^2 + b^2)(a+b)(a-b) - (a+b)(a-b) \\ &= (a^2 + b^2 - 1)(a+b)(a-b). \end{aligned}$$

$$\begin{aligned} 120. \quad x^3 - 6x^2 + 12x - 8 &= x^3 - 8 - 6x^2 + 12x \\ &= (x-2)(x^2 + 2x + 4) - 6x(x-2) \\ &= (x^2 + 2x + 4 - 6x)(x-2) \\ &= (x^2 - 4x + 4)(x-2) \\ &= (x-2)(x-2)(x-2). \end{aligned}$$

$$\begin{aligned} 121. \quad 1000x^3 - 27y^3 &= (10x)^3 - (3y)^3 \\ &= (10x - 3y)(100x^2 + 30xy + 9y^2). \end{aligned}$$

$$\begin{aligned} 122. \quad (a+x)^4 - x^4 &= [(a+x)^2 + x^2][(a+x)^2 - x^2] \\ &= (a^2 + 2ax + 2x^2)[(a+x) + x][(a+x) - x] \\ &= (a^2 + 2ax + 2x^2)(a+2x)a \\ &= a(a+2x)(a^2 + 2ax + 2x^2). \end{aligned}$$

$$\begin{aligned} 123. \quad 1 + (x+1)^3 &= 1^3 + (x+1)^3 \\ \S 132, \quad &= (1+x+1)[1 - 1(x+1) + (x+1)^2] \\ &= (x+2)(x^2 + x + 1). \end{aligned}$$

$$124. \quad ab - bx^n + x^n y^m - ay^m = b(a - x^n) - y^m(a - x^n) = (b - y^m)(a - x^n).$$

$$\begin{aligned} 125. \quad x^9 + 4x &= x(x^8 + 4) \\ &= x(x^8 + 4x^4 + 4 - 4x^4) \\ &= x(x^4 + 2x^2 + 2)(x^4 - 2x^2 + 2). \end{aligned}$$

$$\begin{aligned} 126. \quad x^5 - x^2 - x^4 + x^3 &= x^5 + x^3 - (x^4 + x^2) \\ &= x^3(x^2 + 1) - x^2(x^2 + 1) \\ &= x^2(x-1)(x^2 + 1). \end{aligned}$$

$$\begin{aligned} 127. \quad (a+b)^4 - (b-c)^4 &= [(a+b)^2 + (b-c)^2][(a+b)^2 - (b-c)^2] \\ &= [(a+b)^2 + (b-c)^2](a+b+b-c)(a+b-b+c) \\ &= (a^2 + 2ab + b^2 + b^2 - 2bc + c^2)(a+2b-c)(a+c) \\ &= (a^2 + 2ab + 2b^2 - 2bc + c^2)(a+2b-c)(a+c). \end{aligned}$$

$$\begin{aligned} 128. \quad 3ab(a+b) + a^3 + b^3 &= 3ab(a+b) + (a^3 + ab^2 + b^3) \\ &= (a^2 + 2ab + b^2)(a+b) \\ &= (a+b)(a+b)(a+b). \end{aligned}$$

$$\begin{aligned} 129. \quad (x+y)^3 + (x-y)^3 &= [(x+y) + (x-y)][(x+y)^2 - (x+y)(x-y) + (x-y)^2] \\ \S 132, \quad &= 2x(x^2 + 2xy + y^2 - x^2 + y^2 + x^2 - 2xy + y^2) \\ &= 2x(x^2 + 3y^2). \end{aligned}$$

$$\begin{aligned} 130. \quad a^3 - (a+b)^3 &= [a - (a+b)][a^2 + a(a+b) + (a+b)^2] \\ \S 133, \quad &= -b(3a^2 + 3ab + b^2). \end{aligned}$$

$$\begin{aligned} 131. \quad x^4 - 119x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - 121x^2y^2 \\ &= (x^2 + y^2)^2 - (11xy)^2 \\ &= (x^2 + 11xy + y^2)(x^2 - 11xy + y^2). \end{aligned}$$

$$\begin{aligned} 132. \quad m^3 + m^2 - mn - mn^2 &= m(m^2 + m - n - n^2) \\ &= m(m^2 - n^2 + m - n) \\ &= m[(m+n)(m-n) + (m-n)] \\ &= m(m+n+1)(m-n). \end{aligned}$$

$$\begin{aligned}
 133. \quad & (x^2 - y^2)^2 - (x^2 - xy)^2 \\
 &= (x + y)(x - y) \cdot (x + y)(x - y) - x(x - y) \cdot x(x - y) \\
 &= [(x + y)(x + y) - x^2](x - y)(x - y) \\
 &= (x + y + x)(x + y - x)(x - y)(x - y) \\
 &= y(2x + y)(x - y)(x - y).
 \end{aligned}$$

$$\begin{aligned}
 134. \quad & x^6 - y^6 - 3x^2y^2(x^2 - y^2) \\
 &= (x^2 - y^2)(x^4 + x^2y^2 + y^4) - 3x^2y^2(x^2 - y^2) \\
 &= (x^4 + x^2y^2 + y^4 - 3x^2y^2)(x^2 - y^2) \\
 &= (x^4 - 2x^2y^2 + y^4)(x^2 - y^2) \\
 &= (x^2 - y^2)(x^2 - y^2)(x^2 - y^2) \\
 &= (x + y)(x - y)(x + y)(x - y)(x + y)(x - y).
 \end{aligned}$$

$$\begin{aligned}
 135. \quad & (x^2 + 6x + 9)^2 - (x^2 + 5x + 6)^2 \\
 &= (x + 3)(x + 3)(x + 3) - (x + 2)(x + 3)(x + 3) \\
 &= [x^2 + 6x + 9 - (x^2 + 4x + 4)](x + 3)(x + 3) \\
 &= (2x + 5)(x + 3)(x + 3).
 \end{aligned}$$

$$\begin{aligned}
 136. \quad & \text{If } x = -a, x^3 + (a + b - c)x^2 + (ab - ac - bc)x - abc \\
 &= -a^3 + (a + b - c)a^2 + (ab - ac - bc)(-a) - abc \\
 &= -a^3 + a^3 + a^2b - a^2c - a^2b + a^2c + abc - abc = 0.
 \end{aligned}$$

Hence, $x + a$ is one factor.

In a similar manner the given polynomial reduces to zero when $-b$ and $+c$, respectively, are substituted for x . Hence, $x + b$ and $x - c$ are factors of the given polynomial.

Dividing by $(x + a)(x + b)(x - c)$ the quotient is 1.

Hence, $x^3 + (a + b - c)x^2 + (ab - ac - bc)x - abc = (x + a)(x + b)(x - c)$.

$$137. \text{ Substituting } 2 \text{ for } x, 32 - x^5 = 32 - 32 = 0;$$

$\therefore x - 2$ and consequently $-1(x - 2)$, or $2 - x$, is a factor.

$$\text{Dividing by } 2 - x, 32 - x^5 = (2 - x)(16 + 8x + 4x^2 + 2x^3 + x^4).$$

$$138. \text{ Substituting } -1 \text{ for } x, 16 + 5x - 11x^2 = 16 - 5 - 11 = 0;$$

$\therefore x + 1$ is one factor.

$$\text{Dividing by } x + 1, \text{ or } 1 + x, 16 + 5x - 11x^2 = (1 + x)(16 - 11x).$$

$$139. \text{ Substituting } a \text{ for } x, x^n - a^n = a^n - a^n = 0;$$

$\therefore x - a$ is a factor of $x^n - a^n$.

$$\text{Substituting } -a \text{ for } x, x^n - a^n = (-a)^n - a^n$$

$$\text{If } n \text{ is odd, } = -a^n - a^n = -2a^n;$$

$\therefore x + a$ is not a factor of $x^n - a^n$ in this case. Hence, $x - a$ is the only rational binomial factor of the first degree.

$$\text{Dividing by } x - a, x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1}).$$

$$140. \text{ Substituting } -r \text{ for } x, x^n + r^n = (-r)^n + r^n$$

$$\text{If } n \text{ is odd, } = -r^n + r^n = 0;$$

$\therefore x + r$ is a factor of $x^n + r^n$, n being odd.

$$\text{Substituting } r \text{ for } x, x^n + r^n = r^n + r^n = 2r^n;$$

$\therefore x + r$ is the only rational binomial factor of the first degree.

$$\text{Dividing by } x + r, x^n + r^n = (x + r)(x^{n-1} - x^{n-2}r + x^{n-3}r^2 - \dots + r^{n-1}).$$

$$141. \quad x^3 - 6bx^2 + 12b^2x - 8b^3$$

$$\text{Substituting } 2b \text{ for } x, = 8b^3 - 24b^3 + 24b^3 - 8b^3 = 0;$$

$\therefore x - 2b$ is one factor.

$$\text{Dividing by } x - 2b, = (x - 2b)(x^2 - 4bx + 4b^2)$$

$$\text{Substituting } 2b \text{ for } x, = (x - 2b)(4b^2 - 8b^2 + 4b^2) = 0$$

$$\text{Dividing by } x - 2b, = (x - 2b)(x - 2b)(x - 2b).$$

142. Substituting $+a$ for x , $x^n + a^n = (+a)^n + a^n = 2a^n$ (1)

Substituting $-a$ for x , $x^n + a^n = (-a)^n + a^n$

If n is odd, $= -a^n + a^n = 0$ (2)

If n is even, $= +a^n + a^n = 2a^n$ (3)

By (2), $x^3 + a^3$, $x^5 + a^5$, $x^7 + a^7$, $x^9 + a^9$, $x^{11} + a^{11}$, $x^{13} + a^{13}$, $x^{15} + a^{15}$, $x^{17} + a^{17}$, and $x^{19} + a^{19}$ have the binomial factor $x + a$.

By (3) and (1), $x^2 + a^2$, $x^4 + a^4$, $x^6 + a^6$, $x^8 + a^8$, $x^{10} + a^{10}$, $x^{12} + a^{12}$, $x^{14} + a^{14}$, $x^{16} + a^{16}$, $x^{18} + a^{18}$ and $x^{20} + a^{20}$ have neither $x + a$ nor $x - a$ for a binomial factor.

Testing the latter for other binomial factors,

$$x^6 + a^6 = (x^2)^3 + (a^2)^3 = 0, \text{ if } x^2 = -a^2; \therefore x^2 + a^2 \text{ is a factor;}$$

$$x^{10} + a^{10} = (x^2)^5 + (a^2)^5 = 0, \text{ if } x^2 = -a^2; \therefore x^2 + a^2 \text{ is a factor;}$$

$$x^{12} + a^{12} = (x^4)^3 + (a^4)^3 = 0, \text{ if } x^4 = -a^4; \therefore x^4 + a^4 \text{ is a factor;}$$

$$x^{14} + a^{14} = (x^2)^7 + (a^2)^7 = 0, \text{ if } x^2 = -a^2; \therefore x^2 + a^2 \text{ is a factor;}$$

$$x^{18} + a^{18} = (x^2)^9 + (a^2)^9 = 0, \text{ if } x^2 = -a^2; \therefore x^2 + a^2 \text{ is a factor;}$$

$$x^{20} + a^{20} = (x^4)^5 + (a^4)^5 = 0, \text{ if } x^4 = -a^4; \therefore x^4 + a^4 \text{ is a factor.}$$

The above sums of like even powers of x and a have binomial factors because they can be written as sums of like *odd* powers of two other numbers, as x^2 and a^2 , or x^4 and a^4 ; but $x^2 + a^2$, $x^4 + a^4$, $x^8 + a^8$, and $x^{16} + a^{16}$ cannot be so expressed, and hence have no rational binomial factors.

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99. $x^3 - 15x^2 + 71x - 105 = 0.$

Factoring by the factor law, $(x - 3)(x - 5)(x - 7) = 0.$

$$\therefore x - 3 = 0, \text{ or } x - 5 = 0, \text{ or } x - 7 = 0;$$

whence, $x = 3, \text{ or } 5, \text{ or } 7.$

100. $x^3 + 10x^2 + 11x - 70 = 0.$

Factoring by the factor law, $(x - 2)(x + 5)(x + 7) = 0.$

$$\therefore x - 2 = 0, \text{ or } x + 5 = 0, \text{ or } x + 7 = 0;$$

whence, $x = 2, \text{ or } -5, \text{ or } -7.$

101. $x^3 - 12x + 16 = 0.$

Factoring by the factor law, $(x - 2)(x - 2)(x + 4) = 0.$

$$\therefore x - 2 = 0, \text{ or } x - 2 = 0, \text{ or } x + 4 = 0;$$

whence, $x = 2, \text{ or } 2, \text{ or } -4.$

102. $x^3 - 19x - 30 = 0.$

Factoring by the factor law, $(x + 2)(x + 3)(x - 5) = 0.$

$$\therefore x + 2 = 0, \text{ or } x + 3 = 0, \text{ or } x - 5 = 0;$$

whence, $x = -2, \text{ or } -3, \text{ or } 5.$

103. $x^4 + x^3 - 21x^2 - x + 20 = 0.$

Factoring by the factor law, $(x - 1)(x + 1)(x - 4)(x + 5) = 0.$

$$\therefore x - 1 = 0, \text{ or } x + 1 = 0, \text{ or } x - 4 = 0, \text{ or } x + 5 = 0;$$

whence, $x = 1, \text{ or } -1, \text{ or } 4, \text{ or } -5.$

104. $x^4 - 7x^3 + x^2 + 63x - 90 = 0.$

Factoring by the factor law, $(x - 2)(x - 3)(x + 3)(x - 5) = 0.$

$$\therefore x - 2 = 0, \text{ or } x - 3 = 0, \text{ or } x + 3 = 0, \text{ or } x - 5 = 0;$$

whence, $x = 2, \text{ or } 3, \text{ or } -3, \text{ or } 5.$

105. $x^4 + 8x^3 - x^2 - 68x + 60 = 0.$

Factoring by the factor law, $(x - 1)(x - 2)(x + 5)(x + 6) = 0.$

$$\therefore x - 1 = 0, \text{ or } x - 2 = 0, \text{ or } x + 5 = 0, \text{ or } x + 6 = 0;$$

whence, $x = 1, \text{ or } 2, \text{ or } -5, \text{ or } -6.$

106.

$$x^5 - 11x^4 + 45x^3 - 85x^2 + 74x - 24 = 0.$$

Factoring by the factor law, $(x-1)(x-1)(x-2)(x-3)(x-4) = 0$.

$\therefore x-1 = 0$, or $x-1 = 0$, or $x-2 = 0$, or $x-3 = 0$, or $x-4 = 0$;
whence, $x = 1$, or 1, or 2, or 3, or 4.

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$$\begin{array}{rcl} 28. & x^2 - 6x + 5 & = (x-1)(x-5). \\ & x^3 - 5x^2 + 7x - 3 & = (x-1)(x^2 - 4x + 3). \\ & \hline & \therefore \text{H.C.D.} = x-1. \end{array}$$

$$\begin{array}{rcl} 29. & x^2 - 4 & = (x+2)(x-2). \\ & x^3 - 10x^2 + 31x - 30 & = (x-2)(x-3)(x-5). \\ & \hline & \therefore \text{H.C.D.} = x-2. \end{array}$$

$$\begin{array}{rcl} 30. & x^2 - 9 & = (x+3)(x-3). \\ & x^3 - 12x^2 + 41x - 42 & = (x-3)(x-2)(x-7). \\ & \hline & \therefore \text{H.C.D.} = x-3. \end{array}$$

$$\begin{array}{rcl} 31. & x^3 - 4x + 3 & = (x-1)(x^2 + x - 3). \\ & x^3 + x^2 - 37x + 35 & = (x-1)(x-5)(x+7). \\ & \hline & \therefore \text{H.C.D.} = x-1. \end{array}$$

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$$\begin{array}{rcl} 6. & x^2 + 2x - 24 & | 2x^2 + 7x - 30 \quad | 2 \\ & & \underline{2x^2 + 4x - 48} \\ & & 3 | 3x + 18 \\ & & \underline{3x + 6} \\ & & x + 6 | x^2 + 2x - 24 \quad | x - 4 \\ & & \underline{x^2 + 6x} \\ & & -4x - 24 \\ & & \underline{-4x - 24} \\ & & \therefore \text{H.C.D.} = x + 6. \end{array}$$

$$\begin{array}{rcl} 7. & 2x^2 - x - 21 & | 4x^2 + 4x - 63 \quad | 2 \\ & & \underline{4x^2 - 2x - 42} \\ & & 3 | 6x - 21 \\ & & \underline{2x - 7} \\ & & 2x^2 - x - 21 \quad | x + 3 \\ & & \underline{2x^2 - 7x} \\ & & 6x - 21 \\ & & \underline{6x - 21} \\ & & \therefore \text{H.C.D.} = 2x - 7. \end{array}$$

$$\begin{array}{rcl} 8. & 3x^2 + 10x - 8 & | 6x^2 - 7x + 2 \quad | 2 \\ & & \underline{6x^2 + 20x - 16} \\ & & -9 | -27x + 18 \\ & & \underline{3x - 2} \\ & & 3x^2 + 10x - 8 \quad | x + 4 \\ & & \underline{3x^2 - 2x} \\ & & 12x - 8 \\ & & \underline{12x - 8} \\ & & \therefore \text{H.C.D.} = 3x - 2. \end{array}$$

$$\begin{array}{r}
 9. \quad 2x^3 - 6x^2 + 7x - 6 \overline{) 2x^3 + 4x^2 - 3x + 9} \quad \underline{1} \\
 \quad \quad \quad 2x^3 - 6x^2 + 7x - 6 \\
 \quad \quad \quad \quad \quad \quad 5 \overline{) 10x^2 - 10x + 15} \\
 \quad \quad \quad \quad \quad \quad \quad \quad 2x^2 - 2x + 3 \overline{) 2x^3 - 6x^2 + 7x - 6} \quad \underline{x-2} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 2x^3 - 2x^2 + 3x \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -4x^2 + 4x - 6 \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -4x^2 + 4x - 6 \\
 \hline
 \therefore \text{H.C.D.} = 2x^2 - 2x + 3.
 \end{array}$$

$$\begin{array}{r}
 10. \quad 2x \overline{) 2x^3 + 14x^2 + 20x} \quad x^3 + 9x^2 + 26x + 24 \overline{) x + 2} \\
 \quad \quad \quad x^2 + 7x + 10 \overline{) x^3 + 7x^2 + 10x} \\
 \quad \quad \quad \quad \quad \quad 2x^2 + 16x + 24 \\
 \quad \quad \quad \quad \quad \quad 2x^2 + 14x + 20 \\
 \quad \quad \quad \quad \quad \quad \quad \quad 2 \overline{) 2x + 4} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad x + 2 \overline{) x^2 + 7x + 10} \quad \underline{x+5} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x^2 + 2x \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 5x + 10 \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 5x + 10 \\
 \hline
 \therefore \text{H.C.D.} = x + 2.
 \end{array}$$

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$$\begin{array}{r}
 12. \quad \left| \begin{array}{r} 2x^3 - 7x^2 + 2x + 3 \\ x \overline{) 2x^3 - x^2 - x} \\ \quad \quad -6x^2 + 3x + 3 \\ -3 \overline{) -6x^2 + 3x + 3} \end{array} \right| \quad \left| \begin{array}{r} 2x^3 + 7x^2 - 5x - 4 \\ 2x^3 - 7x^2 + 2x + 3 \quad \underline{1} \\ \quad \quad 7 \overline{) 14x^2 - 7x - 7} \\ \quad \quad \quad 2x^2 - x - 1 \end{array} \right| \\
 \therefore \text{H.C.D.} = 2x^2 - x - 1.
 \end{array}$$

$$\begin{array}{r}
 13. \quad \left| \begin{array}{r} 9x^3 + 18x^2 - x - 10 \\ 3 \overline{) 9x^3 + 39x^2 + 6x - 24} \\ \quad \quad -7 \overline{) -21x^2 - 7x + 14} \\ \quad \quad \quad 3x^2 + x - 2 \end{array} \right| \quad \left| \begin{array}{r} 3x^3 + 13x^2 + 2x - 8 \\ 3x^3 + \quad x^2 - 2x \quad \quad \quad x \\ \quad \quad 12x^2 + 4x - 8 \\ \quad \quad 12x^2 + 4x - 8 \quad \underline{+4} \end{array} \right| \\
 \therefore \text{H.C.D.} = 3x^2 + x - 2.
 \end{array}$$

$$\begin{array}{r}
 14. \quad \left| \begin{array}{r} 1 - 2x - 5x^2 + 6x^3 \\ 1 \overline{) 1 + x - 2x^2} \\ \quad \quad -3x - 3x^2 + 6x^3 \\ -3x \overline{) -3x - 3x^2 + 6x^3} \end{array} \right| \quad \left| \begin{array}{r} 1 + 5x + 2x^2 - 8x^3 \\ 1 - 2x - 5x^2 + 6x^3 \quad \underline{1} \\ \quad \quad 7x \overline{) 7x + 7x^2 - 14x^3} \\ \quad \quad \quad 1 + x - 2x^2 \end{array} \right| \\
 \therefore \text{H.C.D.} = 1 + x - 2x^2.
 \end{array}$$

$$\begin{array}{r}
 15. \quad \left| \begin{array}{r} 1 - 4x + x^2 + 6x^3 \\ 1 \overline{) 1 - x - 2x^2} \\ \quad \quad -3x + 3x^2 + 6x^3 \\ -3x \overline{) -3x + 3x^2 + 6x^3} \end{array} \right| \quad \left| \begin{array}{r} 1 + 3x - 6x^2 - 8x^3 \\ 1 - 4x + x^2 + 6x^3 \quad \underline{1} \\ \quad \quad 7x \overline{) 7x - 7x^2 - 14x^3} \\ \quad \quad \quad 1 - x - 2x^2 \end{array} \right| \\
 \therefore \text{H.C.D.} = 1 - x - 2x^2.
 \end{array}$$

$$\begin{array}{r|l}
 16. & \left| \begin{array}{l} 1 - x - 14x^2 + 24x^3 \\ 1 - 5x + 6x^2 \\ 4x - 20x^2 + 24x^3 \\ 4x - 20x^2 + 24x^3 \end{array} \right| \begin{array}{l} 1 + x - 24x^2 + 36x^3 \\ 1 - x - 14x^2 + 24x^3 \\ 2x | 2x - 10x^2 + 12x^3 \\ 1 - 5x + 6x^2 \end{array} \right| 1 \\
 & + 4x \quad \therefore \text{H.C.D.} = 1 - 5x + 6x^2.
 \end{array}$$

$$\begin{array}{r|l}
 17. & \left| \begin{array}{l} m^3 - 4m^2 - 20m + 48 \\ m^3 + 2m^2 - 8m \\ -6m^2 - 12m + 48 \\ -6m^2 - 12m + 48 \end{array} \right| \begin{array}{l} m^3 - m^2 - 14m + 24 \\ m^3 - 4m^2 - 20m + 48 \\ 3 | 3m^2 + 6m - 24 \\ m^2 + 2m - 8 \end{array} \right| 1 \\
 & m \quad \therefore \text{H.C.D.} = m^2 + 2m - 8. \\
 & - 6
 \end{array}$$

$$\begin{array}{r|l}
 18. & \left| \begin{array}{l} 3a^3 + 20a^2 - a - 2 \\ 3a^3 + 17a^2 + 21a - 9 \\ 3a^2 - 22a + 7 \\ 3a^2 - a \\ -21a + 7 \\ -21a + 7 \end{array} \right| \begin{array}{l} 3a^3 + 17a^2 + 21a - 9 \\ 3a^3 - 22a^2 + 7a \\ 39a^2 + 14a - 9 \\ 39a^2 - 286a + 91 \\ 100 | 300a - 100 \\ 3a - 1 \end{array} \right| a \\
 & a \quad \therefore \text{H.C.D.} = 3a - 1. \\
 & - 7
 \end{array}$$

$$\begin{array}{r|l}
 19. & \left| \begin{array}{l} 8ax^2 + 22ax + 15a \\ 8x^2 + 22x + 15 \\ 8x^2 + 12x \\ 10x + 15 \\ 10x + 15 \end{array} \right| \begin{array}{l} 6bx^2 + 11bx + 3b \\ 6x^2 + 11x + 3 \\ 4 \\ 24x^2 + 44x + 12 \\ 24x^2 + 66x + 45 \\ -11 | -22x - 33 \\ 2x + 3 \end{array} \right| b \\
 & 4x \quad \therefore \text{H.C.D.} = 2x + 3. \\
 & + 5
 \end{array}$$

20. See next page.

21. Reserve the common factor ax as a factor of the H.C.D.

$$\begin{array}{r|l}
 ax & \left| \begin{array}{l} 21ax - 17ax^2 - 5ax^3 + ax^4 \\ 21 - 17x - 5x^2 + x^3 \\ 21 + 102x - 15x^2 \\ -119x + 10x^2 + x^3 \\ -119x - 578x^2 + 85x^3 \\ 84x^2 | 588x^2 - 84x^3 \\ 7 - x \end{array} \right| \begin{array}{l} 7ax + 34ax^2 - 5ax^3 | ax \\ 7 + 34x - 5x^2 \\ 7 - x \end{array} \right| 1 \\
 - 17x & \therefore \text{H.C.D.} = ax(7 - x). \\
 & \quad \quad \quad 35x - 5x^2 \\
 & \quad \quad \quad 35x - 5x^2 + 5x
 \end{array}$$

$$20. \quad \frac{2c}{10b^2 - b - 2} \mid \frac{20b^2c - 2bc - 4c}{8b^3 - 4b + 1} \mid \frac{8a^2b^3c - 4a^2bc + a^2c}{a^2c}$$

Reserve the common factor c as a factor of the H.C.D.

| | | |
|---|----------------------|------|
| $10b^2 - b - 2$ | $8b^3 - 4b + 1$ | |
| $5b$ | 5 | |
| $+ 2$ | $40b^3 - 20b + 5$ | |
| $5b$ | $40b^3 - 4b^2 - 8b$ | $4b$ |
| $10b^2 - 5b$ | $4b^2 - 12b + 5$ | |
| $4b - 2$ | 5 | |
| $4b - 2$ | $20b^2 - 60b + 25$ | |
| $\therefore \text{H.C.D.} = c(2b - 1).$ | $20b^2 - 2b - 4$ | 2 |
| $2b - 1$ | $-29 \mid -58b + 29$ | |

| | | |
|--|------------------------|--------|
| $22. \quad x \mid x^4 - 2x^3 - 9x^2 + 18x$ | $x^3 + x^2 - 4x - 4$ | |
| $x^3 - 2x^2 - 9x + 18$ | $x^3 - 2x^2 - 9x + 18$ | 1 |
| 3 | $3x^2 + 5x - 22$ | |
| $3x^3 - 6x^2 - 27x + 54$ | | |
| $3x^3 + 5x^2 - 22x$ | | |
| $-11x^2 - 5x + 54$ | | |
| 3 | | |
| $-33x^2 - 15x + 162$ | | |
| $-33x^2 - 55x + 242$ | | |
| $40 \mid 40x - 80$ | | |
| $x - 2$ | $3x^2 - 6x$ | $3x$ |
| -11 | $11x - 22$ | |
| $40x - 80$ | $11x - 22$ | $+ 11$ |

By the factor law it is seen that $x - 2$, the H.C.D. of the second and third polynomials, is an exact divisor of the first, $x^3 - 7x + 6$.

$\therefore \text{H.C.D.} = x - 2$.

| | | |
|--------------------------|-------------------------|----------|
| $23. \quad x^3 - 5x + 4$ | $x^4 - 2x^3 + 1$ | |
| 5 | $x^4 - 5x^2 + 4x$ | x |
| $5x^3 - 25x + 20$ | $-2x^3 + 5x^2 - 4x + 1$ | |
| $5x^3 - 14x^2 + 9x$ | $-2x^3 + 10x - 8$ | -2 |
| $14x^2 - 34x + 20$ | $5x^2 - 14x + 9$ | |
| 5 | | |
| $70x^2 - 170x + 100$ | | |
| $70x^2 - 196x + 126$ | | |
| $26 \mid 26x - 26$ | | |
| $x - 1$ | $5x^2 - 14x + 9$ | $5x - 9$ |

By the factor law it is seen that $x - 1$, the H.C.D. of the first two polynomials, is an exact divisor of the third, $x^3 + 4x^2 - 3x - 2$.

$\therefore \text{H.C.D.} = x - 1$.

$$\begin{array}{r|l}
 24. & \begin{array}{r} 1 \\ 4 \end{array} + 4x^2 + 5x^3 \quad \begin{array}{r} 2 + 5x \\ 2 + 8x^2 + 10x^3 + 3x^4 \end{array} \\
 & \hline
 & \begin{array}{r} 4 \\ 4 + 15x + 11x^2 \end{array} + 16x^2 + 20x^3 \\
 1 & \begin{array}{r} 4 \\ 4 + 15x + 11x^2 \end{array} \\
 & \hline
 & -15x + 5x^2 + 20x^3 \\
 & \hline
 & -60x + 20x^2 + 80x^3 \\
 -15x & \begin{array}{r} -60x - 225x^2 - 165x^3 \\ 245x^2 | 245x^2 + 245x^3 \end{array} \\
 & \hline
 & 1 + x \quad \begin{array}{r} 4 + 15x + 11x^2 \end{array} \quad 4 + 11x
 \end{array}$$

By the factor law it is seen that $x + 1$, or $1 + x$, the H.C.D. of the first two polynomials, is an exact divisor of the third, $x^6 - 4x^4 + 5x^2 - 2$.

\therefore H.C.D. = $1 + x$.

25. See next page.

$$\begin{array}{r|l}
 26. & \begin{array}{r} x^3 - 6x^2 - 5x - 14 \\ 4 \end{array} \quad \begin{array}{r} x^3 - 10x^2 + 20x + 7 \\ x^3 - 6x^2 - 5x - 14 \end{array} \\
 & \hline
 -x & \begin{array}{r} 4x^3 - 24x^2 - 20x - 56 \\ 4x^3 - 25x^2 - 21x \end{array} \quad \begin{array}{r} -4x^2 + 25x + 21 \\ -4x^2 - 4x + 224 \end{array} \\
 & \hline
 & \begin{array}{r} x^2 + x - 56 \\ x^2 + x - 56 \end{array} \quad \begin{array}{r} 29 | 29x - 203 \\ x - 7 \end{array} \\
 x + 8 & \hline
 & \begin{array}{r} x^2 + x - 56 \end{array}
 \end{array}$$

By trial it is found that $x - 7$, the H.C.D. of the first two polynomials, is an exact divisor of the third, $x^4 - 310x - 231$.

\therefore H.C.D. = $x - 7$.

$$\begin{array}{r|l}
 28. & \begin{array}{r} 1 - 1 - 2 - 1 + 1 + 2 \\ 4 \end{array} \quad \begin{array}{r} 1 + 3 + 3 + 1 - 1 - 1 \\ 1 - 1 - 2 - 1 + 1 + 2 \end{array} \\
 & \hline
 1 & \begin{array}{r} 4 - 4 - 8 - 4 + 4 + 8 \\ 4 + 5 + 2 - 2 - 3 \end{array} \quad \begin{array}{r} 4 + 5 + 2 - 2 - 3 \\ 9 \end{array} \\
 & \hline
 & -9 - 10 - 2 + 7 + 8 \quad \begin{array}{r} 36 + 45 + 18 - 18 - 27 \\ 36 + 40 + 8 - 28 - 32 \end{array} \\
 & \hline
 -9 & \begin{array}{r} -9 - 18 - 18 - 9 \\ 8 + 16 + 16 + 8 \end{array} \quad \begin{array}{r} 5 | 5 + 10 + 10 + 5 \\ 1 + 2 + 2 + 1 \end{array} \\
 & \hline
 +8 & \begin{array}{r} 8 + 16 + 16 + 8 \\ 8 + 16 + 16 + 8 \end{array} \quad \therefore \text{H.C.D.} = x^3 + 2x^2 + 2x + 1.
 \end{array}$$

$$\begin{array}{r|l}
 29. & \begin{array}{r} 1 + 1 + 0 - 1 - 7 - 4 \\ 1 + 3 + 5 + 7 + 4 \end{array} \quad \begin{array}{r} 2 + 3 + 3 + 3 - 7 - 4 \\ 2 + 2 + 0 - 2 - 14 - 8 \end{array} \\
 & \hline
 1 & \begin{array}{r} 1 + 3 + 5 + 7 + 4 \\ -2 - 5 - 8 - 11 - 4 \end{array} \quad \begin{array}{r} 1 + 3 + 5 + 7 + 4 \\ 1 + 2 + 3 + 4 \end{array} \\
 & \hline
 -2 & \begin{array}{r} -2 - 6 - 10 - 14 - 8 \\ 1 + 2 + 3 + 4 \end{array} \quad \begin{array}{r} 1 + 2 + 3 + 4 \\ 1 + 2 + 3 + 4 \end{array} \\
 & \hline
 & \therefore \text{H.C.D.} = x^3 + 2x^2 + 3x + 4.
 \end{array}$$

$$\begin{array}{r|l}
 25. & \begin{array}{r} 3 + x - 8x^2 + 4x^3 \\ 3 - 8x - 8x^2 + 8x^3 \\ \hline x \mid 9x \quad - 4x^3 \\ \hline 9 \quad - 4x^2 \end{array} & \begin{array}{r} 3 - 8x - 8x^2 + 8x^3 \\ 3 \\ \hline 9 - 24x - 24x^2 + 24x^3 \\ 9 \quad - 4x^2 \\ \hline - 24x - 20x^2 + 24x^3 \\ 3 \\ \hline - 72x - 60x^2 + 72x^3 \\ - 72x \quad + 32x^3 \\ \hline - 20x^2 - 60x^2 + 40x^3 \\ 3 \quad - 2x \end{array} \\
 & 1 & 1 \\
 & 3 + 2x & 9 \quad - 4x^2
 \end{array}$$

By trial it is found that $3 - 2x$, the H.C.D. of the first two polynomials, is an exact divisor of the third, $16x^4 - 48x^3 + 81$, or $81 - 48x^3 + 16x^4$.
 \therefore H.C.D. = $3 - 2x$.

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$$\begin{array}{r|l}
 30. & \begin{array}{r} 1 - 2 - 2 - 11 - 1 - 15 \\ 3 \\ \hline 3 - 6 - 6 - 33 - 3 - 45 \\ - 1 \mid 3 - 8 - 7 - 3 - 20 \\ \hline 2 + 1 - 30 + 17 - 45 \end{array} & \begin{array}{r} 2 - 7 + 4 - 15 + 1 - 10 \\ 2 - 4 - 4 - 22 - 2 - 30 \\ \hline - 3 + 8 + 7 + 3 + 20 \\ 2 \\ \hline - 6 + 16 + 14 + 6 + 40 \\ - 6 - 3 + 90 - 51 + 135 \\ \hline 19 \mid 19 - 76 + 57 - 95 \\ 1 - 4 + 3 - 5 \end{array} \\
 & - 1 & 2 \\
 & 2 & - 3 \\
 & + 9 & \therefore \text{H.C.D.} = x^3 - 4x^2 + 3x - 5.
 \end{array}$$

$$\begin{array}{r|l}
 31. & \begin{array}{r} 1 - 3 - 3 - 3 - 19 - 15 \\ 1 \mid 1 + 0 + 2 + 3 \\ \hline - 3 - 5 - 6 - 19 - 15 \\ - 3 \mid - 3 + 0 \quad 6 - 9 \\ \hline - 5 + 0 - 10 - 15 \\ - 5 \mid - 5 + 0 - 10 - 15 \end{array} & \begin{array}{r} 1 + 3 - 3 + 9 - 1 - 15 \\ 1 - 3 - 3 - 3 - 19 - 15 \\ \hline 6a \mid 6 + 0 + 12 + 18 + 0 \\ 1 + 0 + 2 + 3 \end{array} \\
 & 1 & 1 \\
 & - 3 & \\
 & - 5 & \therefore \text{H.C.D.} = a^3 + 2a + 3.
 \end{array}$$

32. Reject the factor a from the second polynomial.

$$\begin{array}{r|l}
 - 1 & \begin{array}{r} 5a^5 + a^4 - 11a^3 + 9a^2 - 8a + 4 \\ - 2a^4 + a^3 + 5a^2 - 8a + 4 \\ \hline a^2 \mid 5a^5 + 3a^4 - 12a^3 + 4a^2 \\ 5a^3 + 3a^2 - 12a + 4 \end{array} & \begin{array}{r} 2a^4 - a^3 - 5a^2 + 8a - 4 \\ \hline \text{Add } 5a^3 + 3a^2 - 12a + 4 \\ 2a \mid 2a^4 + 4a^3 - 2a^2 - 4a \\ a^3 + 2a^2 - a - 2 \end{array} \\
 5 & \begin{array}{r} 5a^3 + 10a^2 - 5a - 10 \\ - 7 \mid - 7a^2 - 7a + 14 \\ a^2 + a - 2 \end{array} & \begin{array}{r} a^3 + a^2 - 2a \\ a^2 + a - 2 \\ \hline a^2 + a - 2 \end{array} \\
 & \therefore \text{H.C.D.} = a^2 + a - 2. & a + 1
 \end{array}$$

$$\begin{array}{r|l}
 33. & \\
 1 & \begin{array}{r} 1+0+0+0-5+4 \\ 1-1-3-5-12 \\ \hline 1+3+5+7+4 \\ 1-1-3-5-12 \\ \hline 4|4+8+12+16 \\ 1+2+3+4 \end{array} \\
 +1 & \begin{array}{r} 1-1-3-5-12 \\ \hline 4|4+8+12+16 \\ 1+2+3+4 \end{array} \\
 \hline
 \therefore \text{H.C.D.} = x^3 + 2x^2 + 3x + 4. & \begin{array}{r} 1-1-3-5-12 \\ \hline 1+2+3+4 \\ \hline -3-6-9-12 \\ -3-6-9-12 \end{array} \begin{array}{l} 1 \\ -3 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 34. & \\
 a^3 + 3a^2 - 2a - 6 & \begin{array}{r} a^5 + 4a^4 + 4a^3 + 4a^2 - a - 12 \\ a^5 + 3a^4 - 2a^3 - 6a^2 \\ \hline a^4 + 6a^3 + 10a^2 - a \\ a^4 + 3a^3 - 2a^2 - 6a \\ \hline 3a^3 + 12a^2 + 5a - 12 \\ 3a^3 + 9a^2 - 6a - 18 \\ \hline 3a^2 + 11a + 6 \end{array} \\
 -1 & \begin{array}{r} -3a^2 - 11a - 6 \\ a|a^3 + 6a^2 + 9a \\ \hline a^2 + 6a + 9 \end{array} \\
 a+3 & \begin{array}{r} a^2 + 6a + 9 \\ \hline \therefore \text{H.C.D.} = a + 3. \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 35. & \\
 1 & \begin{array}{r} 1+0+0-4+3 \\ 1-1-1+1 \\ \hline 1+1-5+3 \\ 1-1-1+1 \\ \hline 2|2-4+2 \\ 1-2+1 \end{array} \\
 +1 & \begin{array}{r} 1+1-5+3 \\ 1-1-1+1 \\ \hline 2|2-4+2 \\ 1-2+1 \end{array} \\
 \hline
 \therefore \text{H.C.D.} = 1 - 2a + a^2. & \begin{array}{r} 1+1-1-5+4 \\ 1+0+0-4+3 \\ \hline 1-1-1+1 \\ \hline 1-2+1 \\ \hline 1-2+1 \\ 1-2+1 \end{array} \begin{array}{l} 1 \\ 1 \\ 1 \\ +1 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 36. & \\
 -1 & \begin{array}{r} 2-1+3+5-1 \\ 2+5+10+7 \\ \hline -6-7-2-1 \\ -6-15-30-21 \\ \hline 4|8+28+20 \\ 2+7+5 \end{array} \\
 +3 & \begin{array}{r} 2-1+3+5-1 \\ 2+5+10+7 \\ \hline -6-7-2-1 \\ -6-15-30-21 \\ \hline 4|8+28+20 \\ 2+7+5 \end{array} \\
 2+5 & \begin{array}{r} 2-1+3+5-1 \\ 2+5+10+7 \\ \hline -6-7-2-1 \\ -6-15-30-21 \\ \hline 4|8+28+20 \\ 2+7+5 \end{array} \\
 \hline
 \therefore \text{H.C.D.} = 1 + a. & \begin{array}{r} 4-4+1+0-9 \\ 4-2+6+10-2 \\ \hline -2-5-10-7 \\ \hline -2-7-5 \\ \hline 2-5-7 \\ 2+7+5 \\ \hline -12|-12-12 \\ 1+1 \end{array} \begin{array}{l} 2 \\ -1 \\ +1 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 37. & \begin{array}{l} y^5 + 13y^2 + 20y - 14 \\ y^5 - 2y^3 + 20y^2 + 3y - 7 \\ \hline 2y^3 - 7y^2 + 17y - 7 \end{array} & \begin{array}{l} - y^5 + 2y^3 - 20y^2 - 3y + 7 \\ \hline - 2y^3 + 7y^2 - 17y + 7 \\ \hline - y^5 + 4y^3 - 27y^2 + 14y \\ \hline - 4y^4 + 14y^3 - 34y^2 + 14y \\ \hline - y^5 + 4y^4 - 10y^3 + 7y^2 \\ \hline - 2y^5 + 7y^4 - 17y^3 + 7y^2 \\ \hline y^3 | y^5 - 3y^4 + 7y^3 \\ \hline y^2 - 3y + 7 \end{array} & \begin{array}{l} 1 \\ - 2y \\ - y^2 \end{array} \\
 2y & \begin{array}{l} 2y^3 - 6y^2 + 14y \\ \hline - y^2 + 3y - 7 \\ \hline - y^2 + 3y - 7 \end{array} & \begin{array}{l} \therefore \text{H.C.D.} = y^2 - 3y + 7. \end{array} \\
 -1 & &
 \end{array}$$

38. Dividing the first polynomial by x , and adding it to the second,

$$\begin{array}{r}
 30x^2 - 115x + 35 \\
 6x^2 - 11x - 35 \\
 \hline
 \text{sum} = 36x^2 - 126x.
 \end{array}$$

Rejecting $18x$, divisor = $2x - 7$.

$$\begin{array}{r|l}
 6x^2 - 11x - 35 & 2x - 7 \\
 6x^2 - 11x - 35 & \hline 3x + 5 & \\
 \hline 2x^3 - 5x^2 - 5x - 7 & 2x - 7 \\
 2x^3 - 7x^2 & \hline x^2 + (x + 1) \\
 \hline 2x^2 - 5x - 7 & \\
 2x^2 - 5x - 7 & \hline 0 &
 \end{array}$$

$\therefore \text{H.C.D.} = 2x - 7$.

LOWEST COMMON MULTIPLE

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$$\begin{array}{l}
 30. \quad \begin{array}{l} x^2 - y^2 = (x + y)(x - y). \\ x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2). \\ x^3 + y^3 = (x + y)(x^2 - xy + y^2). \\ x^2 + xy + y^2 = x^2 + xy + y^2. \\ \hline \therefore \text{L.C.M.} = (x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2) \\ = (x^3 + y^3)(x^3 - y^3) = x^6 - y^6. \end{array}
 \end{array}$$

$$\begin{array}{l}
 31. \quad \begin{array}{l} x^3 + x^2y + xy^2 + y^3 = (x^2 + y^2)(x + y). \\ x^3 - x^2y + xy^2 - y^3 = (x^2 + y^2)(x - y). \\ \hline \therefore \text{L.C.M.} = (x^2 + y^2)(x + y)(x - y) \\ = x^4 - y^4. \end{array}
 \end{array}$$

$$\begin{array}{l}
 32. \quad \begin{array}{l} a^2 + 4a + 4 = (a + 2)^2. \\ a^2 - 4 = (a + 2)(a - 2). \\ a^4 - 16 = (a^2 + 4)(a + 2)(a - 2). \\ \hline \therefore \text{L.C.M.} = (a + 2)^2(a - 2)(a^2 + 4) \\ = (a^4 - 16)(a + 2). \end{array}
 \end{array}$$

$$\begin{array}{l}
 33. \quad \begin{array}{l} a^2 - (b + c)^2 = (a + b + c)(a - b - c) \\ b^2 - (c + a)^2 = (b + c + a)(b - c - a) \\ c^2 - (a + b)^2 = (c + a + b)(c - a - b) \\ \hline \therefore \text{L.C.M.} = (a + b + c)(a - b - c)(b - c - a)(c - a - b). \end{array}
 \end{array}$$

34. Since $1 + a^2 + a^4$, the third polynomial, is the product of the other polynomials, the L.C.M. is $1 + a^2 + a^4$.

35.
$$\frac{a^4 + 4 = (a^2 + 2a + 2)(a^2 - 2a + 2).}{a^4 - 2a^3 + 4a - 4 = (a^2 - 2)(a^2 - 2a + 2).}$$

$$\therefore \text{L.C.M.} = \frac{(a^2 + 2a + 2)(a^2 - 2a + 2)(a^2 - 2)}{(a^4 + 4)(a^2 - 2)}.$$
36.
$$\frac{a^6 - b^3 = (a^2 - b)(a^4 + a^2b + b^2).}{a^8 + a^4b^2 + b^4 = (a^4 + a^2b + b^2)(a^4 - a^2b + b^2).}$$

$$\therefore \text{L.C.M.} = \frac{(a^2 - b)(a^4 + a^2b + b^2)(a^4 - a^2b + b^2)}{(a^2 - b)(a^8 + a^4b^2 + b^4)}.$$
37.
$$\frac{x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).}{a^2x^2 - b^2y^2 + a^2y^2 - b^2x^2 = (a^2 - b^2)(x^2 + y^2).}$$

$$\therefore \text{L.C.M.} = \frac{(x^2 + y^2)(x^4 - x^2y^2 + y^4)(a^2 - b^2)}{(a^2 - b^2)(x^6 + y^6)}.$$
38.
$$\frac{a^4 - 2a^3b + a^2b^2 - 9b^4 = (a^2 - ab + 3b^2)(a^2 - ab - 3b^2).}{a^4 + 5a^2b^2 + 9b^4 = (a^2 + ab + 3b^2)(a^2 - ab + 3b^2).}$$

$$\therefore \text{L.C.M.} = \frac{(a^2 - ab + 3b^2)(a^2 - ab - 3b^2)(a^2 + ab + 3b^2)}{(a^2 - ab + 3b^2)(a^2 - ab - 3b^2)(a^2 + ab + 3b^2)}.$$
39.
$$\frac{a^4 - a^2 + 1 = a^4 - a^2 + 1.}{a^6 + 1 = (a^2 + 1)(a^4 - a^2 + 1).}$$

$$\frac{a^4 + a^2 + 1 = a^4 + a^2 + 1.}{a^2 - 1 = a^2 - 1.}$$

$$\therefore \text{L.C.M.} = \frac{(a^4 - a^2 + 1)(a^2 + 1)(a^4 + a^2 + 1)(a^2 - 1)}{(a^6 + 1)(a^6 - 1)} = a^{12} - 1.$$

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41.
$$\frac{x^6 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3).}{x^8 - 9x^2 + 26x - 24 = (x - 2)(x - 3)(x - 4).}$$

$$\therefore \text{L.C.M.} = \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{(x - 2)(x - 3)(x - 4)}.$$
42.
$$\frac{x^8 - 5x^2 - 4x + 20 = (x - 2)(x + 2)(x - 5).}{x^8 + 2x^2 - 25x - 50 = (x + 2)(x - 5)(x + 5).}$$

$$\therefore \text{L.C.M.} = \frac{(x - 2)(x + 2)(x - 5)(x + 5)}{(x^2 - 4)(x^2 - 25)}.$$
43.
$$\frac{x^3 + 3x^2 - 4 = (x - 1)(x + 2)^2.}{x^3 + x^2 - x - 1 = (x - 1)(x + 1)^2.}$$

$$\therefore \text{L.C.M.} = \frac{(x - 1)(x + 2)^2(x + 1)^2}{(x - 1)(x + 2)^2(x + 1)^2}.$$
44.
$$\frac{x^8 - 4x^2 + 5x - 2 = (x - 1)(x - 1)(x - 2).}{x^8 - 8x^2 + 21x - 18 = (x - 2)(x - 3)^2.}$$

$$\therefore \text{L.C.M.} = \frac{(x - 1)^2(x - 2)(x - 3)^2}{(x - 1)^2(x - 2)(x - 3)^2}.$$
45.
$$\frac{x^3 + 5x^2 + 7x + 3 = (x + 1)(x + 1)(x + 3).}{x^3 - 7x^2 - 5x + 75 = (x - 5)(x - 5)(x + 3).}$$

$$\therefore \text{L.C.M.} = \frac{(x + 1)^2(x + 3)(x - 5)^2}{(x + 1)^2(x + 3)(x - 5)^2}.$$
46.
$$\frac{x^3 + 2x^2 - 4x - 8 = (x - 2)(x + 2)^2.}{x^3 - x^2 - 8x + 12 = (x - 2)(x - 2)(x + 3).}$$

$$\frac{x^3 + 4x^2 - 3x - 18 = (x - 2)(x + 3)^2.}{\therefore \text{L.C.M.} = \frac{(x - 2)^2(x + 2)^2(x + 3)^2}{(x - 2)^2(x + 2)^2(x + 3)^2}}.$$

$$\begin{aligned}
 47. \quad & x^3 - 9x^2 + 23x - 15 = (x-1)(x-3)(x-5). \\
 & x^3 + x^2 - 17x + 15 = (x-1)(x-3)(x+5). \\
 & x^3 + 7x^2 + 7x - 15 = (x-1)(x+3)(x+5). \\
 & \therefore \text{L.C.M.} = \frac{(x-1)(x-3)(x-5)(x+3)(x+5)}{(x-1)(x^2-9)(x^2-25)}.
 \end{aligned}$$

$$\begin{aligned}
 48. \quad & x^3 + 7x^2 + 14x + 8 = (x+1)(x+2)(x+4). \\
 & x^3 + 3x^2 - 6x - 8 = (x+1)(x-2)(x+4). \\
 & x^3 + x^2 - 10x + 8 = (x-1)(x-2)(x+4). \\
 & \therefore \text{L.C.M.} = \frac{(x+1)(x+2)(x+4)(x-2)(x-1)}{(x^2-1)(x^2-4)(x+4)}.
 \end{aligned}$$

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$$\begin{aligned}
 2. \quad & \text{H.C.D.} = a^2 + 2a + 3. \\
 & 4a^3 + 7a^2 + 10a - 3 = (a^2 + 2a + 3)(4a - 1). \\
 & 4a^3 + 9a^2 + 14a + 3 = (a^2 + 2a + 3)(4a + 1). \\
 & \therefore \text{L.C.M.} = (a^2 + 2a + 3)(4a - 1)(4a + 1). \\
 3. \quad & \text{H.C.D.} = a^2 - 2a + 2. \\
 & 2a^3 - 11a^2 + 18a - 14 = (a^2 - 2a + 2)(2a - 7). \\
 & 2a^3 + 3a^2 - 10a + 14 = (a^2 - 2a + 2)(2a + 7). \\
 & \therefore \text{L.C.M.} = (a^2 - 2a + 2)(2a - 7)(2a + 7). \\
 4. \quad & \text{H.C.D.} = x^2 - 3x + 3. \\
 & 5x^3 - 11x^2 + 3x + 12 = (x^2 - 3x + 3)(5x + 4). \\
 & 5x^3 - 19x^2 + 27x - 12 = (x^2 - 3x + 3)(5x - 4). \\
 & \therefore \text{L.C.M.} = (x^2 - 3x + 3)(5x + 4)(5x - 4).
 \end{aligned}$$

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$$\begin{aligned}
 5. \quad & \text{H.C.D.} = x^2 - 3x + 4. \\
 & 4x^3 - 14x^2 + 22x - 8 = (x^2 - 3x + 4) \cdot 2(2x - 1). \\
 & 2x^4 - 3x^3 - x^2 + 12x = (x^2 - 3x + 4) \cdot x(2x + 3). \\
 & \therefore \text{L.C.M.} = 2x(x^2 - 3x + 4)(2x - 1)(2x + 3). \\
 6. \quad & \text{H.C.D.} = a^2 + 3a + 5. \\
 & 6a^3 + 3a^2 - 15a - 75 = (a^2 + 3a + 5) \cdot 3(2a - 5). \\
 & 2a^3 + 11a^2 + 25a + 25 = (a^2 + 3a + 5)(2a + 5). \\
 & \therefore \text{L.C.M.} = 3(a^2 + 3a + 5)(2a - 5)(2a + 5). \\
 7. \quad & \text{H.C.D.} = a^2 - 6a - 5. \\
 & 4a^3 - 27a^2 - 2a + 15 = (a^2 - 6a - 5)(4a - 3). \\
 & 2a^4 - 9a^3 - 28a^2 - 15a = (a^2 - 6a - 5) \cdot a(2a + 3). \\
 & \therefore \text{L.C.M.} = a(a^2 - 6a - 5)(4a - 3)(2a + 3). \\
 8. \quad & \text{H.C.D.} = c^2 - 5c - 4. \\
 & 3c^3 - 11c^2 - 32c - 16 = (c^2 - 5c - 4)(3c + 4). \\
 & 3c^3 - 19c^2 + 8c + 16 = (c^2 - 5c - 4)(3c - 4). \\
 & \therefore \text{L.C.M.} = (c^2 - 5c - 4)(3c + 4)(3c - 4). \\
 9. \quad & \text{H.C.D.} = x^3 - x^2 + x - 2. \\
 & 4x^4 - 7x^3 + 7x^2 - 11x + 6 = (x^3 - x^2 + x - 2)(4x - 3). \\
 & 2x^4 + x^3 - x^2 - x - 6 = (x^3 - x^2 + x - 2)(2x + 3). \\
 & \therefore \text{L.C.M.} = (x^3 - x^2 + x - 2)(4x - 3)(2x + 3). \\
 10. \quad & \text{H.C.D.} = x^2 - 3x + 3. \\
 & x^4 - x^3 - 3x + 9 = (x^2 - 3x + 3)(x^2 + 2x + 3). \\
 & 3ax^4 - 3ax^3 - 18ax^2 + 45ax - 27a = (x^2 - 3x + 3) \cdot 3a(x^2 + 2x - 3). \\
 & \therefore \text{L.C.M.} = 3a(x^2 - 3x + 3)(x^2 + 2x + 3)(x^2 + 2x - 3).
 \end{aligned}$$

11. H.C.D. = $2x^2 - x + 5$.
 $20x^3 + 40x^2 + 25x + 125 = (2x^2 - x + 5) \cdot 5(2x + 5)$.
 $6x^3 + 7x^2 + 10x + 25 = (2x^2 - x + 5)(3x + 5)$.
 \therefore L.C.M. = $5(2x^2 - x + 5)(2x + 5)(3x + 5)$.
12. H.C.D. = $3m^2 - 3m + 5$.
 $12m^3 - 18m^2 + 26m - 10 = (3m^2 - 3m + 5) \cdot 2(2m - 1)$.
 $15m^3 - 9m^2 + 19m + 10 = (3m^2 - 3m + 5)(5m + 2)$.
 \therefore L.C.M. = $2(3m^2 - 3m + 5)(2m - 1)(5m + 2)$.
13. H.C.D. = $2a^2 - 3a - 4$.
 $6a^3x - 5a^2x - 18ax - 8x = (2a^2 - 3a - 4) \cdot x(3a + 2)$.
 $6a^3b - 13a^2b - 6ab + 8b = (2a^2 - 3a - 4) \cdot b(3a - 2)$.
 \therefore L.C.M. = $bx(2a^2 - 3a - 4)(3a + 2)(3a - 2)$.
14. H.C.D. = $2x^2 - 3xy + 5y^2$.
 $4x^3 + 4x^2y - 5xy^2 + 25y^3 = (2x^2 - 3xy + 5y^2)(2x + 5y)$.
 $4x^3 - 16x^2y + 25xy^2 - 25y^3 = (2x^2 - 3xy + 5y^2)(2x - 5y)$.
 \therefore L.C.M. = $(2x^2 - 3xy + 5y^2)(2x + 5y)(2x - 5y)$.
15. H.C.D. = $2a^2 + 7a - 3$.
 $10a^3 + 29a^2 - 36a + 9 = (2a^2 + 7a - 3)(5a - 3)$.
 $8a^3 + 34a^2 + 9a - 9 = (2a^2 + 7a - 3)(4a + 3)$.
 \therefore L.C.M. = $(2a^2 + 7a - 3)(5a - 3)(4a + 3)$.
16. H.C.D. = $2x^2 - 3xy - 2y^2$.
 $4x^4 - 17x^2y^2 + 4y^4 = (2x^2 - 3xy - 2y^2)(2x^2 + 3xy - 2y^2)$.
 $2x^4 - x^3y - 3x^2y^2 - 5xy^3 - 2y^4 = (2x^2 - 3xy - 2y^2)(x^2 + xy + y^2)$.
 \therefore L.C.M. = $(2x^2 - 3xy - 2y^2)(2x^2 + 3xy - 2y^2)(x^2 + xy + y^2)$.
17. H.C.D. = $x^2 + 2x - 5$.
 $5x^4 + 8x^3 - 27x^2 + 14x - 10 = (x^2 + 2x - 5)(5x^2 - 2x + 2)$.
 $3x^4 + 4x^3 - 17x^2 + 14x - 10 = (x^2 + 2x - 5)(3x^2 - 2x + 2)$.
 \therefore L.C.M. = $(x^2 + 2x - 5)(5x^2 - 2x + 2)(3x^2 - 2x + 2)$.
18. H.C.D. = $x^2 - 3x + 3$.
 $2x^4 - 9x^3 + 18x^2 - 18x + 9 = (x^2 - 3x + 3)(2x^2 - 3x + 3)$.
 $3x^4 - 11x^3 + 17x^2 - 12x + 6 = (x^2 - 3x + 3)(3x^2 - 2x + 2)$.
 \therefore L.C.M. = $(x^2 - 3x + 3)(2x^2 - 3x + 3)(3x^2 - 2x + 2)$.
19. H.C.D. = $a^3 + 5a^2 - 3a + 2$.
 $3a^4 + 13a^3 - 19a^2 + 12a - 4 = (a^3 + 5a^2 - 3a + 2)(3a - 2)$.
 $4a^4 + 22a^3 - 2a^2 + 2a + 4 = (a^3 + 5a^2 - 3a + 2) \cdot 2(2a + 1)$.
 \therefore L.C.M. = $2(a^3 + 5a^2 - 3a + 2)(3a - 2)(2a + 1)$.
20. H.C.D. of first two expressions = $2x + 3$.
 $6x^2 + 5x - 6 = (2x + 3)(3x - 2)$.
 $8x^2 + 10x - 3 = (2x + 3)(4x - 1)$.
 \therefore their L.C.M. = $(2x + 3)(3x - 2)(4x - 1)$.
 $10x^2 + 9x - 9 = (2x + 3)(5x - 3)$.
 \therefore L.C.M. = $(2x + 3)(3x - 2)(4x - 1)(5x - 3)$.
21. H.C.D. of first two expressions = $x^2 - x + 1$.
 $\therefore x^4 - 2x^3 + x^2 - 1 = (x^2 - x + 1)(x^2 - x - 1)$
and $x^4 - x^2 + 2x - 1 = (x^2 - x + 1)(x^2 + x - 1)$.
Trying these factors in the third expression,
 $x^4 - 3x^2 + 1 = (x^2 + x - 1)(x^2 - x - 1)$.
 \therefore L.C.M. = $(x^2 - x + 1)(x^2 - x - 1)(x^2 + x - 1)$

22. $x^4 - 7x^2 + 9 = x^4 - 6x^2 + 9 - x^2 = (x^2 + x - 3)(x^2 - x - 3)$.
 $x^4 + 2x^3 + x^2 - 9 = (x^2 + x)^2 - 3^2 = (x^2 + x + 3)(x^2 + x - 3)$.
 $x^4 - x^2 - 6x - 9 = (x^2)^2 - (x + 3)^2 = (x^2 + x + 3)(x^2 - x - 3)$.
 $\therefore \text{L.C.M.} = (x^2 + x - 3)(x^2 - x - 3)(x^2 + x + 3)$.
23. $x^4 - 4x^3 + 4x^2 - 16 = (x^2 - 2x)^2 - 4^2 = (x^2 - 2x + 4)(x^2 - 2x - 4)$.
 $x^4 - 12x^2 + 16 = x^4 - 8x^2 + 16 - 4x^2 = (x^2 + 2x - 4)(x^2 - 2x - 4)$.
 $x^4 - 4x^2 + 16x - 16 = (x^2)^2 - (2x - 4)^2 = (x^2 + 2x - 4)(x^2 - 2x + 4)$.
 $\therefore \text{L.C.M.} = (x^2 - 2x + 4)(x^2 - 2x - 4)(x^2 + 2x - 4)$.
24. $x^4 - 4x^3 + 4x^2 - 25 = (x^2 - 2x)^2 - 5^2 = (x^2 - 2x + 5)(x^2 - 2x - 5)$.
 $x^4 - 4x^2 + 20x - 25 = (x^2)^2 - (2x - 5)^2 = (x^2 + 2x - 5)(x^2 - 2x + 5)$.
 $x^4 - 14x^2 + 25 = x^4 - 10x^2 + 25 - 4x^2 = (x^2 + 2x - 5)(x^2 - 2x - 5)$.
 $\therefore \text{L.C.M.} = (x^2 - 2x + 5)(x^2 - 2x - 5)(x^2 + 2x - 5)$.
25. H.C.D. of first two expressions $= 2x^3 + x^2 + 3x + 1$.
 $4x^4 + 5x^2 - x - 1 = (2x^3 + x^2 + 3x + 1)(2x - 1)$.
and $6x^4 + x^3 + 8x^2 - 1 = (2x^3 + x^2 + 3x + 1)(3x - 1)$.
 $\therefore \text{L.C.M. of first two expressions} = (2x^3 + x^2 + 3x + 1)(2x - 1)(3x - 1)$.
 $36x^4 - 13x^2 + 1 = 36x^4 - 12x^2 + 1 - x^2$
 $= (6x^2 + x - 1)(6x^2 - x - 1)$
 $= (2x + 1)(3x - 1)(2x - 1)(3x + 1)$.
 $\therefore \text{L.C.M.} = (2x^3 + x^2 + 3x + 1)(2x - 1)(3x - 1)(2x + 1)(3x + 1)$
 $= (2x^3 + x^2 + 3x + 1)(36x^4 - 13x^2 + 1)$.
26. H.C.D. $= 2x^2 + 3x - 5$.
 $10x^4 + 7x^3 - 33x^2 + 26x - 10 = (2x^2 + 3x - 5)(5x^2 - 4x + 2)$.
 $2x^4 + 7x^3 + 5x^2 - 4x - 10 = (2x^2 + 3x - 5)(x^2 + 2x + 2)$.
 $\therefore \text{L.C.M.} = (2x^2 + 3x - 5)(5x^2 - 4x + 2)(x^2 + 2x + 2)$.
27. H.C.D. $= 4x^2 - 9$.
 $16x^4 + 16x^3 - 48x^2 - 36x + 27 = (4x^2 - 9)(4x^2 + 4x - 3)$.
 $24x^4 + 20x^3 - 74x^2 - 45x + 45 = (4x^2 - 9)(6x^2 + 5x - 5)$.
 $\therefore \text{L.C.M.} = (4x^2 - 9)(4x^2 + 4x - 3)(6x^2 + 5x - 5)$.
28. H.C.D. $= 2x^2 + x + 1$.
 $10x^4 + 7x^3 + 2x^2 - x - 2 = (2x^2 + x + 1)(5x^2 + x - 2)$.
 $6x^3 + 5x^2 + 4x + 1 = (2x^2 + x + 1)(3x + 1)$.
 $\therefore \text{L.C.M.} = (2x^2 + x + 1)(5x^2 + x - 2)(3x + 1)$.
29. H.C.D. $= 5x^2 - 2x + 3$.
 $5x^4 + 3x^3 + 6x^2 + x + 3 = (5x^2 - 2x + 3)(x^2 + x + 1)$.
 $15x^3 + 14x^2 + x + 12 = (5x^2 - 2x + 3)(3x + 4)$.
 $\therefore \text{L.C.M.} = (5x^2 - 2x + 3)(x^2 + x + 1)(3x + 4)$.
30. H.C.D. $= 2x^2 + x - 2$.
 $2x^3 - x^2 - 3x + 2 = (2x^2 + x - 2)(x - 1)$.
 $4x^3 + 6x^2 - 2x - 4 = (2x^2 + x - 2) \cdot 2(x + 1)$.
 $4x^3 - 5x + 2 = (2x^2 + x - 2)(2x - 1)$.
 $\therefore \text{L.C.M.} = 2(2x^2 + x - 2)(x - 1)(x + 1)(2x - 1)$.
31. H.C.D. $= x - 1$.
 $x^3 - 1 = (x - 1)(x^2 + x + 1)$.
 $2x^3 + 2x^2 - 5x + 1 = (x - 1)(2x^2 + 4x - 1)$.
 $x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$.
 $\therefore \text{L.C.M.} = (x - 1)^2(x^2 + x + 1)(2x^2 + 4x - 1)(x + 2)$.

FRACTIONS

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31. $\frac{a^2 - b^2}{a^2 + 2ab + b^2} = \frac{(a+b)(a-b)}{(a+b)(a+b)} = \frac{a-b}{a+b}$.
32. $\frac{a^2 - 2ab + b^2}{a^2 - b^2} = \frac{(a-b)(a-b)}{(a+b)(a-b)} = \frac{a-b}{a+b}$.
33. $\frac{4a^2 - 9x^2}{8a^3 + 27x^3} = \frac{(2a+3x)(2a-3x)}{(2a+3x)(4a^2-6ax+9x^2)} = \frac{2a-3x}{4a^2-6ax+9x^2}$.
34. $\frac{3a^2 + 3ab}{a^4 + ab^3} = \frac{3a(a+b)}{a(a+b)(a^2-ab+b^2)} = \frac{3}{a^2-ab+b^2}$.
35. $\frac{3x^2y - 6xy}{x^4y - 8xy} = \frac{3xy(x-2)}{xy(x-2)(x^2+2x+4)} = \frac{3}{x^2+2x+4}$.
36. $\frac{3a^2b - 3b^3}{2a^3b - 2b^4} = \frac{3b(a+b)(a-b)}{2b(a-b)(a^2+ab+b^2)} = \frac{3(a+b)}{2(a^2+ab+b^2)}$.
37. $\frac{4a^3 - ab^2}{8a^4 + ab^3} = \frac{a(2a+b)(2a-b)}{a(2a+b)(4a^2-2ab+b^2)} = \frac{2a-b}{4a^2-2ab+b^2}$.
38. $\frac{2x^2y^2 - 8y^4}{4x^3y - 32y^4} = \frac{2y^2(x+2y)(x-2y)}{4y(x-2y)(x^2+2xy+4y^2)} = \frac{y(x+2y)}{2(x^2+2xy+4y^2)}$.
39. $\frac{a^4bc - b^5c}{3a^6b + 3b^7} = \frac{bc(a^2+b^2)(a^2-b^2)}{3b(a^2+b^2)(a^4-a^2b^2+b^4)} = \frac{c(a^2-b^2)}{3(a^4-a^2b^2+b^4)}$.
40. $\frac{10nx + 10ny}{25nx^2 - 25ny^2} = \frac{10n(x+y)}{25n(x+y)(x-y)} = \frac{2}{5(x-y)}$.
41. $\frac{x^{n+2} - x^n}{x^{n+3} - x^n} = \frac{x^n(x+1)(x-1)}{x^n(x-1)(x^2+x+1)} = \frac{x+1}{x^2+x+1}$.
42. $\frac{a^{n+4} - a^ny^4}{a^{n+8} + a^{n+1}y^2} = \frac{a^n(a^2+y^2)(a^2-y^2)}{a^{n+1}(a^2+y^2)} = \frac{a^2-y^2}{a}$.
43. $\frac{x^4y + x^2y^3 + y^5}{x^6 - y^6} = \frac{y(x^4 + x^2y^2 + y^4)}{(x^2 - y^2)(x^4 + x^2y^2 + y^4)} = \frac{y}{x^2 - y^2}$.
44. $\frac{x^4y - x^2y^3 + y^5}{x^6 + y^6} = \frac{y(x^4 - x^2y^2 + y^4)}{(x^2 + y^2)(x^4 - x^2y^2 + y^4)} = \frac{y}{x^2 + y^2}$.
45. $\frac{a^2 - 11a + 24}{a^2 - a - 6} = \frac{(a-3)(a-8)}{(a-3)(a+2)} = \frac{a-8}{a+2}$.
46. $\frac{x^3 - 6x^2 + 5x}{x^3 + 2x^2 - 35x} = \frac{x(x-1)(x-5)}{x(x+7)(x-5)} = \frac{x-1}{x+7}$.
47. $\frac{7x - 2x^2 - 3}{2x^2 + 7x - 4} = \frac{(3-x)(2x-1)}{(4+x)(2x-1)} = \frac{3-x}{4+x}$.
48. $\frac{a(a+2b)^4}{b(a^2-4b^2)^2} = \frac{a(a+2b)(a+2b)(a+2b)(a+2b)(a+2b)}{b(a+2b)(a-2b)(a+2b)(a-2b)} = \frac{a(a+2b)^2}{b(a-2b)^2}$.

$$49. \frac{a^3 + 2a^2b + ab^2}{a^5 - 2a^3b^2 + ab^4} = \frac{a(a+b)(a+b)}{a(a+b)(a-b)(a+b)(a-b)} = \frac{1}{(a-b)^2}.$$

$$50. \frac{x^2 - 2x^4 + x^6}{x^2 - x^6} = \frac{x^2(1-x^2)(1-x^2)}{x^2(1+x^2)(1-x^2)} = \frac{1-x^2}{1+x^2}.$$

$$51. \frac{x^3 + 5x^2 - 6x}{2x^2 - 2} = \frac{x(x+6)(x-1)}{2(x+1)(x-1)} = \frac{x(x+6)}{2(x+1)}.$$

$$52. \frac{x^3 - 7x + 6}{x^4 - 10x^2 + 9} = \frac{(x-1)(x-2)(x+3)}{(x+1)(x-1)(x+3)(x-3)} = \frac{x-2}{x^2 - 2x - 3}.$$

$$53. \frac{x^3 - 21x + 20}{x^4 - 26x^2 + 25} = \frac{(x+5)(x-4)(x-1)}{(x+5)(x-5)(x+1)(x-1)} = \frac{x-4}{x^2 - 4x - 5}.$$

$$54. \frac{x^3 + 3x^2 + 3x + 1}{x^3 + x^2 - 4x - 4} = \frac{(x+1)(x+1)(x+1)}{(x+1)(x+2)(x-2)} = \frac{x^2 + 2x + 1}{x^2 - 4}.$$

$$55. \frac{a^3 - 3a^2b + 3ab^2 - b^3}{3a^2b - 3ab^2} = \frac{(a-b)(a-b)(a-b)}{3ab(a-b)} = \frac{a^2 - 2ab + b^2}{3ab}.$$

$$56. \frac{3a^2 + 4ax - 4x^2}{9a^2 - 12ax + 4x^2} = \frac{(3a-2x)(a+2x)}{(3a-2x)(3a-2x)} = \frac{a+2x}{3a-2x}.$$

$$57. \frac{2ax - ay - 4bx + 2by}{4ax - 2ay - 2bx + by} = \frac{(a-2b)(2x-y)}{(2a-b)(2x-y)} = \frac{a-2b}{2a-b}.$$

$$58. \frac{9x^3 - 13a^2x - 4a^3}{3bx + 3xy - 4ab - 4ay} = \frac{(3x+a)(x+a)(3x-4a)}{(3x-4a)(b+y)} = \frac{3x^2 + 4ax + a^2}{b+y}.$$

$$59. \frac{m - m^2 - n + mn}{m - mn + n^2 - n} = \frac{(m-n)(1-m)}{(m-n)(1-n)} = \frac{1-m}{1-n}.$$

$$60. \frac{am - an - m + n}{am - an + m - n} = \frac{(a-1)(m-n)}{(a+1)(m-n)} = \frac{a-1}{a+1}.$$

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$$62. \frac{x^3 + 5x^2 - 9x - 45}{x^3 + 3x^2 - 25x - 75}.$$

Subtracting the denominator from the numerator,

$$2x^2 + 16x + 30 = 2(x^2 + 8x + 15).$$

By trial, $x^2 + 8x + 15$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 + 5x^2 - 9x - 45}{x^3 + 3x^2 - 25x - 75} = \frac{(x^2 + 8x + 15)(x-3)}{(x^2 + 8x + 15)(x-5)} = \frac{x-3}{x-5}.$$

$$63. \frac{x^3 + 2x^2 - 23x - 60}{x^3 - 11x^2 - 10x + 200}.$$

Subtracting the denominator from the numerator,

$$13x^2 - 13x - 260 = 13(x^2 - x - 20).$$

By trial, $x^2 - x - 20$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 + 2x^2 - 23x - 60}{x^3 - 11x^2 - 10x + 200} = \frac{(x^2 - x - 20)(x+3)}{(x^2 - x - 20)(x-10)} = \frac{x+3}{x-10}.$$

$$64. \quad \frac{4x^3 + 7x^2 + 10x - 3}{4x^3 + 9x^2 + 14x + 3}.$$

Subtracting the numerator from the denominator,

$$2x^2 + 4x + 6 = 2(x^2 + 2x + 3).$$

By trial, $x^2 + 2x + 3$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{4x^3 + 7x^2 + 10x - 3}{4x^3 + 9x^2 + 14x + 3} = \frac{(x^2 + 2x + 3)(4x - 1)}{(x^2 + 2x + 3)(4x + 1)} = \frac{4x - 1}{4x + 1}.$$

$$65. \quad \frac{x^3 + 5x^2 + 8x + 6}{x^3 + 3x^2 + 4x + 2}.$$

Subtracting the denominator from the numerator,

$$2x^2 + 4x + 4 = 2(x^2 + 2x + 2).$$

By trial, $x^2 + 2x + 2$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 + 5x^2 + 8x + 6}{x^3 + 3x^2 + 4x + 2} = \frac{(x^2 + 2x + 2)(x + 3)}{(x^2 + 2x + 2)(x + 1)} = \frac{x + 3}{x + 1}.$$

$$66. \quad \frac{3x^3 - 7x^2 + 4}{5x^3 - 17x^2 + 16x - 4}.$$

Adding the numerator and denominator,

$$8x^3 - 24x^2 + 16x = 8x(x^2 - 3x + 2).$$

By trial, $x^2 - 3x + 2$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{3x^3 - 7x^2 + 4}{5x^3 - 17x^2 + 16x - 4} = \frac{(x^2 - 3x + 2)(3x + 2)}{(x^2 - 3x + 2)(5x - 2)} = \frac{3x + 2}{5x - 2}.$$

$$67. \quad \frac{5x^3 - 14x^2 + 22x + 5}{5x^3 - 18x^2 + 34x - 15}.$$

Subtracting the denominator from the numerator,

$$4x^2 - 12x + 20 = 4(x^2 - 3x + 5).$$

By trial, $x^2 - 3x + 5$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{5x^3 - 14x^2 + 22x + 5}{5x^3 - 18x^2 + 34x - 15} = \frac{(x^2 - 3x + 5)(5x + 1)}{(x^2 - 3x + 5)(5x - 3)} = \frac{5x + 1}{5x - 3}.$$

$$68. \quad \frac{x^3 - 6x^2y + 2xy^2 + 3y^3}{x^3 + 6x^2y - 2xy^2 - 5y^3}.$$

By § 148, $x - y$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 - 6x^2y + 2xy^2 + 3y^3}{x^3 + 6x^2y - 2xy^2 - 5y^3} = \frac{(x - y)(x^2 - 5xy - 3y^2)}{(x - y)(x^2 + 7xy + 5y^2)} = \frac{x^2 - 5xy - 3y^2}{x^2 + 7xy + 5y^2}.$$

$$69. \quad \frac{a^2 + b^2 + 2c^2 + 2ab + 3ac + 3bc}{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}.$$

Subtracting the denominator from the numerator,

$$c^2 + ac + bc = c(a + b + c).$$

By trial, $a + b + c$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{a^2 + b^2 + 2c^2 + 2ab + 3ac + 3bc}{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc} = \frac{(a + b + c)(a + b + 2c)}{(a + b + c)(a + b + c)} = \frac{a + b + 2c}{a + b + c}.$$

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$$70. \quad \frac{a^2 + b^2 + c^2 + 2ab - 2ac - 2bc}{a^2 + b^2 - c^2 + 2ab}.$$

Subtracting the numerator from the denominator,

$$2ac + 2bc - 2c^2 = 2c(a + b - c).$$

By trial, $a + b - c$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{a^2 + b^2 + c^2 + 2ab - 2ac - 2bc}{a^2 + b^2 - c^2 + 2ab} = \frac{(a + b - c)(a + b - c)}{(a + b - c)(a + b + c)} = \frac{a + b - c}{a + b + c}.$$

$$71. \quad \frac{a^2 + b^2 + c^2 - 2ab - 2ac + 2bc}{a^2 + b^2 + 5c^2 - 2ab - 6ac + 6bc}.$$

Subtracting the denominator from the numerator,

$$4ac - 4bc - 4c^2 = 4c(a - b - c).$$

By trial, $a - b - c$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{a^2 + b^2 + c^2 - 2ab - 2ac + 2bc}{a^2 + b^2 + 5c^2 - 2ab - 6ac + 6bc} = \frac{(a - b - c)(a - b - c)}{(a - b - c)(a - b - 5c)} = \frac{a - b - c}{a - b - 5c}.$$

$$72. \quad \frac{4a^2 + 9b^2 + 16c^2 + 12ab + 16ac + 24bc}{4a^2 - 9b^2 + 16c^2 + 16ac}.$$

Subtracting the denominator from the numerator,

$$12ab + 18b^2 + 24bc = 6b(2a + 3b + 4c).$$

By trial, $2a + 3b + 4c$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{4a^2 + 9b^2 + 16c^2 + 12ab + 16ac + 24bc}{4a^2 - 9b^2 + 16c^2 + 16ac} = \frac{(2a + 3b + 4c)(2a + 3b + 4c)}{(2a + 3b + 4c)(2a - 3b + 4c)} = \frac{2a + 3b + 4c}{2a - 3b + 4c}.$$

$$73. \quad \frac{ab(x^2 + y^2) + xy(a^2 + b^2)}{ab(x^2 - y^2) + xy(a^2 - b^2)}.$$

Adding the numerator and denominator,

$$2abx^2 + 2a^2xy = 2ax(bx + ay).$$

By trial, $bx + ay$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{ab(x^2 + y^2) + xy(a^2 + b^2)}{ab(x^2 - y^2) + xy(a^2 - b^2)} = \frac{(bx + ay)(ax + by)}{(bx + ay)(ax - by)} = \frac{ax + by}{ax - by}.$$

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8. The L.C.D. is $a(m + n)$.

$$\therefore \frac{m - n}{a} = \frac{(m - n)(m + n)}{a(m + n)} = \frac{m^2 - n^2}{a(m + n)},$$

$$2 = \frac{2 \cdot a(m + n)}{a(m + n)} = \frac{2am + 2an}{a(m + n)},$$

and

$$\frac{a}{m + n} = \frac{a \cdot a}{a(m + n)} = \frac{a^2}{a(m + n)}.$$

13. The L.C.D. is
- $3xy$
- .

$$\therefore \frac{a-2b}{x} = \frac{3y(a-2b)}{3xy} = \frac{3ay-6by}{3xy},$$

$$\frac{2b-a}{y} = \frac{3x(2b-a)}{3xy} = \frac{6bx-3ax}{3xy},$$

and
$$a - \frac{c}{3x} = \frac{3ax-c}{3x} = \frac{y(3ax-c)}{3xy} = \frac{3axy-cy}{3xy}.$$

14. The L.C.D. is
- x^2-1
- .

$$\therefore \frac{x^2}{x^2-1} = \frac{x^2}{x^2-1},$$

$$\frac{x}{x+1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x^2-x}{x^2-1},$$

and
$$\frac{x}{x-1} = \frac{x(x+1)}{(x-1)(x+1)} = \frac{x^2+x}{x^2-1}.$$

15. The L.C.D. is
- a^4-16
- .

$$\therefore \frac{a^3}{a^4-16} = \frac{a^3}{a^4-16},$$

$$\frac{a}{a^2+4} = \frac{a(a^2-4)}{(a^2+4)(a^2-4)} = \frac{a^3-4a}{a^4-16},$$

and
$$\frac{2a}{a^2-4} = \frac{2a(a^2+4)}{(a^2-4)(a^2+4)} = \frac{2a^3+8a}{a^4-16}.$$

16. The L.C.D. is
- a^2-b^2
- .

$$\therefore \frac{4a}{a-b} = \frac{4a(a+b)}{(a-b)(a+b)} = \frac{4a^2+4ab}{a^2-b^2},$$

$$\frac{3b}{b+a} = \frac{3b(a-b)}{(b+a)(a-b)} = \frac{3ab-3b^2}{a^2-b^2},$$

and
$$\frac{1}{a^2-b^2} = \frac{1}{a^2-b^2}.$$

17. The L.C.D. is
- $1-a^2x^2$
- .

$$\therefore \frac{a}{1-ax} = \frac{a(1+ax)}{(1-ax)(1+ax)} = \frac{a+a^2x}{1-a^2x^2},$$

$$\frac{x}{1+ax} = \frac{x(1-ax)}{(1+ax)(1-ax)} = \frac{x-ax^2}{1-a^2x^2},$$

and
$$\frac{-ax}{ax+1} = \frac{-ax(1-ax)}{(ax+1)(1-ax)} = \frac{a^2x^2-ax}{1-a^2x^2}.$$

18. The L.C.D. is
- $(x+5)(x+2)(x-1)$
- .

$$\therefore \frac{1}{x^2+7x+10} = \frac{1}{(x+5)(x+2)} = \frac{x-1}{(x+5)(x+2)(x-1)},$$

$$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{x+5}{(x+5)(x+2)(x-1)},$$

and
$$\frac{1}{x^2+4x-5} = \frac{1}{(x+5)(x-1)} = \frac{x+2}{(x+5)(x+2)(x-1)}.$$

19. The L.C.D. is $(a-1)(a-3)(a-5)$.

$$\therefore \frac{a+5}{a^2-4a+3} = \frac{a+5}{(a-1)(a-3)} = \frac{(a+5)(a-5)}{(a-1)(a-3)(a-5)} = \frac{a^2-25}{(a-1)(a-3)(a-5)},$$

$$\frac{a-2}{a^2-8a+15} = \frac{a-2}{(a-3)(a-5)} = \frac{(a-2)(a-1)}{(a-1)(a-3)(a-5)} = \frac{a^2-3a+2}{(a-1)(a-3)(a-5)},$$

and $\frac{a+1}{a^2-6a+5} = \frac{a+1}{(a-1)(a-5)} = \frac{(a+1)(a-3)}{(a-1)(a-3)(a-5)} = \frac{a^2-2a-3}{(a-1)(a-3)(a-5)}.$

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$$4. \frac{a-b}{ab} + \frac{b-c}{bc} = \frac{ac-bc}{abc} + \frac{ab-ac}{abc} = \frac{ab-bc}{abc} = \frac{a-c}{ac}.$$

$$5. \frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{a^2+2ab+b^2}{a^2-b^2} + \frac{a^2-2ab+b^2}{a^2-b^2} = \frac{2a^2+2b^2}{a^2-b^2}.$$

$$6. \frac{ax}{a-x} + a = \frac{ax}{a-x} + \frac{a^2-ax}{a-x} = \frac{a^2}{a-x}.$$

$$7. a+b + \frac{a^2+b^2}{a-b} = \frac{a^2-b^2}{a-b} + \frac{a^2+b^2}{a-b} = \frac{2a^2}{a-b}.$$

$$8. \frac{x+1}{x^2+x+1} + \frac{1}{x-1} = \frac{x^2-1}{x^3-1} + \frac{x^2+x+1}{x^3-1} = \frac{2x^2+x}{x^3-1}.$$

$$9. \frac{b-c}{bc} - \frac{a-c}{ac} = \frac{ab-ac-(ab-bc)}{abc} = \frac{ab-ac-ab+bc}{abc} = \frac{bc-ac}{abc} = \frac{b-a}{ab}.$$

$$10. \frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{a^2-b^2} = \frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2} \\ = \frac{4ab}{a^2-b^2}.$$

$$11. \frac{5a^2+b^2}{a^2-b^2} - 2 = \frac{5a^2+b^2-(2a^2-2b^2)}{a^2-b^2} = \frac{5a^2+b^2-2a^2+2b^2}{a^2-b^2} = \frac{3a^2+3b^2}{a^2-b^2}.$$

$$12. x+y - \frac{x^2+y^2}{x-y} = \frac{x^2-y^2-(x^2+y^2)}{x-y} = \frac{x^2-y^2-x^2-y^2}{x-y} = \frac{-2y^2}{x-y} = \frac{2y^2}{y-x}.$$

$$13. \frac{x}{x-2} - \frac{x-2}{x+2} = \frac{x^2+2x-(x^2-4x+4)}{x^2-4} = \frac{x^2+2x-x^2+4x-4}{x^2-4} = \frac{6x-4}{x^2-4}.$$

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$$14. \frac{2x+1}{3} + \frac{x-2}{4} - \frac{x-3}{6} + \frac{5-x}{2} = \frac{8x+4+3x-6-(2x-6)+30-6x}{12} \\ = \frac{8x+4+3x-6-2x+6+30-6x}{12} = \frac{3x+34}{12}.$$

$$15. \frac{x-2}{6} - \frac{x-4}{9} + \frac{2-3x}{4} - \frac{2x+1}{12} = \frac{6x-12-(4x-16)+18-27x-(6x+3)}{36} \\ = \frac{6x-12-4x+16+18-27x-6x-3}{36} = \frac{19-31x}{36}.$$

$$16. \frac{x-1}{3} - \frac{x-2}{18} - \frac{4x-3}{27} + \frac{1-x}{6} = \frac{18x-18-(3x-6)-(8x-6)+9-9x}{54} \\ = \frac{18x-18-3x+6-8x+6+9-9x}{54} = \frac{3-2x}{54}.$$

$$17. \frac{2-6x}{5} + \frac{4x-1}{2} - \frac{5x-3}{6} - \frac{1-x}{3} = \frac{12-36x+60x-15-(25x-15)-(10-10x)}{30} \\ = \frac{12-36x+60x-15-25x+15-10+10x}{30} = \frac{9x+2}{30}.$$

$$18. \frac{x+3}{4} - \frac{x-2}{5} + \frac{x-4}{10} - \frac{x+3}{6} = \frac{15x+45-(12x-24)+6x-24-(10x+30)}{60} \\ = \frac{15x+45-12x+24+6x-24-10x-30}{60} = \frac{15-x}{60}.$$

$$19. \frac{x-4}{3} - \frac{x-6}{8} + 2 - \frac{x+8}{6} = \frac{8x-32-(3x-18)+48-(4x+32)}{24} \\ = \frac{8x-32-3x+18+48-4x-32}{24} = \frac{x+2}{24}.$$

$$20. \frac{2-3x}{3} - \frac{3-2x}{4} + x - \frac{1-4x}{5} = \frac{40-60x-(45-30x)+60x-(12-48x)}{60} \\ = \frac{40-60x-45+30x+60x-12+48x}{60} = \frac{78x-17}{60}.$$

$$21. \frac{1-2a}{5} + \frac{2a-1}{4} - \frac{2a-a^2+1}{8} = \frac{8-16a+20a-10-(10a-5a^2+5)}{40} \\ = \frac{8-16a+20a-10-10a+5a^2-5}{40} = \frac{5a^2-6a-7}{40}.$$

$$22. \frac{3+x-x^2}{4} - \frac{1-x+x^2}{6} - \frac{1-2x-2x^2}{8} = \frac{9+3x-3x^2-(2-2x+2x^2)-(4-8x-8x^2)}{12} \\ = \frac{9+3x-3x^2-2+2x-2x^2-4+8x+8x^2}{12} = \frac{3+13x+3x^2}{12}.$$

$$24. \frac{a-b}{a+b} - \frac{a+b}{a-b} + \frac{6ab}{a^2-b^2} = \frac{a^2-2ab+b^2-(a^2+2ab+b^2)+6ab}{a^2-b^2} \\ = \frac{a^2-2ab+b^2-a^2-2ab-b^2+6ab}{a^2-b^2} = \frac{2ab}{a^2-b^2}.$$

$$25. \frac{a+x}{a-x} + \frac{a-x}{a+x} + \frac{2ax}{a^2-x^2} = \frac{a^2+2ax+x^2+a^2-2ax+x^2+2ax}{a^2-x^2} \\ = \frac{2a^2+2ax+2x^2}{a^2-x^2}.$$

$$26. x+1 + \frac{x^3-3}{x-1} = \frac{x^2-1}{x-1} + \frac{x^3-3}{x-1} = \frac{x^3+x^2-4}{x-1}.$$

$$27. 1 - \frac{ax-bx+ab}{x^2} = \frac{x^2-(ax-bx+ab)}{x^2} = \frac{x^2-ax+bx-ab}{x^2}.$$

$$28. \frac{a+1}{a^3+a+1} + \frac{a-1}{a^3-a+1} = \frac{a^3+1}{a^4+a^2+1} + \frac{a^3-1}{a^4+a^2+1} = \frac{2a^3}{a^4+a^2+1}.$$

$$29. \quad 3x + \frac{5}{ax} - \left(2x + \frac{3}{ax}\right) = 3x + \frac{5}{ax} - 2x - \frac{3}{ax} = x + \frac{2}{ax} = \frac{ax^2 + 2}{ax}.$$

$$30. \quad \frac{a-b}{2(a+b)} + \frac{a^2+b^2}{a^2-b^2} - \frac{a}{a-b} = \frac{a^2-2ab+b^2+2a^2+2b^2-(2a^2+2ab)}{2(a^2-b^2)} \\ = \frac{a^2-2ab+b^2+2a^2+2b^2-2a^2-2ab}{2(a^2-b^2)} \\ = \frac{a^2-4ab+3b^2}{2(a^2-b^2)} = \frac{(a-b)(a-3b)}{2(a+b)(a-b)} = \frac{a-3b}{2(a+b)}.$$

$$31. \quad \frac{a+1}{a^2-9} - \frac{6}{a+5} + \frac{10}{a+3} = \frac{a^2+6a+5-(6a^2-54)+10a^2+20a-150}{(a+3)(a-3)(a+5)} \\ = \frac{a^2+6a+5-6a^2+54+10a^2+20a-150}{(a+3)(a-3)(a+5)} \\ = \frac{5a^2+26a-91}{(a+3)(a-3)(a+5)}.$$

$$32. \quad 2a-3b - \frac{4a^2+9b^2}{2a+3b} = \frac{4a^2-9b^2-(4a^2+9b^2)}{2a+3b} \\ = \frac{4a^2-9b^2-4a^2-9b^2}{2a+3b} = \frac{-18b^2}{2a+3b}.$$

$$33. \quad 3a-2x - \frac{8a^2-4x}{3a+2x} = \frac{9a^2-4x^2-(8a^2-4x)}{3a+2x} \\ = \frac{9a^2-4x^2-8a^2+4x}{3a+2x} = \frac{a^2-4x^2+4x}{3a+2x}.$$

$$34. \quad m - \frac{m^2+n^2}{m-n} + n = m+n - \frac{m^2+n^2}{m-n} \\ = \frac{m^2-n^2-(m^2+n^2)}{m-n} = \frac{m^2-n^2-m^2-n^2}{m-n} \\ = \frac{-2n^2}{m-n} = \frac{2n^2}{n-m}.$$

$$35. \quad \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{x^2} = \frac{x^3+x^2-(x^3-x^2)+2x^2-2}{2x^2(x^2-1)} \\ = \frac{x^3+x^2-x^3+x^2+2x^2-2}{2x^2(x^2-1)} = \frac{4x^2-2}{2x^2(x^2-1)} = \frac{2x^2-1}{x^2(x^2-1)}.$$

$$36. \quad \frac{1}{x} + 1 + \frac{2x}{1+x} - 2 = \frac{1}{x} + \frac{2x}{1+x} - 1 \\ = \frac{1+x+2x^2-(x+x^2)}{x+x^2} = \frac{1+x+2x^2-x-x^2}{x+x^2} = \frac{1+x^2}{x+x^2}.$$

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$$37. \quad \frac{a}{a-2} - \frac{a-2}{a+2} + \frac{3}{4-a^2} = \frac{a}{a-2} - \frac{a-2}{a+2} + \frac{-3}{a^2-4} \\ = \frac{a^2+2a-(a^2-4a+4)-3}{a^2-4} \\ = \frac{a^2+2a-a^2+4a-4-3}{a^2-4} = \frac{6a-7}{a^2-4}.$$

$$\begin{aligned}
 38. \quad \frac{a+1}{a-1} + \frac{a-1}{a+1} + \frac{2a^2}{1-a^2} &= \frac{a+1}{a-1} + \frac{a-1}{a+1} + \frac{-2a^2}{a^2-1} \\
 &= \frac{a^2+2a+1+a^2-2a+1-2a^2}{a^2-1} = \frac{2}{a^2-1}.
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \frac{5x+2}{x^2-4} + \frac{2}{x-2} - \frac{3}{2-x} &= \frac{5x+2}{x^2-4} + \frac{2}{x-2} + \frac{3}{x-2} \\
 &= \frac{5x+2}{x^2-4} + \frac{5}{x-2} = \frac{10x+12}{x^2-4}.
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \frac{x(a+x)}{a-x} - \frac{3ax-x^2}{x-a} + 4a &= \frac{x(a+x)}{a-x} + \frac{3ax-x^2}{a-x} + 4a \\
 &= \frac{ax+x^2+3ax-x^2+4a^2-4ax}{a-x} = \frac{4a^2}{a-x}.
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \frac{a+b}{a-b} - \frac{a^2+b^2}{b^2-a^2} + \frac{b-a}{a+b} &= \frac{a+b}{a-b} + \frac{a^2+b^2}{a^2-b^2} - \frac{a-b}{a+b} \\
 &= \frac{a^2+2ab+b^2+a^2+b^2-(a^2-2ab+b^2)}{a^2-b^2} \\
 &= \frac{a^2+2ab+b^2+a^2+b^2-a^2+2ab-b^2}{a^2-b^2} = \frac{a^2+4ab+b^2}{a^2-b^2}.
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \frac{a}{x(x-a)} + \frac{2x}{a(a-x)} + \frac{1}{x-a} &= \frac{a}{x(x-a)} - \frac{2x}{a(x-a)} + \frac{1}{x-a} \\
 &= \frac{a^2-2x^2+ax}{ax(x-a)} = \frac{(a+2x)(a-x)}{ax(x-a)} \\
 &= \frac{-(a+2x)(a-x)}{ax(a-x)} = \frac{-a-2x}{ax}.
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \frac{1}{a^3+8} - \frac{1}{8-a^3} + \frac{1}{4-a^2} &= \frac{(8-a^3)-(8+a^3)+(16+4a^2+a^4)}{64-a^6} \\
 &= \frac{8-a^3-8-a^3+16+4a^2+a^4}{64-a^6} = \frac{16+4a^2-2a^3+a^4}{64-a^6}.
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \frac{x-1}{x^2-5x+6} + \frac{x}{x-2} - \frac{3}{x+1} \\
 &= \frac{(x-1)(x+1)+x(x-3)(x+1)-3(x-3)(x-2)}{(x-3)(x-2)(x+1)} \\
 &= \frac{x^2-1+x^3-2x^2-3x-3x^2+15x-18}{(x-3)(x-2)(x+1)} \\
 &= \frac{x^3-4x^2+12x-19}{(x-3)(x-2)(x+1)}.
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \frac{2}{x} + \frac{3}{1-2x} - \frac{2x-3}{4x^2-1} + \frac{1}{1+2x} &= \frac{2}{x} + \frac{3}{1-2x} + \frac{2x-3}{1-4x^2} + \frac{1}{1+2x} \\
 &= \frac{2-8x^2+3x+6x^2+2x^2-3x+x-2x^2}{x(1-4x^2)} = \frac{2+x-2x^2}{x(1-4x^2)}.
 \end{aligned}$$

$$\begin{aligned}
 46. \quad \frac{x^2 + a^2}{x^3 - a^3} + \frac{x + a}{x^2 + ax + a^2} - \frac{2}{x - a} &= \frac{x^2 + a^2 + x^2 - a^2 - (2x^2 + 2ax + 2a^2)}{x^3 - a^3} \\
 &= \frac{x^2 + a^2 + x^2 - a^2 - 2x^2 - 2ax - 2a^2}{x^3 - a^3} = \frac{-2a^2 - 2ax}{x^3 - a^3} \\
 &= \frac{2a^2 + 2ax}{a^3 - x^3}.
 \end{aligned}$$

$$\begin{aligned}
 47. \quad \frac{3m}{(m-2x)^2} + \frac{m+2x}{(m+x)(m-2x)} - \frac{5}{m+x} \\
 &= \frac{3m^2 + 3mx + m^2 - 4x^2 - (5m^2 - 20mx + 20x^2)}{(m+x)(m-2x)^2} \\
 &= \frac{3m^2 + 3mx + m^2 - 4x^2 - 5m^2 + 20mx - 20x^2}{(m+x)(m-2x)^2} \\
 &= \frac{-m^2 + 23mx - 24x^2}{(m+x)(m-2x)^2}.
 \end{aligned}$$

$$\begin{aligned}
 48. \quad \frac{3}{y^2 - my - 12m^2} - \frac{2}{y^2 - 5my + 4m^2} &= \frac{3}{(y-4m)(y+3m)} - \frac{2}{(y-4m)(y-m)} \\
 &= \frac{3y - 3m - (2y + 6m)}{(y+3m)(y-4m)(y-m)} = \frac{3y - 3m - 2y - 6m}{(y+3m)(y-4m)(y-m)} \\
 &= \frac{y - 9m}{(y+3m)(y-4m)(y-m)}.
 \end{aligned}$$

$$\begin{aligned}
 49. \quad \frac{ab}{a^2 - ab + b^2} - \frac{ab}{a^2 + ab + b^2} - 1 \\
 &= \frac{a^3b + a^2b^2 + ab^3 - (a^3b - a^2b^2 + ab^3) - (a^4 + a^2b^2 + b^4)}{a^4 + a^2b^2 + b^4} \\
 &= \frac{a^3b + a^2b^2 + ab^3 - a^3b + a^2b^2 - ab^3 - a^4 - a^2b^2 - b^4}{a^4 + a^2b^2 + b^4} \\
 &= \frac{a^2b^2 - a^4 - b^4}{a^2b^2 + a^4 + b^4}.
 \end{aligned}$$

$$\begin{aligned}
 50. \quad \frac{1}{x^2 - 3x + 2} - \frac{1}{x^2 + 2x - 3} + \frac{1}{x^2 + x - 6} &= \frac{1}{(x-1)(x-2)} - \frac{1}{(x+3)(x-1)} + \frac{1}{(x+3)(x-2)} \\
 &= \frac{x+3 - (x-2) + x-1}{(x-1)(x-2)(x+3)} = \frac{x+3 - x+2 + x-1}{(x-1)(x-2)(x+3)} \\
 &= \frac{x+4}{(x-1)(x-2)(x+3)}.
 \end{aligned}$$

$$\begin{aligned}
 51. \quad \frac{2}{x^2 + 5x + 6} - \frac{1}{x^2 + 6x + 8} - \frac{1}{x^2 + 7x + 12} &= \frac{2}{(x+2)(x+3)} - \frac{1}{(x+2)(x+4)} - \frac{1}{(x+3)(x+4)} \\
 &= \frac{2x+8 - (x+3) - (x+2)}{(x+2)(x+3)(x+4)} = \frac{2x+8 - x-3 - x-2}{(x+2)(x+3)(x+4)} \\
 &= \frac{3}{(x+2)(x+3)(x+4)}.
 \end{aligned}$$

52.

$$\begin{aligned}
 \frac{5(x-3)}{x^2-x-2} - \frac{2(x+2)}{x^2+4x+3} - \frac{x-1}{6-x-x^2} &= \frac{5(x-3)}{(x+1)(x-2)} - \frac{2(x+2)}{(x+1)(x+3)} + \frac{x-1}{(x-2)(x+3)} \\
 &= \frac{5(x-3)(x+3) - 2(x+2)(x-2) + (x-1)(x+1)}{(x+1)(x-2)(x+3)} \\
 &= \frac{5x^2 - 45 - 2x^2 + 8 + x^2 - 1}{(x+1)(x-2)(x+3)} = \frac{4x^2 - 38}{(x+1)(x-2)(x+3)}.
 \end{aligned}$$

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$$\begin{aligned}
 53. \quad \frac{a-b-c}{a+b+c} - \frac{a-b+c}{a+b-c} + \frac{4ac}{(a+b)^2 - c^2} &= \frac{a^2 - 2ac + c^2 - b^2 - (a^2 + 2ac + c^2 - b^2) + 4ac}{(a+b)^2 - c^2} \\
 &= \frac{a^2 - 2ac + c^2 - b^2 - a^2 - 2ac - c^2 + b^2 + 4ac}{(a+b)^2 - c^2} \\
 &= \frac{0}{(a+b)^2 - c^2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 55. \quad \frac{a^2 + 2ab + b^2}{a^2 + b^2} - 1 + \frac{2ab}{a^2 - b^2} &= 1 + \frac{2ab}{a^2 + b^2} - 1 + \frac{2ab}{a^2 - b^2} \\
 &= \frac{2ab}{a^2 + b^2} + \frac{2ab}{a^2 - b^2} \\
 &= \frac{2a^3b - 2ab^3 + 2a^3b + 2ab^3}{a^4 - b^4} = \frac{4a^3b}{a^4 - b^4}.
 \end{aligned}$$

$$\begin{aligned}
 56. \quad \frac{a^2 + 3ab + 2b^2}{a^2 + 3ab - 4b^2} - \frac{a^2 - 13b^2}{a^2 - 16b^2} &= \left(1 + \frac{6b^2}{a^2 + 3ab - 4b^2}\right) - \left(1 + \frac{3b^2}{a^2 - 16b^2}\right) \\
 &= \frac{6b^2}{a^2 + 3ab - 4b^2} - \frac{3b^2}{a^2 - 16b^2} \\
 &= \frac{6b^2}{(a+4b)(a-b)} - \frac{3b^2}{(a+4b)(a-4b)} \\
 &= \frac{6ab^2 - 24b^3 - (3ab^2 - 3b^3)}{(a+4b)(a-4b)(a-b)} \\
 &= \frac{6ab^2 - 24b^3 - 3ab^2 + 3b^3}{(a+4b)(a-4b)(a-b)} \\
 &= \frac{3ab^2 - 21b^3}{(a+4b)(a-4b)(a-b)}.
 \end{aligned}$$

$$\begin{aligned}
 57. \quad \frac{x^2 + x + 1}{x^2 - x + 1} - \frac{x^2 - x + 1}{x^2 - 2x + 1} &= \left(1 + \frac{2x}{x^2 - x + 1}\right) - \left(1 + \frac{x}{x^2 - 2x + 1}\right) \\
 &= \frac{2x}{x^2 - x + 1} - \frac{x}{x^2 - 2x + 1} \\
 &= \frac{2x^3 - 4x^2 + 2x - (x^3 - x^2 + x)}{(x^2 - x + 1)(x-1)^2} \\
 &= \frac{2x^3 - 4x^2 + 2x - x^3 + x^2 - x}{(x^2 - x + 1)(x-1)^2} \\
 &= \frac{x^3 - 3x^2 + x}{(x^2 - x + 1)(x-1)^2}.
 \end{aligned}$$

$$58. \frac{x^3 + x^2 + x + 1}{x^3 - x^2 + x - 1} - 1 - \frac{3}{x-1} = 1 + \frac{2x^2 + 2}{x^3 - x^2 + x - 1} - 1 - \frac{3}{x-1}$$

$$= \frac{2x^2 + 2}{x^3 - x^2 + x - 1} - \frac{3}{x-1}$$

$$\S 165, \quad = \frac{2}{x-1} - \frac{3}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}.$$

$$59. \frac{x+1}{x-1} + \frac{x-1}{x+1} - \frac{x+2}{x-2} - \frac{x-2}{x+2} = 1 + \frac{2}{x-1} + 1 - \frac{2}{x+1} - \left(1 + \frac{4}{x-2}\right) - \left(1 - \frac{4}{x+2}\right)$$

$$= \frac{2}{x-1} - \frac{2}{x+1} - \frac{4}{x-2} + \frac{4}{x+2}$$

$$= \frac{2(x+1)(x^2-4) - 2(x-1)(x^2-4) - 4(x+2)(x^2-1) + 4(x-2)(x^2-1)}{(x^2-1)(x^2-4)}$$

$$= \frac{(2x+2-2x+2)(x^2-4) + (-4x-8+4x-8)(x^2-1)}{(x^2-1)(x^2-4)}$$

$$= \frac{4(x^2-4) - 16(x^2-1)}{(x^2-1)(x^2-4)}$$

$$= \frac{4x^2 - 16 - 16x^2 + 16}{(x^2-1)(x^2-4)} = \frac{-12x^2}{(x^2-1)(x^2-4)}.$$

$$60. \frac{x+3}{x-3} - \frac{x-3}{x+3} + \frac{x+4}{x-4} - \frac{x-4}{x+4} = 1 + \frac{6}{x-3} - \left(1 - \frac{6}{x+3}\right) + 1 + \frac{8}{x-4} - \left(1 - \frac{8}{x+4}\right)$$

$$= \frac{6}{x-3} + \frac{6}{x+3} + \frac{8}{x-4} + \frac{8}{x+4}$$

$$= \frac{6(x+3)(x^2-16) + 6(x-3)(x^2-16) + 8(x+4)(x^2-9) + 8(x-4)(x^2-9)}{(x^2-9)(x^2-16)}$$

$$= \frac{(6x+18+6x-18)(x^2-16) + (8x+32+8x-32)(x^2-9)}{(x^2-9)(x^2-16)}$$

$$= \frac{12x(x^2-16) + 16x(x^2-9)}{(x^2-9)(x^2-16)}$$

$$= \frac{12x^3 - 192x + 16x^3 - 144x}{(x^2-9)(x^2-16)} = \frac{28x^3 - 336x}{(x^2-9)(x^2-16)}.$$

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$$62. \frac{a+b}{a-b} - \frac{a-b}{a+b} - \frac{4ab}{a^2+b^2} + \frac{8ab^3}{a^4+b^4} = \frac{4ab}{a^2-b^2} - \frac{4ab}{a^2+b^2} + \frac{8ab^3}{a^4+b^4}$$

$$= \frac{8ab^3}{a^4-b^4} + \frac{8ab^3}{a^4+b^4} = \frac{16a^5b^3}{a^8-b^8}.$$

$$63. \frac{1}{a-b} - \frac{1}{a+b} - \frac{2b}{a^2+b^2} + \frac{2b^3}{a^4+b^4} = \frac{2b}{a^2-b^2} - \frac{2b}{a^2+b^2} + \frac{2b^3}{a^4+b^4}$$

$$= \frac{4b^3}{a^4-b^4} + \frac{2b^3}{a^4+b^4} = \frac{6a^4b^3 + 2b^7}{a^8-b^8}.$$

$$\begin{aligned}
 64. \quad \frac{1}{x-1} + \frac{1}{x+1} - \frac{x}{x^2+1} - \frac{x^3+3x}{x^4+1} &= \frac{2x}{x^2-1} - \frac{x}{x^2+1} - \frac{x^3+3x}{x^4+1} \\
 &= \frac{x^3+3x}{x^4-1} - \frac{x^3+3x}{x^4+1} = \frac{2x^3+6x}{x^8-1}.
 \end{aligned}$$

$$\begin{aligned}
 65. \quad \frac{a+x}{a-x} + \frac{a^2+x^2}{a^2-x^2} - \frac{a-x}{a+x} - \frac{a^2-x^2}{a^2+x^2} - \frac{4a^3x+4ax^3}{a^4-x^4} \\
 = \left(\frac{a+x}{a-x} - \frac{a-x}{a+x} \right) + \left(\frac{a^2+x^2}{a^2-x^2} - \frac{a^2-x^2}{a^2+x^2} \right) - \frac{4ax(a^2+x^2)}{(a^2-x^2)(a^2+x^2)} \\
 = \frac{4ax}{a^2-x^2} + \frac{4a^2x^2}{a^4-x^4} - \frac{4ax}{a^2-x^2} \\
 = \frac{4a^2x^2}{a^4-x^4}.
 \end{aligned}$$

$$\begin{aligned}
 67. \quad \frac{1}{(b-c)(a-c)} + \frac{1}{(c-a)(a-b)} + \frac{1}{(b-a)(b-c)} \\
 = \frac{1}{(b-c)(a-c)} - \frac{1}{(a-c)(a-b)} - \frac{1}{(a-b)(b-c)} \\
 = \frac{a-b-(b-c)-(a-c)}{(a-b)(a-c)(b-c)} \\
 = \frac{a-b-b+c-a+c}{(a-b)(a-c)(b-c)} = \frac{-2(b-c)}{(a-b)(a-c)(b-c)} = \frac{2}{(a-b)(c-a)}.
 \end{aligned}$$

$$\begin{aligned}
 68. \quad \frac{a+1}{(a-b)(a-c)} + \frac{b+1}{(b-c)(b-a)} + \frac{c+1}{(a-c)(b-c)} \\
 = \frac{a+1}{(a-b)(a-c)} - \frac{b+1}{(b-c)(a-b)} + \frac{c+1}{(a-c)(b-c)} \\
 = \frac{(a+1)(b-c) - (b+1)(a-c) + (c+1)(a-b)}{(a-b)(a-c)(b-c)} \\
 = \frac{ab+b-ac-c-ab-a+bc+c+ac+a-bc-b}{(a-b)(a-c)(b-c)} \\
 = \frac{0}{(a-b)(a-c)(b-c)} = 0.
 \end{aligned}$$

$$\begin{aligned}
 69. \quad \frac{c^2ab}{(c-a)(b-c)} - \frac{b^2ca}{(b-a)(b-c)} - \frac{a^2bc}{(a-b)(a-c)} \\
 = \frac{-c^2ab}{(a-c)(b-c)} + \frac{b^2ca}{(a-b)(b-c)} - \frac{a^2bc}{(a-b)(a-c)} \\
 = \frac{-c^2ab(a-b) + b^2ca(a-c) - a^2bc(b-c)}{(a-b)(a-c)(b-c)} \\
 = \frac{-a^2bc^2 + ab^2c^2 + a^2b^2c - ab^2c^2 - a^2b^2c + a^2bc^2}{(a-b)(a-c)(b-c)} \\
 = \frac{0}{(a-b)(a-c)(b-c)} = 0.
 \end{aligned}$$

$$\begin{aligned}
 70. \quad & \frac{b-c}{(b-a)(a-c)} - \frac{c-a}{(b-c)(a-b)} - \frac{a+b}{(a-c)(b-c)} \\
 &= \frac{-(b-c)}{(a-b)(a-c)} + \frac{a-c}{(b-c)(a-b)} - \frac{a+b}{(a-c)(b-c)} \\
 &= \frac{-(b-c)(b-c) + (a-c)(a-c) - (a+b)(a-b)}{(a-b)(a-c)(b-c)} \\
 &= \frac{-b^2 + 2bc - c^2 + a^2 - 2ac + c^2 - a^2 + b^2}{(a-b)(a-c)(b-c)} \\
 &= \frac{-2c(a-b)}{(a-b)(a-c)(b-c)} = \frac{2c}{(c-a)(b-c)}.
 \end{aligned}$$

$$\begin{aligned}
 71. \quad & \frac{c+a}{(a-b)(b-c)} - \frac{b+c}{(c-a)(b-a)} + \frac{a+b}{(c-b)(a-c)} \\
 &= \frac{a+c}{(a-b)(b-c)} - \frac{b+c}{(a-c)(a-b)} - \frac{a+b}{(b-c)(a-c)} \\
 &= \frac{(a+c)(a-c) - (b+c)(b-c) - (a+b)(a-b)}{(a-b)(a-c)(b-c)} \\
 &= \frac{a^2 - c^2 - b^2 + c^2 - a^2 + b^2}{(a-b)(a-c)(b-c)} = \frac{0}{(a-b)(a-c)(b-c)} = 0.
 \end{aligned}$$

$$\begin{aligned}
 72. \quad & \frac{c+a}{(a+b)(b-c)} - \frac{c-b}{(c-a)(a+b)} - \frac{a-b}{(b-c)(a-c)} \\
 &= \frac{a+c}{(a+b)(b-c)} - \frac{b-c}{(a-c)(a+b)} - \frac{a-b}{(b-c)(a-c)} \\
 &= \frac{(a+c)(a-c) - (b-c)(b-c) - (a-b)(a+b)}{(a+b)(a-c)(b-c)} \\
 &= \frac{a^2 - c^2 - b^2 + 2bc - c^2 - a^2 + b^2}{(a+b)(a-c)(b-c)} \\
 &= \frac{2c(b-c)}{(a+b)(a-c)(b-c)} = \frac{2c}{(a+b)(a-c)}.
 \end{aligned}$$

$$\begin{aligned}
 73. \quad & \frac{c+a-b}{(a-b)(b-c)} + \frac{b+c-a}{(c-a)(a-b)} - \frac{a+b+c}{(b-c)(a-c)} \\
 &= \frac{a-b+c}{(a-b)(b-c)} + \frac{a-b-c}{(a-c)(a-b)} - \frac{a+b+c}{(b-c)(a-c)} \\
 &= \frac{(a-b+c)(a-c) + (a-b-c)(b-c) - (a+b+c)(a-b)}{(a-b)(a-c)(b-c)} \\
 &= \frac{a^2 - ab + bc - c^2 + ab - b^2 - ac + c^2 - a^2 - ac + b^2 + bc}{(a-b)(a-c)(b-c)} \\
 &= \frac{2c(b-a)}{(a-b)(a-c)(b-c)} = \frac{-2c(a-b)}{(a-b)(a-c)(b-c)} = \frac{2c}{(c-a)(b-c)}.
 \end{aligned}$$

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$$8. \quad \frac{a^m b^n}{4x} \times \frac{6x^2}{a^{m-1} b^{2n}} = \frac{a^{m-1} a^1 b^n}{4x} \times \frac{6x^2}{a^{m-1} b^n} = \frac{3ax}{2b^n}.$$

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9. $\frac{a^{m+1}}{b^{m+2}} \times \frac{b^{m+1}}{a^m} = \frac{a^m a^1}{b^{m+1} b^1} \times \frac{b^{m+1}}{a^m} = \frac{a}{b}$.
11. $\frac{xy^2}{20-8x} \times \frac{25-10x}{x^2y} = \frac{xy^2}{4(5-2x)} \times \frac{5(5-2x)}{x^2y} = \frac{5y}{4x}$.
12. $\frac{1-5x+6x^2}{2-3x+x^2} \times \frac{2-x}{1-x} = \frac{(1-2x)(1-3x)}{(2-x)(1-x)} \times \frac{2-x}{1-x} = \frac{1-5x+6x^2}{(1-x)^2}$.
13. $\frac{(a-b)^2}{a+b} \times \frac{b}{a^2-ab} \times \frac{(a+b)^2}{a^2-b^2} = \frac{(a-b)^2}{a+b} \times \frac{b}{a(a-b)} \times \frac{(a+b)^2}{(a+b)(a-b)} = \frac{b}{a}$.
14. $\frac{a^4-x^4}{a^3+x^3} \times \frac{a+x}{a^2-x^2} \times \frac{a^2-ax+x^2}{(a+x)^2}$
 $= \frac{(a^2+x^2)(a+x)(a-x)}{(a+x)(a^2-ax+x^2)} \times \frac{a+x}{(a+x)(a-x)} \times \frac{a^2-ax+x^2}{(a+x)^2} = \frac{a^2+x^2}{(a+x)^2}$.
15. $\frac{4a-b}{2x+y} \times \frac{2a}{4a^2-ab} \times \frac{4x^2-y^2}{4} = \frac{4a-b}{2x+y} \times \frac{2a}{a(4a-b)} \times \frac{(2x+y)(2x-y)}{4}$
 $= \frac{2x-y}{2}$.
16. $\frac{p+2}{x-3} \times \frac{3x^2-27}{2p^2-8} \times \frac{4}{px+3p} = \frac{p+2}{x-3} \times \frac{3(x+3)(x-3)}{2(p+2)(p-2)} \times \frac{4}{p(x+3)}$
 $= \frac{6}{p(p-2)}$.
17. $\frac{p^4-q^4}{(p-q)^2} \times \frac{p-q}{p^2+pq} \times \frac{p^2}{p^2+q^2}$
 $= \frac{(p^2+q^2)(p+q)(p-q)}{(p-q)^2} \times \frac{p-q}{p(p+q)} \times \frac{p^2}{p^2+q^2} = p$.
18. $\frac{a^3+8}{a^3-8} \times \frac{a^2+2a+4}{a^2-2a+4} = \frac{(a+2)(a^2-2a+4)}{(a-2)(a^2+2a+4)} \times \frac{a^2+2a+4}{a^2-2a+4} = \frac{a+2}{a-2}$.
19. $\frac{a^4+a^2x^2+x^4}{a^4-ax^3} \times \frac{x}{a^2-ax+x^2}$
 $= \frac{(a^2+ax+x^2)(a^2-ax+x^2)}{a(a-x)(a^2+ax+x^2)} \times \frac{x}{a^2-ax+x^2} = \frac{x}{a(a-x)}$.
20. $\frac{a^4+4}{a^4+a^2+1} \times \frac{a^2+a+1}{a^2+2a+2} = \frac{(a^2+2a+2)(a^2-2a+2)}{(a^2+a+1)(a^2-a+1)} \times \frac{a^2+a+1}{a^2+2a+2} = \frac{a^2-2a+2}{a^2-a+1}$.
21. $\frac{x^2+5x+6}{x^2+6x+5} \times \frac{x^2+7x+10}{x^2+7x+12} = \frac{(x+3)(x+2)}{(x+5)(x+1)} \times \frac{(x+5)(x+2)}{(x+4)(x+3)} = \frac{x^2+4x+4}{x^2+5x+6}$.
22. $\frac{x^2+3x-10}{x^2-4x-21} \times \frac{x^2-10x+21}{x^2+7x+10} = \frac{(x+5)(x-2)}{(x-7)(x+3)} \times \frac{(x-7)(x-3)}{(x+5)(x+2)} = \frac{x^2-5x+6}{x^2+5x+6}$.
23. $\frac{a^2+ab+ac+bc}{ax-ay-x^2+xy} \times \frac{a^2-ax+ay-xy}{a^2+ac+ax+cx} \times \frac{x^2-x(y-a)-ay}{a^2-a(y-b)-by}$
 $= \frac{(a+c)(a+b)}{(a-x)(x-y)} \times \frac{(a+y)(a-x)}{(a+x)(a+c)} \times \frac{(a+x)(x-y)}{(a+b)(a-y)} = \frac{a+y}{a-y}$.

24. $\frac{x^3 - 5x^2 + 8x - 4}{x^3 - 8x^2 + 19x - 12} \times \frac{x^3 - 10x^2 + 33x - 36}{x^3 - 6x^2 + 11x - 6}$
 $= \frac{(x-1)(x-2)(x-2)}{(x-1)(x-3)(x-4)} \times \frac{(x-3)(x-3)(x-4)}{(x-1)(x-2)(x-3)} = \frac{x-2}{x-1}$
25. $\frac{x^4 - 3x^3 - 23x^2 + 75x - 50}{x^4 - 5x^3 - 21x^2 + 125x - 100} \times \frac{x^3 - 10x^2 + 29x - 20}{x^3 - 12x^2 + 45x - 50}$
 $= \frac{(x-1)(x-2)(x-5)(x+5)}{(x-1)(x-4)(x-5)(x+5)} \times \frac{(x-1)(x-4)(x-5)}{(x-2)(x-5)(x-5)} = \frac{x-1}{x-5}$

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17. $\frac{my - y^2}{(m+y)^2} \div \frac{y^2}{m^2 - y^2} = \frac{y(m-y)}{(m+y)^2} \times \frac{(m+y)(m-y)}{y^2} = \frac{(m-y)^2}{y(m+y)}$
18. $\frac{(a-b)^2}{a+b} \div \frac{a^2 - ab}{b} = \frac{(a-b)^2}{a+b} \times \frac{b}{a(a-b)} = \frac{b(a-b)}{a(a+b)}$
19. $(4a+2) \div \frac{2a+1}{5a} = 2(2a+1) \times \frac{5a}{2a+1} = 10a$
20. $\frac{x^2 - y^2}{x+2y} \div (x^2 - 3xy + 2y^2) = \frac{(x+y)(x-y)}{x+2y} \times \frac{1}{(x-y)(x-2y)} = \frac{x+y}{x^2 - 4y^2}$
21. $\frac{x^2 + x - 2}{x^2 - 5x + 4} \div \frac{x^2 - x - 6}{x^2 + x - 20} = \frac{(x-1)(x+2)}{(x-1)(x-4)} \times \frac{(x-4)(x+5)}{(x+2)(x-3)} = \frac{x+5}{x-3}$
22. $\frac{a^4 - b^4}{a^2 - 2ab + b^2} \div \frac{a^2 + b^2}{a^2 - ab} = \frac{(a^2 + b^2)(a+b)(a-b)}{(a-b)^2} \times \frac{a(a-b)}{a^2 + b^2} = a(a+b)$
23. $\frac{x^3 + y^3}{x^2 - y^2} \div \frac{x^2 + xy + y^2}{x-y} = \frac{(x+y)(x^2 - xy + y^2)}{(x+y)(x-y)} \times \frac{x-y}{x^2 + xy + y^2}$
 $= \frac{x^2 - xy + y^2}{x^2 + xy + y^2}$
24. $\frac{a^3 + b^3}{a^2 - 4b^2} \div \frac{a^2 - ab + b^2}{a - 2b} = \frac{(a+b)(a^2 - ab + b^2)}{(a+2b)(a-2b)} \times \frac{a-2b}{a^2 - ab + b^2}$
 $= \frac{a+b}{a+2b}$
25. $\frac{m^3 - y^3}{m^2y^2 - y^4} \div \frac{m^3 + m^2y + my^2}{my^2 + y^3} = \frac{(m-y)(m^2 + my + y^2)}{y^2(m+y)(m-y)} \times \frac{y^2(m+y)}{m(m^2 + my + y^2)} = \frac{1}{m}$
26. $\frac{m^4x + m^5}{m^3x - mx^3} \div \frac{m^3x^2 - mx^4}{m^3x^3 + x^6} = \frac{m^4(x+m)}{mx(m+x)(m-x)} \times \frac{x^3(m+x)(m^2 - mx + x^2)}{m^3x^2(m+x)(m-x)}$
 $= \frac{m^2(m^2 - mx + x^2)}{(m-x)^2}$
27. $\left(x \div \frac{1}{y}\right) \div \left(y^2 \div \frac{1}{x^2}\right) = xy \div x^2y^2 = xy \times \frac{1}{x^2y^2} = \frac{1}{xy}$
28. $\left(\frac{a^3}{b} \div b^2\right) \div \left(\frac{a^2}{b^2} \times ab\right) = \left(\frac{a^3}{b} \times \frac{1}{b^2}\right) \div \frac{a^3}{b}$
 $= \frac{a^3}{b} \times \frac{1}{b^2} \times \frac{b}{a^3} = \frac{1}{b^2}$

$$\begin{aligned}
 29. \quad (a+c) \div \left(\frac{a^2-c^2}{1+x} \div \frac{a-c}{1-x^2} \right) &= (a+c) \div \left(\frac{a^2-c^2}{1+x} \times \frac{1-x^2}{a-c} \right) \\
 &= (a+c) \times \frac{1+x}{(a+c)(a-c)} \times \frac{a-c}{(1+x)(1-x)} = \frac{1}{1-x}.
 \end{aligned}$$

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$$\begin{aligned}
 30. \quad &\frac{x^3-6x^2+11x-6}{x^3+2x^2-19x-20} \div \frac{x^3-13x+12}{x^3+10x^2+29x+20} \\
 &= \frac{x^3-6x^2+11x-6}{x^3+2x^2-19x-20} \times \frac{x^3+10x^2+29x+20}{x^3-13x+12} \\
 &= \frac{(x-1)(x-2)(x-3)}{(x+1)(x-4)(x+5)} \times \frac{(x+1)(x+4)(x+5)}{(x-1)(x-3)(x+4)} = \frac{x-2}{x-4}.
 \end{aligned}$$

$$\begin{aligned}
 31. \quad &\frac{x^3-15x^2+74x-120}{x^3-5x^2-x+5} \div \frac{x^3-9x^2+26x-24}{x^3-6x^2+11x-6} \\
 &= \frac{x^3-15x^2+74x-120}{x^3-5x^2-x+5} \times \frac{x^3-6x^2+11x-6}{x^3-9x^2+26x-24} \\
 &= \frac{(x-4)(x-5)(x-6)}{(x+1)(x-1)(x-5)} \times \frac{(x-1)(x-2)(x-3)}{(x-2)(x-3)(x-4)} = \frac{x-6}{x+1}.
 \end{aligned}$$

$$\begin{aligned}
 32. \quad &\frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc} \div \frac{a^2-b^2+c^2-2ac}{a^2-b^2+c^2+2ac} \\
 &= \frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc} \times \frac{a^2-b^2+c^2+2ac}{a^2-b^2+c^2-2ac} \\
 &= \frac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)} \times \frac{(a+b+c)(a-b+c)}{(a+b-c)(a-b-c)} = \frac{(a+b+c)^2}{(a+b-c)(a-b-c)}.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad &\frac{a^2+x^2-y^2+2ax}{a^2-x^2+y^2-2ay} \div \frac{a^2-x^2+y^2+2ay}{a^2-x^2-y^2-2xy} \\
 &= \frac{a^2+x^2-y^2+2ax}{a^2-x^2+y^2-2ay} \times \frac{a^2-x^2-y^2-2xy}{a^2-x^2+y^2+2ay} \\
 &= \frac{(a+x+y)(a+x-y)}{(a+x-y)(a-x-y)} \times \frac{(a+x+y)(a-x-y)}{(a+x+y)(a-x+y)} = \frac{a+x+y}{a-x+y}.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad &\left(x-4+\frac{9}{x+2} \right) \div \left(1-\frac{4x-7}{x^2-4} \right) \\
 &= \frac{x^2-2x+1}{x+2} \div \frac{x^2-4x+3}{x^2-4} \\
 &= \frac{(x-1)^2}{x+2} \times \frac{(x+2)(x-2)}{(x-1)(x-3)} = \frac{(x-1)(x-2)}{x-3}.
 \end{aligned}$$

$$\begin{aligned}
 36. \quad &\left(x+\frac{3x+6}{x^2-1}+2 \right) \div \left(x+3+\frac{1}{x+1} \right) \\
 &= \frac{x^3+2x^2+2x+4}{x^2-1} \div \frac{x^2+4x+4}{x+1} \\
 &= \frac{(x^2+2)(x+2)}{(x+1)(x-1)} \times \frac{x+1}{(x+2)^2} = \frac{x^2+2}{(x-1)(x+2)}.
 \end{aligned}$$

$$37. \quad \left(x^3 - \frac{1}{x^3}\right) \div \left(x - \frac{1}{x}\right) = \frac{x^6 - 1}{x^3} \div \frac{x^2 - 1}{x} \\ = \frac{x^6 - 1}{x^3} \times \frac{x}{x^2 - 1} = \frac{(x^2 - 1)(x^4 + x^2 + 1)}{x^3} \times \frac{x}{x^2 - 1} = \frac{x^4 + x^2 + 1}{x^2}.$$

$$38. \quad \left(1 + \frac{1}{y^2} + \frac{1}{y^4}\right) \div \left(1 + \frac{1}{y} + \frac{1}{y^2}\right) = \frac{y^4 + y^2 + 1}{y^4} \div \frac{y^2 + y + 1}{y^2} \\ = \frac{(y^2 + y + 1)(y^2 - y + 1)}{y^4} \times \frac{y^2}{y^2 + y + 1} = \frac{y^2 - y + 1}{y^2}.$$

$$39. \quad \left(1 - \frac{y^2}{x^2}\right) \div \left(1 - \frac{2x}{y} + \frac{x^2}{y^2}\right) = \frac{x^2 - y^2}{x^2} \div \frac{y^2 - 2xy + x^2}{y^2} \\ = \frac{(x + y)(x - y)}{x^2} \times \frac{y^2}{(x - y)^2} = \frac{y^2(x + y)}{x^2(x - y)}.$$

$$40. \quad \left(1 - \frac{2y^3}{x^3 + y^3}\right) \div \left(1 - \frac{2y}{x + y}\right) = \frac{x^3 - y^3}{x^3 + y^3} \div \frac{x - y}{x + y} \\ = \frac{(x - y)(x^2 + xy + y^2)}{(x + y)(x^2 - xy + y^2)} \times \frac{x + y}{x - y} = \frac{x^2 + xy + y^2}{x^2 - xy + y^2}.$$

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$$2. \quad \frac{\frac{x + y}{ab}}{\frac{x^2 - y^2}{ab^2}} = \frac{x + y}{ab} \div \frac{x^2 - y^2}{ab^2} = \frac{x + y}{ab} \times \frac{ab^2}{(x + y)(x - y)} = \frac{b}{x - y}.$$

$$3. \quad \frac{a + \frac{b}{c}}{b + \frac{c}{a}} = \frac{ac + b}{c} \div \frac{ab + c}{a} = \frac{ac + b}{c} \times \frac{a}{ab + c} = \frac{a(ac + b)}{c(ab + c)}.$$

$$4. \quad \frac{\frac{m - 3m}{x}}{\frac{x - m}{m}} = \frac{mx - 3m}{x} \div \frac{mx - x}{m} = \frac{m(x - 3)}{x} \times \frac{m}{x(m - 1)} = \frac{m^2(x - 3)}{x^2(m - 1)}.$$

$$5. \quad \frac{2 + \frac{3a}{4b}}{a + \frac{8b}{3}} = \frac{8b + 3a}{4b} \div \frac{3a + 8b}{3} = \frac{3a + 8b}{4b} \times \frac{3}{3a + 8b} = \frac{3}{4b}.$$

$$6. \quad \frac{1 - \frac{y^2}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{x^2 - y^2}{x^2} \div \frac{x^2 + y^2}{x^2} = \frac{x^2 - y^2}{x^2} \times \frac{x^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2}.$$

$$7. \quad \frac{x - \frac{1}{x}}{1 + \frac{1}{x}} = \frac{x^2 - 1}{x} \div \frac{x + 1}{x} = \frac{(x + 1)(x - 1)}{x} \times \frac{x}{x + 1} = x - 1.$$

$$8. \frac{\frac{b-c}{2}}{\frac{b-c}{2}} = \frac{2b-c}{2} \div \frac{b-2c}{2} = \frac{2b-c}{2} \times \frac{2}{b-2c} = \frac{2b-c}{b-2c}.$$

$$9. \frac{\frac{ax-x^2}{2}}{\frac{a^2-ax}{2}} = \frac{2ax-x^2}{2} \div \frac{a^2-2ax}{2} = \frac{x(2a-x)}{2} \times \frac{2}{a(a-2x)} = \frac{x(2a-x)}{a(a-2x)}.$$

$$10. \frac{\frac{x+y}{y} - \frac{x+y}{x}}{\frac{1}{y} - \frac{1}{x}} = \frac{\frac{x^2-y^2}{xy}}{\frac{x-y}{xy}} = \frac{(x+y)(x-y)}{xy} \times \frac{xy}{x-y} = x+y.$$

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$$12. \frac{\frac{x^2-1}{x}}{x+1} = \frac{x(x^2-1)}{x+1} = \frac{x(x+1)(x-1)}{x+1} = x(x-1).$$

$$13. \frac{\frac{1}{x} + \frac{1}{y+z}}{\frac{1}{x} - \frac{1}{y+z}} = \frac{y+z+x}{y+z-x}.$$

$$14. \frac{\frac{x^3+y^3}{xy}}{\frac{x^2-xy+y^2}{xy}} = \frac{x^3+y^3}{x^2-xy+y^2} = \frac{(x+y)(x^2-xy+y^2)}{x^2-xy+y^2} = x+y.$$

$$15. \frac{\frac{1}{a+1}}{1 - \frac{1}{a+1}} = \frac{1}{a+1-1} = \frac{1}{a}.$$

$$16. \frac{\frac{x^2+y^2}{2y} - \frac{x}{y}}{\frac{x-y}{y}} = \frac{\frac{x^3+xy^2-2x^2y}{2(x^2-y^2)}}{\frac{x-y}{y}} = \frac{x(x-y)^2}{2(x+y)(x-y)} = \frac{x(x-y)}{2(x+y)}.$$

$$17. \frac{\frac{1}{1-a}}{\frac{1}{1+a}} = \frac{1+a}{a(1-a)}.$$

$$18. \frac{\frac{x-2}{x+2} + \frac{1}{x-2}}{x+2 + \frac{1}{x-2}} = \frac{\frac{x^2-3}{x+2}}{x+2} \div \frac{x^2-3}{x-2} = \frac{x^2-3}{x+2} \times \frac{x-2}{x^2-3} = \frac{x-2}{x+2}.$$

$$19. \frac{\frac{1}{x} + \frac{4}{x^2} + \frac{4}{x^3}}{1 + \frac{5}{x} + \frac{6}{x^2}} = \frac{x^2 + 4x + 4}{x^3 + 5x^2 + 6x} = \frac{(x+2)^2}{x(x+2)(x+3)} = \frac{x+2}{x(x+3)}.$$

$$20. \frac{6a-1-\frac{1}{a}}{\frac{2a-1}{3a}} = \frac{18a^2-3a-3}{2a-1} = \frac{3(3a+1)(2a-1)}{2a-1} = 3(3a+1).$$

$$21. \frac{\frac{x-5}{2} - 7 + \frac{24}{x}}{\frac{3x-9}{x}} = \frac{x^2-5x-14x+48}{6x-18} = \frac{x^2-19x+48}{6x-18} = \frac{(x-3)(x-16)}{6(x-3)} = \frac{x-16}{6}.$$

$$22. \frac{\frac{1}{x+1}}{1-\frac{1}{1+x}} + \frac{\frac{1}{x+1}}{1-\frac{1}{1-x}} + \frac{\frac{1}{1-x}}{1+\frac{1}{x}} = \frac{1}{x} + \frac{1-x}{x(1+x)} + \frac{1+x}{x(1-x)} \\ = \frac{1-x^2+1-2x+x^2+1+2x+x^2}{x(1-x^2)} = \frac{3+x^2}{x(1-x^2)}.$$

$$23. \frac{3xyz}{yz+zx+xy} - \frac{\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z}}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \\ = \frac{3xyz}{yz+zx+xy} - \frac{xyz-yz+xyz-xz+xyz-xy}{yz+zx+xy} \\ = \frac{3xyz - (3xyz - yz - xz - xy)}{yz+zx+xy} \\ = \frac{3xyz - 3xyz + yz + xz + xy}{yz+zx+xy} = 1.$$

$$24. \frac{\frac{1}{x+y} + \frac{2}{x-y} - \frac{9}{3x-y}}{\frac{-8y}{y^2-9x^2}} = \frac{\frac{8y^2}{(x+y)(x-y)(3x-y)}}{\frac{8y}{9x^2-y^2}} \\ = \frac{8y^2}{(x+y)(x-y)(3x-y)} \times \frac{(3x+y)(3x-y)}{8y} = \frac{y(3x+y)}{x^2-y^2}.$$

$$25. \frac{\frac{x^2+(a+b)x+ab}{x^2-b^2}}{\frac{x^2-a^2}{(x+a)(x-a)}} = \frac{\frac{(x+a)(x+b)}{(x-a)(x-b)}}{\frac{(x+b)(x-b)}{(x+a)(x-a)}} = \frac{(x+a)(x+b)(x+a)}{(x+b)(x-b)(x-b)} = \frac{(x+a)^2}{(x-b)^2}.$$

$$26. \frac{\frac{1}{a} + \frac{1}{b+c}}{\frac{1}{a} - \frac{1}{b+c}} \div \frac{1}{1 + \frac{b^2+c^2-a^2}{2bc}} = \frac{b+c+a}{b+c-a} \div \frac{2bc}{2bc+b^2+c^2-a^2} \\ = \frac{b+c+a}{b+c-a} \times \frac{(b+c+a)(b+c-a)}{2bc} = \frac{(a+b+c)^2}{2bc}.$$

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$$\begin{aligned}
 27. \quad & \frac{\frac{x+y}{x-y} + \frac{x^2+y^2}{x^2-y^2}}{\frac{x-y}{x+y} - \frac{x^3-y^3}{x^3+y^3}} \times \frac{(x-y)^2}{x^4+x^2y^2+y^4} \\
 &= \frac{2(x^2+xy+y^2)}{(x+y)(x-y)} \div \frac{-2xy(x-y)}{(x+y)(x^2-xy+y^2)} \times \frac{(x-y)^2}{(x^2+xy+y^2)(x^2-xy+y^2)} \\
 &= \frac{2(x^2+xy+y^2)}{(x+y)(x-y)} \times \frac{(x+y)(x^2-xy+y^2)}{-2xy(x-y)} \times \frac{(x-y)^2}{(x^2+xy+y^2)(x^2-xy+y^2)} \\
 &= -\frac{1}{xy}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & \frac{\frac{a-b}{1+ab} + \frac{b-c}{1+bc}}{1 - \frac{(a-b)(b-c)}{(1+ab)(1+bc)}} \div \frac{1-\frac{c}{a}}{\frac{1}{a} + c} = \frac{a+ab^2-c-b^2c}{1+ac+b^2+ab^2c} \div \frac{a-c}{1+ac} \\
 &= \frac{(a-c)(1+b^2)}{(1+b^2)(1+ac)} \times \frac{1+ac}{a-c} = 1.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \frac{x^2 - \frac{(x^2+y^2-z^2)^2}{4y^2}}{\frac{(x+y)^2-z^2}{y^2} \times \frac{(x-y+z)^2}{4}} = \frac{4x^2y^2 - (x^2+y^2-z^2)^2}{(x+y+z)(x+y-z)(x-y+z)^2} \\
 &= \frac{(x+y+z)(x+y-z)(x-y+z)(z-x+y)}{(x+y+z)(x+y-z)(x-y+z)^2} = \frac{z-x+y}{x-y+z}.
 \end{aligned}$$

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$$31. \quad \frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}} = \frac{1}{x + \frac{1}{\frac{4}{3-x}}} = \frac{1}{x + \frac{3-x}{4}} = \frac{1}{\frac{3x+3}{4}} = \frac{4}{3x+3}.$$

$$\begin{aligned}
 32. \quad & \frac{a}{a+1 + \frac{a}{a+1 - \frac{1}{a}}} = \frac{a}{a+1 + \frac{a}{a^2 + \frac{a-1}{a}}} = \frac{a}{a+1 + \frac{a^2}{a^2+a-1}} \\
 &= \frac{a}{\frac{a^3+3a^2-1}{a^2+a-1}} = \frac{a(a^2+a-1)}{a^3+3a^2-1}.
 \end{aligned}$$

$$33. \quad \frac{2}{2 - \frac{2}{2 - \frac{2}{2-x}}} = \frac{2}{2 - \frac{2}{\frac{2-2x}{2-x}}} = \frac{2}{2 - \frac{2-x}{1-x}} = \frac{2}{\frac{x}{x-1}} = \frac{2(x-1)}{x}.$$

$$\begin{aligned}
 34. \quad \frac{x-2}{x-2-\frac{x}{x-\frac{x-1}{x-2}}} &= \frac{x-2}{x-2-\frac{x}{x^2-3x+1}} = \frac{x-2}{x-2-\frac{x^2-2x}{x^2-3x+1}} \\
 &= \frac{x-2}{\frac{x^3-6x^2+9x-2}{x^2-3x+1}} = \frac{(x-2)(x^2-3x+1)}{(x-2)(x^2-4x+1)} = \frac{x^2-3x+1}{x^2-4x+1}
 \end{aligned}$$

$$35. \quad \frac{1}{a+\frac{1}{a+\frac{1}{a}}} = \frac{1}{a+\frac{1}{\frac{a^2+1}{a}}} = \frac{1}{a+\frac{a}{a^2+1}} = \frac{1}{\frac{a^3+2a}{a^2+1}} = \frac{a^2+1}{a^3+2a}$$

$$\begin{aligned}
 36. \quad 1 + \frac{c}{1+c+\frac{2c}{1+\frac{1}{c}}} &= 1 + \frac{c}{1+c+\frac{2c}{\frac{c+1}{c}}} = 1 + \frac{c}{1+c+\frac{2c^2}{c+1}} = 1 + \frac{c}{\frac{3c^2+2c+1}{c+1}} \\
 &= 1 + \frac{c^2+c}{3c^2+2c+1} = \frac{3c^2+2c+1+c^2+c}{3c^2+2c+1} = \frac{4c^2+3c+1}{3c^2+2c+1}
 \end{aligned}$$

$$1. \quad \frac{x^3+x^2+x-3}{x^3+3x^2+5x+3}$$

Subtracting the numerator from the denominator,

$$2x^2+4x+6=2(x^2+2x+3).$$

By trial, x^2+2x+3 is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3+x^2+x-3}{x^3+3x^2+5x+3} = \frac{(x^2+2x+3)(x-1)}{(x^2+2x+3)(x+1)} = \frac{x-1}{x+1}$$

$$2. \quad \frac{x^3-x^2-x-2}{x^3+3x^2+3x+2}$$

Subtracting the numerator from the denominator,

$$4x^2+4x+4=4(x^2+x+1).$$

By trial, x^2+x+1 is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3-x^2-x-2}{x^3+3x^2+3x+2} = \frac{(x^2+x+1)(x-2)}{(x^2+x+1)(x+2)} = \frac{x-2}{x+2}$$

$$3. \quad \frac{x^3+4x^2+8x+5}{x^3+3x^2+7x+5}$$

Subtracting the denominator from the numerator,

$$x^2+x=x(x+1).$$

By trial, $x+1$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3+4x^2+8x+5}{x^3+3x^2+7x+5} = \frac{(x+1)(x^2+3x+5)}{(x+1)(x^2+2x+5)} = \frac{x^2+3x+5}{x^2+2x+5}$$

$$4. \quad \frac{x^3 + 3x^2 + 4x + 2}{x^3 - 3x^2 - 8x - 10}.$$

Subtracting the denominator from the numerator,

$$6x^2 + 12x + 12 = 6(x^2 + 2x + 2).$$

By trial, $x^2 + 2x + 2$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 + 3x^2 + 4x + 2}{x^3 - 3x^2 - 8x - 10} = \frac{(x^2 + 2x + 2)(x + 1)}{(x^2 + 2x + 2)(x - 5)} = \frac{x + 1}{x - 5}.$$

$$5. \quad \frac{x^3 + x^2 - 22x - 40}{x^3 - 7x^2 + 2x + 40}.$$

Subtracting the denominator from the numerator,

$$8x^2 - 24x - 80 = 8(x^2 - 3x - 10).$$

By trial, $x^2 - 3x - 10$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 + x^2 - 22x - 40}{x^3 - 7x^2 + 2x + 40} = \frac{(x^2 - 3x - 10)(x + 4)}{(x^2 - 3x - 10)(x - 4)} = \frac{x + 4}{x - 4}.$$

$$6. \quad \frac{x^3 + 10x^2 + 7x - 18}{x^3 - 8x^2 - 11x + 18}.$$

Subtracting the denominator from the numerator,

$$18x^2 + 18x - 36 = 18(x^2 + x - 2).$$

By trial, $x^2 + x - 2$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{x^3 + 10x^2 + 7x - 18}{x^3 - 8x^2 - 11x + 18} = \frac{(x^2 + x - 2)(x + 9)}{(x^2 + x - 2)(x - 9)} = \frac{x + 9}{x - 9}.$$

$$7. \quad \frac{2x^3 + 7x^2 - 9x - 9}{2x^3 + 9x^2 + x - 3}.$$

Subtracting the numerator from the denominator,

$$2x^2 + 10x + 6 = 2(x^2 + 5x + 3)$$

By trial, $x^2 + 5x + 3$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{2x^3 + 7x^2 - 9x - 9}{2x^3 + 9x^2 + x - 3} = \frac{(x^2 + 5x + 3)(2x - 3)}{(x^2 + 5x + 3)(2x - 1)} = \frac{2x - 3}{2x - 1}.$$

$$8. \quad \frac{8x^3 - 22x^2 + 17x - 3}{6x^3 - 17x^2 + 14x - 3}.$$

Subtracting the denominator from the numerator,

$$2x^3 - 5x^2 + 3x = x(2x^2 - 5x + 3).$$

By trial, $2x^2 - 5x + 3$ is found to be the H.C.D. of numerator and denominator.

$$\therefore \frac{8x^3 - 22x^2 + 17x - 3}{6x^3 - 17x^2 + 14x - 3} = \frac{(2x^2 - 5x + 3)(4x - 1)}{(2x^2 - 5x + 3)(3x - 1)} = \frac{4x - 1}{3x - 1}.$$

$$\begin{aligned} 9. \quad \frac{-x}{2y-1} + \frac{y}{2y+1} - \frac{y-x}{1-4y^2} &= \frac{x}{2y-1} + \frac{y}{2y+1} - \frac{x-y}{4y^2-1} \\ &= \frac{2xy+x+2y^2-y+(x-y)}{4y^2-1} \\ &= \frac{2xy+2y^2}{4y^2-1} = \frac{2y(x+y)}{4y^2-1}. \end{aligned}$$

$$\begin{aligned}
 10. \quad & \frac{a^2}{4(1-a)^2} - \left(\frac{3}{8(1-a)} + \frac{1}{8(a+1)} - \frac{1-a}{4(a+1)} \right) \\
 &= \frac{a^2}{4(1-a)^2} - \left(\frac{3}{8(1-a)} + \frac{-1+2a}{8(1+a)} \right) \\
 &= \frac{a^2}{4(1-a)^2} - \frac{3+3a-1+3a-2a^2}{8(1-a^2)} = \frac{a^2}{4(1-a)^2} - \frac{1+3a-a^2}{4(1-a^2)} \\
 &= \frac{a^2+a^3-(1+2a-4a^2+a^3)}{4(1-a)^2(1+a)} = \frac{5a^2-2a-1}{4(1-a)^2(1+a)}.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \frac{2x-1}{x-1} - \frac{3x+1}{x+1} + \frac{3x-1}{x-2} - \frac{2x+1}{x+2} \\
 &= \frac{-x^2+3x}{x^2-1} + \frac{x^2+8x}{x^2-4} \\
 &= \frac{-x^4+3x^3+4x^2-12x+x^4+8x^3-x^2-8x}{(x^2-1)(x^2-4)} \\
 &= \frac{11x^3+3x^2-20x}{(x^2-1)(x^2-4)}.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & \frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \\
 &= \frac{a^3}{(a-b)(a-c)} - \frac{b^3}{(b-c)(a-b)} + \frac{c^3}{(a-c)(b-c)} \\
 &= \frac{a^3b - a^3c - (ab^3 - b^3c) + ac^3 - bc^3}{(a-b)(a-c)(b-c)} \\
 &= \frac{a^3b - a^3c - ab^3 + b^3c + ac^3 - bc^3}{a^2b - ab^2 + b^2c - a^2c + ac^2 - bc^2} = a + b + c.
 \end{aligned}$$

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$$13. \quad \left(1 + \frac{2}{m-1}\right) \left(\frac{m^2+m-2}{m^2+m}\right) = \frac{m+1}{m-1} \times \frac{(m-1)(m+2)}{m(m+1)} = \frac{m+2}{m}.$$

$$14. \quad (1-a+a^2) \left(\frac{1}{a^2} + \frac{1}{a} + 1\right) = (1-a+a^2) \left(\frac{1+a+a^2}{a^2}\right) = \frac{1+a^2+a^4}{a^2}.$$

$$15. \quad \left(\frac{x^2}{y^2} + \frac{x}{y} + 1\right) \left(\frac{x^2}{y^2} - \frac{x}{y} + 1\right) = \left[\left(\frac{x^2}{y^2} + 1\right)^2 - \left(\frac{x}{y}\right)^2\right] = \frac{x^4}{y^4} + \frac{x^2}{y^2} + 1.$$

$$\begin{aligned}
 16. \quad & \left(\frac{a+b}{a-b} - \frac{a-b}{a+b} - \frac{4ab}{a^2+b^2}\right) \frac{a^2-b^2}{8b^2} = \left(\frac{4ab}{a^2-b^2} - \frac{4ab}{a^2+b^2}\right) \frac{a^2-b^2}{8b^2} \\
 &= \frac{8ab^3}{(a^2-b^2)(a^2+b^2)} \times \frac{a^2-b^2}{8b^2} = \frac{ab}{a^2+b^2}.
 \end{aligned}$$

$$17. \quad \left(1 + \frac{x}{y}\right) \left(\frac{x}{y^2} - \frac{1}{y} + \frac{1}{x}\right) = \frac{x+y}{y} \times \frac{x^2-xy+y^2}{xy^2} = \frac{x^3+y^3}{xy^3}.$$

$$\begin{aligned}
 18. \quad & \left(x+1 + \frac{1}{x} + \frac{1}{x^2}\right) \div \left(x+1 - \frac{1}{x} - \frac{1}{x^2}\right) \\
 &= \frac{x^3+x^2+x+1}{x^2} \div \frac{x^3+x^2-x-1}{x^2} \\
 &= \frac{(x^2+1)(x+1)}{x^2} \times \frac{x^2}{(x^2-1)(x+1)} = \frac{x^2+1}{x^2-1}.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & \left(\frac{a+b}{a-b} + \frac{a^2+b^2}{a^2-b^2} \right) \left(\frac{a+b}{a-b} - \frac{a^3+b^3}{a^3-b^3} \right) \\
 &= \frac{2a^2+2ab+2b^2}{a^2-b^2} \times \frac{2a^2b+2ab^2}{a^3-b^3} \\
 &= \frac{2(a^2+ab+b^2)}{(a+b)(a-b)} \times \frac{2ab(a+b)}{(a-b)(a^2+ab+b^2)} = \frac{4ab}{(a-b)^2}.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & \left(1 - \frac{2xy}{x^2+y^2} \right) \div \left(\frac{x^3-y^3}{x-y} - 3xy \right) \\
 &= \frac{x^2-2xy+y^2}{x^2+y^2} \div \frac{x^3-3x^2y+3xy^2-y^3}{x-y} \\
 &= \frac{(x-y)^2}{x^2+y^2} \times \frac{x-y}{(x-y)^3} = \frac{1}{x^2+y^2}.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & (x^3-y^3-z^3-2yz) \div \frac{x+y+z}{x+y-z} \\
 &= (x+y+z)(x-y-z) \times \frac{x+y-z}{x+y+z} \\
 &= (x-y-z)(x+y-z) = x^2-y^2+z^2-2xz.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & \left(\frac{1+a}{1-a} + \frac{4a}{1+a^2} + \frac{8a}{1-a^2} - \frac{1-a}{1+a} \right) \div \left(\frac{1+a^2}{1-a^2} + \frac{4a^2}{1+a^2} - \frac{1-a^2}{1+a^2} \right) \\
 &= \frac{16a+8a^3}{(1+a^2)(1-a^2)} \div \frac{8a^2-4a^4}{(1-a^2)(1+a^2)} \\
 &= \frac{8a(2+a^2)}{(1+a^2)(1-a^2)} \times \frac{(1-a^2)(1+a^2)}{4a^2(2-a^2)} = \frac{2(2+a^2)}{a(2-a^2)}.
 \end{aligned}$$

$$\begin{aligned}
 23. \quad & \frac{\frac{a^2x^3}{bd} + \frac{abx^2}{c^2d} - \frac{acx^2}{d^2} - \frac{b^2x}{cd^2} + \frac{a^2x}{bc} - \frac{a}{d}}{\frac{\frac{ax}{c} - \frac{b}{d}}{c} - \frac{b}{d}} \\
 &= \frac{a^2c^2dx^3 + ab^2dx^2 - abc^3x^2 - b^3cx + a^2cd^2x - abc^2d}{abc^2d^2x - b^2c^2d} \\
 &= \frac{(adx - bc)(ac^2x^2 + b^2x + acd)}{bcd(adx - bc)} = \frac{ac^2x^2 + b^2x + acd}{bcd}.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & \left(x^2 - 3xy - 2y^2 + \frac{12y^3}{x+3y} \right) \div \left(3x - 6y - \frac{2x^2}{x+3y} \right) \\
 &= \frac{x^3 - 11xy^2 + 6y^3}{x+3y} \div \frac{x^2 + 3xy - 18y^2}{x+3y} \\
 &= \frac{(x-3y)(x^2+3xy-2y^2)}{x+3y} \times \frac{x+3y}{(x+6y)(x-3y)} = \frac{x^2+3xy-2y^2}{x+6y}.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \left(\frac{m-3n}{m+n} \right) \left(1 + \frac{4n}{m+n} \right) \div \left(\frac{m}{n} + 2 - \frac{15n}{m} \right) \\
 &= \left(\frac{m-3n}{m+n} \right) \left(\frac{m+5n}{m+n} \right) \div \left(\frac{m^2+2mn-15n^2}{mn} \right) \\
 &= \frac{m-3n}{m+n} \times \frac{m+5n}{m+n} \times \frac{mn}{(m-3n)(m+5n)} = \frac{mn}{(m+n)^2}.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & 1 - \frac{2x + 5x^2}{2(x+1)^2} - \left\{ \frac{x^2 + x}{2x - \frac{2}{x}} \right\} \left(\frac{3 - 3x}{(x+1)^2} \right) \\
 &= 1 - \frac{2x + 5x^2}{2(x+1)^2} - \frac{x^3 + x^2}{2x^2 - 2} \times \frac{3 - 3x}{(x+1)^2} \\
 &= 1 - \frac{2x + 5x^2}{2(x+1)^2} + \frac{3x^2}{2(x+1)^2} \\
 &= 1 - \frac{2x^2 + 2x}{2(x+1)^2} = 1 - \frac{x}{x+1} = \frac{1}{x+1}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & \left(1 + \frac{x}{a-x} \right) \left(\frac{x}{x+a} - \frac{2x^2 + 2ax - a^2}{x^2 + 3ax + 2a^2} \right) \\
 &= \frac{a}{a-x} \times \frac{(a+x)(a-x)}{(x+a)(x+2a)} = \frac{a}{x+2a}.
 \end{aligned}$$

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$$28. \quad \S 96, \quad \left(\frac{x}{y} + \frac{y}{x} \right) \left(\frac{x}{y} - \frac{y}{x} \right) = \frac{x^2}{y^2} - \frac{y^2}{x^2}.$$

$$29. \quad \S 90, \quad \left(\frac{x}{y} + \frac{y}{x} \right) \left(\frac{x}{y} + \frac{y}{x} \right) = \frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}.$$

$$30. \quad \S 92, \quad \left(2x - \frac{1}{2x} \right) \left(2x - \frac{1}{2x} \right) = 4x^2 - 2 + \frac{1}{4x^2}.$$

$$31. \quad \left(\frac{a}{b} + 1 + \frac{b}{a} \right) \left(\frac{a}{b} - 1 + \frac{b}{a} \right) = \left(\frac{a}{b} + \frac{b}{a} + 1 \right) \left(\frac{a}{b} + \frac{b}{a} - 1 \right)$$

$$\S 96, \quad = \left(\frac{a}{b} + \frac{b}{a} \right)^2 - 1$$

$$\S 90, \quad = \frac{a^2}{b^2} + 2 + \frac{b^2}{a^2} - 1 = \frac{a^2}{b^2} + 1 + \frac{b^2}{a^2}.$$

$$\begin{aligned}
 32. \quad & (x^2y + xy^2) \left(\frac{1}{x^3} - \frac{2y}{x^4} + \frac{y^2}{x^5} \right) \\
 &= (x^2y + xy^2) \left(\frac{x^2 - 2xy + y^2}{x^5} \right) \\
 &= xy(x+y) \times \frac{(x-y)^2}{x^5} = \frac{y(x+y)(x-y)^2}{x^4}.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad & (x^2 + x + 1) \left(1 - \frac{1}{x} + \frac{1}{x^2} \right) = (x^2 + x + 1) \left(\frac{x^2 - x + 1}{x^2} \right) \\
 &= \frac{(x^2 + x + 1)(x^2 - x + 1)}{x^2} = \frac{x^4 + x^2 + 1}{x^2}.
 \end{aligned}$$

$$\begin{aligned}
 34. \quad & \frac{1 + \frac{1}{x^2} + \frac{1}{x^4}}{1 + \frac{1}{x} + \frac{1}{x^2}} = \frac{x^4 + x^2 + 1}{x^4 + x^3 + x^2} = \frac{(x^2 + x + 1)(x^2 - x + 1)}{x^2(x^2 + x + 1)} = \frac{x^2 - x + 1}{x^2}.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \frac{\left(\frac{m^2 + n^2}{n} - m\right) \div \left(\frac{1}{n} - \frac{1}{m}\right)}{\frac{m^3 + n^3}{m^2 - n^2}} &= \frac{m^2 - mn + n^2}{n} \div \frac{m - n}{mn} \div \frac{(m + n)(m^2 - mn + n^2)}{(m + n)(m - n)} \\
 &= \frac{m^2 - mn + n^2}{n} \times \frac{mn}{m - n} \times \frac{m - n}{m^2 - mn + n^2} = m.
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \frac{\frac{c}{(a+1)^2} + \frac{d}{(a+1)^2}}{\frac{a}{(a+1)^4} + \frac{1}{(a+1)^4}} &= \frac{c + d}{(a+1)^2} \div \frac{a+1}{(a+1)^4} = \frac{c + d}{(a+1)^2} \times \frac{(a+1)^3}{1} \\
 &= (c + d)(a+1).
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \frac{1}{1 - \frac{1}{1 - \frac{1}{1 - x}}} &= \frac{1}{1 - \frac{1}{-x}} = \frac{1}{1 + \frac{1}{x}} = \frac{1}{\frac{x+1}{x}} = x.
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \frac{\frac{6a}{x+y} - \frac{4}{x+y}}{\frac{1}{(x+y)^2}} &= \frac{6ax + 6ay - (4x + 4y)}{1} = 2(3a - 2)(x + y).
 \end{aligned}$$

$$\begin{aligned}
 39. \quad 1 - \frac{\frac{a^2 + 3a + 2}{a^2 + 2a + 1}}{\frac{a^2 + 7a + 12}{a^2 + 5a + 4}} &= 1 - \frac{(a+1)(a+2)}{(a+1)(a+1)} \cdot \frac{(a+1)(a+1)}{(a+3)(a+4)} = 1 - \frac{(a+1)^2(a+2)(a+4)}{(a+1)^2(a+3)(a+4)} \\
 &= 1 - \frac{a+2}{a+3} = \frac{1}{a+3}.
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \frac{\frac{m^3 - n^3}{m^3 + n^3} \left(1 - \frac{2n}{m+n}\right)}{1 + \frac{2mn}{m^2 - mn + n^2}} &= \frac{m^3 - n^3}{m^3 + n^3} \times \frac{m - n}{m + n} \div \frac{m^2 + mn + n^2}{m^2 - mn + n^2} \\
 &= \frac{(m - n)(m^2 + mn + n^2)}{(m + n)(m^2 - mn + n^2)} \times \frac{m - n}{m + n} \times \frac{m^2 - mn + n^2}{m^2 + mn + n^2} \\
 &= \frac{(m - n)^2}{(m + n)^2}.
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \frac{1}{2 - \frac{3}{4 - \frac{5}{6 - x}}} &= \frac{1}{2 - \frac{3}{\frac{19 - 4x}{6 - x}}} = \frac{1}{2 - \frac{3(6 - x)}{19 - 4x}} = \frac{1}{\frac{20 - 5x}{19 - 4x}} = \frac{19 - 4x}{20 - 5x}.
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \left(\frac{x+y}{x-y} + \frac{x-y}{x+y}\right) \div \left(\frac{x+y}{2(x-y)} - \frac{x-y}{2x+2y}\right) \\
 &= \frac{2x^2 + 2y^2}{x^2 - y^2} \div \frac{4xy}{2(x^2 - y^2)} \\
 &= \frac{2(x^2 + y^2)}{x^2 - y^2} \times \frac{2(x^2 - y^2)}{4xy} = \frac{x^2 + y^2}{xy}.
 \end{aligned}$$

$$\begin{aligned}
 43. \quad & \frac{\left(\frac{a}{x^2} + \frac{1}{x} + \frac{1}{a} + \frac{x}{a^2}\right) \left(\frac{a^2}{x^3} - \frac{1}{x} + \frac{x}{a^2}\right)}{\frac{a^3}{x^5} \left(1 + \frac{x}{a}\right)} = \frac{\frac{a^3 + a^2x + ax^2 + x^3}{a^2x^2} \times \frac{a^4 - a^2x^2 + x^4}{a^2x^3}}{\frac{a^3}{x^5} \left(\frac{a+x}{a}\right)} \\
 & = \frac{a^3 + a^2x + ax^2 + x^3}{a^2x^2} \times \frac{a^4 - a^2x^2 + x^4}{a^2x^3} \div \frac{a^3}{x^5} \left(\frac{a+x}{a}\right) \\
 & = \frac{(a+x)(a^2+x^2)}{a^2x^2} \times \frac{a^4 - a^2x^2 + x^4}{a^2x^3} \times \frac{x^5}{a^2(a+x)} \\
 & = \frac{(a^2+x^2)(a^4 - a^2x^2 + x^4)}{a^6} = \frac{a^6 + x^6}{a^6}.
 \end{aligned}$$

$$\begin{aligned}
 44. \quad & \frac{x^3 - \frac{8}{y^3}}{x^3y^3 - x^2y^2} \times \frac{\frac{1}{xy} + \frac{1}{x^2y^2}}{1 + \frac{2}{xy} + \frac{4}{x^2y^2}} \times \frac{xy - 1}{xy + 1} \\
 & = \frac{x^3y^3 - 8}{x^3y^6 - x^2y^6} \times \frac{xy + 1}{x^2y^2 + 2xy + 4} \times \frac{xy - 1}{xy + 1} \\
 & = \frac{(xy - 2)(x^2y^2 + 2xy + 4)}{x^2y^6(xy - 1)} \times \frac{xy + 1}{x^2y^2 + 2xy + 4} \times \frac{xy - 1}{xy + 1} \\
 & = \frac{xy - 2}{x^2y^6}.
 \end{aligned}$$

SIMPLE EQUATIONS

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$$22. \quad \frac{x+1}{3} - \frac{x+4}{5} + \frac{x+3}{4} = 16.$$

Clearing, $20(x+1) - 12(x+4) + 15(x+3) = 60 \cdot 16.$
 $20x + 20 - 12x - 48 + 15x + 45 = 960.$
 $\therefore x = 41.$

Verifying, $\frac{42}{3} - \frac{45}{5} + \frac{44}{4} = 16,$
 which reduces to $16 = 16.$

$$23. \quad \frac{7x+2}{6} - \frac{12-x}{4} + \frac{x+2}{2} = 6.$$

Clearing, $2(7x+2) - 3(12-x) + 6(x+2) = 12 \cdot 6.$
 $14x + 4 - 36 + 3x + 6x + 12 = 72.$
 $\therefore x = 4.$

Verifying, $\frac{30}{6} - \frac{8}{4} + \frac{6}{2} = 6,$
 which reduces to $6 = 6.$

$$24. \quad \frac{x-3}{7} + \frac{x+5}{3} - \frac{x+2}{6} = 4.$$

Clearing, $6(x-3) + 14(x+5) - 7(x+2) = 42 \cdot 4.$
 $6x - 18 + 14x + 70 - 7x - 14 = 168.$
 $\therefore x = 10.$

Verifying, $\frac{7}{7} + \frac{15}{3} - \frac{12}{6} = 4,$
 which reduces to $4 = 4.$

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$$\frac{3x-5}{4} - \frac{7x-13}{6} = 3 - \frac{x+3}{2}.$$

Clearing,

$$3(3x-5) - 2(7x-13) = 12 \cdot 3 - 6(x+3).$$

$$9x - 15 - 14x + 26 = 36 - 6x - 18.$$

$$\therefore x = 7.$$

Verifying,
which reduces to

$$\frac{16}{4} - \frac{36}{6} = 3 - \frac{10}{2},$$

$$-2 = -2.$$

26.

$$\frac{2x-5}{5} - \frac{3x-2}{7} = 1 - \frac{x+2}{6}.$$

Clearing,

$$42(2x-5) - 30(3x-2) = 210 - 35(x+2).$$

$$84x - 210 - 90x + 60 = 210 - 35x - 70.$$

$$\therefore x = 10.$$

Verifying,
which reduces to

$$\frac{15}{5} - \frac{28}{7} = 1 - \frac{12}{6},$$

$$-1 = -1.$$

27.

$$\frac{1-2x}{3} - \frac{7-2x}{4} + \frac{11-2x}{6} = -\frac{7}{12}.$$

Clearing,

$$4(1-2x) - 3(7-2x) + 2(11-2x) = -7.$$

$$4 - 8x - 21 + 6x + 22 - 4x = -7.$$

$$\therefore x = 2.$$

Verifying,
which reduces to

$$-\frac{3}{3} - \frac{3}{4} + \frac{7}{6} = -\frac{7}{12},$$

$$-\frac{7}{12} = -\frac{7}{12}.$$

28.

$$\frac{x+4}{3} + \frac{2-2x}{6} = \frac{x+1}{2} - 3\frac{1}{3}.$$

Clearing,

$$2(x+4) + 2-2x = 3(x+1) - 20.$$

$$2x + 8 + 2 - 2x = 3x + 3 - 20.$$

$$\therefore x = 9.$$

Verifying,
which reduces to

$$\frac{13}{3} + \frac{-16}{6} = \frac{10}{2} - \frac{10}{3},$$

$$\frac{5}{3} = \frac{5}{3}.$$

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29.

$$\frac{9x+5}{14} + \frac{8x-7}{6x+2} = \frac{36x+15}{56} + \frac{10\frac{1}{4}}{14}.$$

Dist. Law,

$$\frac{9x}{14} + \frac{5}{14} + \frac{8x-7}{6x+2} = \frac{36x}{56} + \frac{15}{56} + \frac{41}{56}.$$

Canceling, etc.,

$$\frac{8x-7}{6x+2} = \frac{36}{56} = \frac{9}{14}.$$

Clearing,

$$112x - 98 = 54x + 18.$$

$$\therefore x = 2.$$

Verifying,
which reduces to

$$\frac{23}{14} + \frac{9}{14} = \frac{37}{56} + \frac{41}{56},$$

$$\frac{32}{14} = \frac{32}{14}.$$

30.

$$\frac{3x-2}{2x-5} + \frac{3x-21}{5} = \frac{6x-22}{10}.$$

Dist. Law,

$$\frac{3x-2}{2x-5} + \frac{3x-21}{5} = \frac{3x-11}{5}.$$

Canceling, etc.,

$$\frac{3x-2}{2x-5} = 2.$$

Clearing,

$$3x-2 = 4x-10.$$

Verifying,

$$\therefore x = 8.$$

which reduces to

$$\frac{22}{11} + \frac{3}{5} = \frac{28}{10},$$

$$2\frac{3}{5} = 2\frac{3}{5}.$$

31.

$$\frac{4x+3}{9} = \frac{8x+19}{18} - \frac{7x-29}{5x-12}.$$

Dist. Law,

$$\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}.$$

Canceling, etc.,

$$\frac{7x-29}{5x-12} = \frac{13}{18}.$$

Clearing,

$$126x - 522 = 65x - 156.$$

Verifying,

$$\therefore x = 6.$$

which reduces to

$$\frac{27}{9} = \frac{67}{18} - \frac{13}{18},$$

$$3 = 3.$$

32.

$$\frac{6x+1}{15} - \frac{2x-4}{7x-13} = \frac{2x-1}{5}.$$

Dist. Law,

$$\frac{6x+1}{15} + \frac{1}{15} - \frac{2x-4}{7x-13} = \frac{2x-1}{5}.$$

Canceling, etc.,

$$\frac{4}{15} = \frac{2x-4}{7x-13}.$$

Clearing,

$$28x - 52 = 30x - 60.$$

Verifying,

$$\therefore x = 4.$$

which reduces to

$$\frac{25}{15} - \frac{4}{15} = \frac{7}{5},$$

$$\frac{7}{5} = \frac{7}{5}.$$

33.

$$\frac{10x+17}{18} - \frac{5x-2}{9} = \frac{12x-1}{11x-8}.$$

Uniting terms in the first member,

$$\frac{7}{6} = \frac{12x-1}{11x-8}.$$

Clearing,

$$77x - 56 = 72x - 6.$$

Verifying,

$$\therefore x = 10.$$

which reduces to

$$\frac{117}{18} - \frac{48}{9} = \frac{119}{108},$$

$$\frac{7}{6} = \frac{7}{6}.$$

34.

$$\frac{6x+3}{15} - \frac{3x-1}{5x-25} = \frac{2x-9}{5}.$$

$$\frac{6x+3}{15} + \frac{3}{15} - \frac{3x-1}{5x-25} = \frac{2x-9}{5}.$$

Canceling, etc.,

$$2 = \frac{3x-1}{5x-25}.$$

Clearing,

$$10x - 50 = 3x - 1.$$

Verifying,

$$\therefore x = 7.$$

which reduces to

$$\frac{15}{15} - \frac{20}{10} = \frac{5}{5},$$

$$1 = 1.$$

$$35. \quad \frac{2x+1}{2x-1} - \frac{8}{4x^2-1} = \frac{2x-1}{2x+1}.$$

Clearing, $(2x+1)(2x+1) - 8 = (2x-1)(2x-1).$
 $4x^2 + 4x + 1 - 8 = 4x^2 - 4x + 1.$

Verifying, $\therefore x = 1$
 $\frac{3}{1} - \frac{8}{3} = \frac{1}{3},$
 which reduces to $\frac{1}{3} = \frac{1}{3}.$

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$$37. \quad \frac{x-1}{x-2} + \frac{x-7}{x-8} = \frac{x-5}{x-6} + \frac{x-3}{x-4}.$$

Transposing, $\frac{x-1}{x-2} - \frac{x-3}{x-4} = \frac{x-5}{x-6} - \frac{x-7}{x-8}.$

Uniting terms in each member,

$$\frac{x^2 - 5x + 4 - (x^2 - 5x + 6)}{(x-2)(x-4)} = \frac{x^2 - 13x + 40 - (x^2 - 13x + 42)}{(x-6)(x-8)},$$

or $\frac{-2}{x^2 - 6x + 8} = \frac{-2}{x^2 - 14x + 48}.$
 $\therefore x^2 - 6x + 8 = x^2 - 14x + 48,$

whence, $x = 5.$

Verifying, $\frac{4}{3} + \frac{-2}{-3} = \frac{0}{-1} + \frac{2}{1}, \text{ or } 2 = 2.$

$$38. \quad \frac{x-3}{x-4} + \frac{x-7}{x-8} = \frac{x-6}{x-7} + \frac{x-4}{x-5}.$$

Transposing, $\frac{x-7}{x-8} - \frac{x-6}{x-7} = \frac{x-4}{x-5} - \frac{x-3}{x-4}.$

Uniting terms in each member,

$$\frac{x^2 - 14x + 49 - (x^2 - 14x + 48)}{(x-8)(x-7)} = \frac{x^2 - 8x + 16 - (x^2 - 8x + 15)}{(x-5)(x-4)},$$

or $\frac{1}{x^2 - 15x + 56} = \frac{1}{x^2 - 9x + 20}.$
 $\therefore x^2 - 9x + 20 = x^2 - 15x + 56.$

$$\therefore x = 6.$$

Verifying, $\frac{3}{2} + \frac{-1}{-2} = \frac{0}{-1} + \frac{2}{1}, \text{ or } 2 = 2.$

$$39. \quad \frac{x+2}{x+1} - \frac{x+3}{x+2} = \frac{x+5}{x+4} - \frac{x+6}{x+5}.$$

Uniting terms in each member,

$$\frac{x^2 + 4x + 4 - (x^2 + 4x + 3)}{(x+1)(x+2)} = \frac{x^2 + 10x + 25 - (x^2 + 10x + 24)}{(x+4)(x+5)},$$

or $\frac{1}{x^2 + 3x + 2} = \frac{1}{x^2 + 9x + 20}.$

Clearing, $x^2 + 9x + 20 = x^2 + 3x + 2.$

$$\therefore x = -3.$$

Verifying, $\frac{-1}{-2} - \frac{0}{-1} = \frac{2}{1} - \frac{3}{2}, \text{ or } \frac{1}{2} = \frac{1}{2}.$

$$40. \quad \frac{x+1}{x+2} + \frac{x+6}{x+7} = \frac{x+2}{x+3} + \frac{x+5}{x+6}$$

$$\text{Transposing,} \quad \frac{x+6}{x+7} - \frac{x+5}{x+6} = \frac{x+2}{x+3} - \frac{x+1}{x+2}$$

Uniting terms in each member,

$$\frac{x^2 + 12x + 36 - (x^2 + 12x + 35)}{(x+7)(x+6)} = \frac{x^2 + 4x + 4 - (x^2 + 4x + 3)}{(x+3)(x+2)},$$

$$\text{or} \quad \frac{1}{x^2 + 13x + 42} = \frac{1}{x^2 + 5x + 6}$$

$$\therefore x^2 + 13x + 42 = x^2 + 5x + 6,$$

whence, $x = -\frac{9}{2}$.

$$\text{Verifying,} \quad -\frac{7}{2} + \frac{3}{2} = -\frac{5}{2} + \frac{1}{2}, \text{ or } 2 = 2.$$

$$41. \quad \frac{x-5}{x+5} - \frac{x-10}{x+10} = \frac{x-4}{x+4} - \frac{x-9}{x+9}$$

Uniting terms in each member,

$$\frac{x^2 + 5x - 50 - (x^2 - 5x - 50)}{(x+5)(x+10)} = \frac{x^2 + 5x - 36 - (x^2 - 5x - 36)}{(x+4)(x+9)},$$

$$\text{or} \quad \frac{10x}{x^2 + 15x + 50} = \frac{10x}{x^2 + 13x + 36}$$

$$\therefore x^2 + 15x + 50 = x^2 + 13x + 36,$$

whence, $x = -\frac{7}{2}$.

$$\text{Verifying,} \quad -\frac{12}{2} - \frac{17}{3} = -\frac{11}{3} - \frac{16}{2}, \text{ or } 11\frac{2}{3} = 11\frac{2}{3}.$$

$$42. \quad \frac{7x-2}{3} - \frac{5x-\frac{1}{2}}{2} = \frac{x-2}{4} - 3\frac{1}{3}.$$

$$\text{Simplifying,} \quad \frac{7x}{12} - \frac{1}{2} - \frac{5x}{3} + \frac{1}{6} = \frac{x}{20} - \frac{2}{5} - \frac{10}{3}.$$

$$\text{Clearing,} \quad 35x - 30 - 100x + 10 = 3x - 24 - 200.$$

$$\therefore x = 3.$$

$$\text{Verifying,} \quad \frac{7-2}{4} - \frac{2\frac{1}{2}}{3} = \frac{-\frac{5}{2}}{5} - \frac{10}{3}, \text{ or } -\frac{43}{12} = -\frac{43}{12}.$$

$$43. \quad \frac{x-1}{x-2} + \frac{x-2}{x-3} + \frac{x-6}{x-4} - 3 = 0.$$

Reducing to mixed numbers and canceling,

$$\frac{1}{x-2} + \frac{1}{x-3} - \frac{2}{x-4} = 0.$$

$$\text{Clearing,} \quad x^2 - 7x + 12 + x^2 - 6x + 8 - 2(x^2 - 5x + 6) = 0.$$

$$\text{Uniting terms and transposing,} \quad -3x = -8.$$

$$\therefore x = \frac{8}{3}.$$

$$\text{Verifying,} \quad \frac{\frac{5}{3}}{\frac{2}{3}} + \frac{\frac{2}{3}}{\frac{-1}{3}} + \frac{-\frac{10}{3}}{\frac{-4}{3}} - 3 = 0, \text{ or } 0 = 0.$$

$$44. \quad \frac{x^3 + 1}{x - 1} - \frac{x^3 - 1}{x + 1} = \frac{8}{x^2 - 1} + 2x.$$

Reducing to mixed numbers,

$$x^2 + x + 1 + \frac{2}{x - 1} - \left(x^2 - x + 1 - \frac{2}{x + 1} \right) = \frac{8}{x^2 - 1} + 2x.$$

$$\text{Uniting terms,} \quad \frac{2}{x - 1} + \frac{2}{x + 1} = \frac{8}{x^2 - 1}.$$

$$\text{Clearing,} \quad 2x + 2 + 2x - 2 = 8. \\ \therefore x = 2.$$

$$\text{Verifying,} \quad \frac{9}{1} - \frac{7}{3} = \frac{8}{3} + 4, \text{ or } 6\frac{2}{3} = 6\frac{2}{3}.$$

$$45. \quad \frac{x^3 + 2}{x + 1} - \frac{x^3 - 2}{x - 1} = \frac{10}{x^2 - 1} - 2x.$$

Reducing to mixed numbers,

$$x^2 - x + 1 + \frac{1}{x + 1} - \left(x^2 + x + 1 - \frac{1}{x - 1} \right) = \frac{10}{x^2 - 1} - 2x.$$

$$\text{Uniting terms,} \quad \frac{1}{x + 1} + \frac{1}{x - 1} = \frac{10}{x^2 - 1}.$$

$$\text{Clearing,} \quad x - 1 + x + 1 = 10. \\ \therefore x = 5.$$

$$\text{Verifying,} \quad \frac{127}{6} - \frac{123}{4} = \frac{10}{24} - 10, \text{ or } -\frac{115}{12} = -\frac{115}{12}.$$

$$46. \quad \frac{2x + 4}{3} = \frac{7\frac{1}{2} - x}{3} + \frac{x}{2} \left(\frac{6}{x} - 1 \right).$$

$$\text{Simplifying each term,} \quad \frac{x}{3} + 2 = \frac{5}{2} - \frac{x}{3} + 3 - \frac{x}{2}.$$

$$\text{Transposing and uniting terms,} \quad \frac{7x}{6} = \frac{7}{2}.$$

$$\text{Multiplying by } \frac{6}{7}, \quad x = 3.$$

$$\text{Verifying,} \quad \frac{2 + 4}{2} = \frac{4\frac{1}{2}}{3} + \frac{3}{2} \text{ of } 1, \text{ or } 3 = 3.$$

$$47. \quad \frac{\frac{4}{5x} - 16}{24} - \frac{\frac{2}{5x} + 6}{60} = \frac{4\frac{1}{5}}{5}.$$

$$\text{Simplifying each term,} \quad \frac{1}{30x} - \frac{2}{3} - \frac{1}{150x} - \frac{1}{10} = \frac{5}{6}.$$

$$\text{Clearing,} \quad 5 - 100x - 1 - 15x = 125x. \\ \therefore x = \frac{1}{60}.$$

$$\text{Verifying, since } \frac{1}{x} = 60, \frac{4}{5x} = \frac{4}{5} \text{ of } 60, \text{ and } \frac{2}{5x} = \frac{2}{5} \text{ of } 60,$$

$$\frac{48 - 16}{24} - \frac{24 + 6}{60} = \frac{5}{6}, \text{ or } \frac{5}{6} = \frac{5}{6}.$$

$$48. \quad \frac{x}{2}(2-x) - \frac{x}{4}(3-2x) = \frac{x+10}{6}.$$

Simplifying each term,

$$x - \frac{x^2}{2} - \frac{3x}{4} + \frac{x^2}{2} = \frac{x}{6} + \frac{5}{3}.$$

Clearing, etc., $12x - 9x = 2x + 20.$

$$\therefore x = 20.$$

Verifying, $10(2-20) - 5(3-40) = 5, \text{ or } 5 = 5.$

$$49. \quad \frac{2x\left(1-\frac{5}{x}\right)}{3} + \frac{3x\left(1-\frac{4}{x}\right)}{4} = \frac{x-4}{\frac{4}{5}}.$$

Simplifying each term,

$$\frac{2x}{3} - \frac{10}{3} + \frac{3x}{4} - 3 = \frac{5x}{4} - 5.$$

Clearing, $8x - 40 + 9x - 36 = 15x - 60.$

$$\therefore x = 8.$$

Verifying, $\frac{16 \cdot \frac{3}{8}}{3} + \frac{24 \cdot \frac{1}{2}}{4} = \frac{4}{\frac{4}{5}},$

$$2 + 3 = 5, \text{ or } 5 = 5.$$

that is,

$$50. \quad \frac{1}{2}x - 2\left(\frac{4x}{5} - 3\right) = 4 - \frac{3}{2}\left(\frac{x}{2} + 1\right).$$

Expanding, $\frac{1}{2}x - \frac{8x}{5} + 6 = 4 - \frac{3x}{4} - \frac{3}{2}.$ Clearing, $10x - 32x + 120 = 80 - 15x - 30.$

$$\therefore x = 10.$$

Verifying, $5 - 2(8-3) = 4 - \frac{3}{2} \text{ of } 6, \text{ or } -5 = -5.$

$$51. \quad \frac{(2x+1)^2}{5} - \frac{(4x-1)^2}{20} = \frac{15}{8} + \frac{3(4x+1)}{40}.$$

Clearing, $8(2x+1)^2 - 2(4x-1)^2 = 75 + 3(4x+1).$ Expanding, $32x^2 + 32x + 8 - 32x^2 + 16x - 2 = 75 + 12x + 3.$

$$\therefore x = 2.$$

Verifying, $\frac{2 \cdot 5}{5} - \frac{4 \cdot 9}{20} = \frac{15}{8} + \frac{27}{40}, \text{ or } \frac{51}{40} = \frac{51}{40}.$

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$$52. \quad \frac{1}{2} \cdot \frac{x+1}{x-1} + 2 = \frac{1}{2} + \frac{x-1}{x+1} - \frac{x^2}{1-x^2}.$$

Reducing to mixed numbers,

$$\frac{1}{2}\left(1 + \frac{2}{x-1}\right) + 2 = \frac{1}{2} + 1 - \frac{2}{x+1} + 1 + \frac{1}{x^2-1}.$$

Canceling, etc., $\frac{1}{x-1} = \frac{1}{x^2-1} - \frac{2}{x+1}.$ Clearing, $x+1 = 1-2x+2.$

$$\therefore x = \frac{2}{3}.$$

Verifying, $\frac{1}{2} \text{ of } \frac{5}{3} + 2 = \frac{1}{2} + \frac{-1}{\frac{5}{3}} - \frac{\frac{4}{9}}{\frac{5}{9}},$ that is, $\frac{-5}{2} + 2 = \frac{1}{2} - \frac{1}{5} - \frac{4}{5}, \text{ or } -\frac{1}{2} = -\frac{1}{2}.$

53. See next page.

$$54. \quad \frac{\frac{1}{2}(x-4)}{\frac{3}{2}} - \frac{4x-16}{6} = \frac{3}{5} - \frac{\frac{2x+5}{5}}{\frac{5}{2}}.$$

Simplifying each term,

$$\frac{x}{6} - \frac{2}{3} - \frac{2x}{3} + \frac{8}{3} = \frac{3}{5} - \frac{4x}{25} - 2.$$

Transposing and uniting,

$$-\frac{17x}{50} = -\frac{17}{5}.$$

$$\therefore x = 10.$$

Verifying,

$$\frac{\frac{1}{2} \text{ of } 6}{\frac{3}{2}} - \frac{24}{6} = \frac{3}{5} - \frac{9}{5},$$

that is,

$$1 - 4 = \frac{3}{5} - \frac{18}{5}, \text{ or } -3 = -3.$$

$$55. \quad 1 + \frac{1}{1 + \frac{1}{x}} = \frac{2 + \frac{10}{x}}{1 + \frac{6}{x}}.$$

Multiplying both terms of each fraction by x ,

$$1 + \frac{x}{x+1} = \frac{2x+10}{x+6}.$$

Clearing, $x^2 + 7x + 6 + x^2 + 6x = 2x^2 + 12x + 10$.

Transposing and uniting, $x = 4$.

Verifying, $1 + \frac{1}{1 + \frac{1}{4}} = \frac{2 + \frac{5}{2}}{1 + \frac{3}{2}}, \text{ or } \frac{9}{5} = \frac{9}{5}.$

$$56. \quad \frac{1}{\frac{x}{3} + \frac{3}{3x-1}} = \frac{3}{x+5}.$$

Multiplying both terms of the second fraction in the denominator of the first member by 3,

$$\frac{1}{\frac{x}{3} + \frac{x-3}{3(3x-1)}} = \frac{3}{x+5}.$$

Multiplying both terms of the first member by $3(3x-1)$,

$$\frac{9x-3}{3x^2-x+x-3} = \frac{3}{x+5}.$$

Simplifying,

$$\frac{3x-1}{x^2-1} = \frac{3}{x+5}.$$

Clearing,

$$3x^2 + 14x - 5 = 3x^2 - 3, \therefore x = \frac{1}{2}.$$

Verifying,

$$\frac{1}{\frac{1}{21} + \frac{-\frac{20}{21}}{-\frac{1}{2}}} = \frac{3}{\frac{36}{7}}, \text{ or } \frac{7}{12} = \frac{7}{12}.$$

$$53. \quad \frac{17 + \frac{3}{x}}{3} + \frac{1 + \frac{18}{x}}{5} = \frac{21}{x} - 1 + \frac{100}{x} + \frac{5}{3}. \quad (1)$$

$$\frac{17}{3} + \frac{1}{x} + \frac{1}{5} + \frac{18}{5} \cdot \frac{1}{x} = \frac{7}{3} \cdot \frac{1}{x} - \frac{1}{9} + \frac{20}{3} \cdot \frac{1}{x} + \frac{1}{9}. \quad (2)$$

$$\text{Canceling and transposing,} \quad \frac{17}{3} + \frac{1}{5} = \left(\frac{7}{3} + \frac{20}{3} - 1 - \frac{18}{5} \right) \frac{1}{x}. \quad (3)$$

$$\text{Uniting terms,} \quad \frac{88}{15} = \frac{22}{5} \cdot \frac{1}{x}. \quad (4)$$

$$\text{Dividing by } \frac{22}{5}, \quad \frac{4}{3} = \frac{1}{x}. \quad (5)$$

$$\therefore x = \frac{3}{4}. \quad (6)$$

$$\text{Substituting (5) in (1),} \quad \frac{17 + \frac{4}{3}}{3} + \frac{1 + \frac{24}{5}}{5} = \frac{21}{9} - 1 + \frac{100}{15} + \frac{5}{3},$$

that is, $7 + 5 = 3 + 9$, or $12 = 12$.

$$2. \quad \frac{c^2 - x}{nx} + \frac{n^2}{cx} = \frac{1}{c}.$$

$$\begin{aligned} \text{Clearing,} & \quad c^3 - cx + n^3 = nx. \\ \text{Transposing, etc.,} & \quad cx + nx = c^3 + n^3. \\ \text{Dividing by } c + n, & \quad x = c^2 - cn + n^2. \end{aligned}$$

$$3. \quad 1 - \frac{ab}{x} = \frac{7}{ab} - \frac{49}{abx}.$$

$$\begin{aligned} \text{Clearing,} & \quad abx - a^2b^2 = 7x - 49. \\ \text{Transposing,} & \quad abx - 7x = a^2b^2 - 49. \\ \text{Dividing by } ab - 7, & \quad x = ab + 7. \end{aligned}$$

$$4. \quad \frac{a^3}{ab^2} - \frac{2a^2}{b^2x} = 1 - \frac{2b^2}{a^2x}.$$

$$\begin{aligned} \text{Clearing,} & \quad a^4x - 2a^4 = a^2b^2x - 2b^4. \\ \text{Transposing,} & \quad a^4x - a^2b^2x = 2a^4 - 2b^4. \\ \text{Dividing by } a^2 - b^2, & \quad a^2x = 2(a^2 + b^2). \\ & \quad \therefore x = \frac{2(a^2 + b^2)}{a^2}. \end{aligned}$$

$$5. \quad \frac{x}{b} - \frac{x + 2b}{a} = \frac{a}{b} - 3.$$

$$\begin{aligned} \text{Clearing,} & \quad ax - bx - 2b^2 = a^2 - 3ab. \\ \text{Transposing,} & \quad ax - bx = a^2 - 3ab + 2b^2. \\ \text{Dividing by } a - b, & \quad x = a - 2b. \end{aligned}$$

$$6. \quad \frac{x - 2ab}{cx} - \frac{1}{x} = \frac{x - 3c}{abx}.$$

$$\begin{aligned} \text{Clearing,} & \quad abx - 2a^2b^2 - abc = cx - 3c^2. \\ \text{Transposing,} & \quad abx - cx = 2a^2b^2 + abc - 3c^2. \\ \text{Dividing by } ab - c, & \quad x = 2ab + 3c. \end{aligned}$$

$$7. \quad \frac{x - a}{b} + \frac{2x}{a} = 5 + \frac{6b}{a}.$$

$$\begin{aligned} \text{Clearing,} & \quad ax - a^2 + 2bx = 5ab + 6b^2. \\ \text{Transposing,} & \quad ax + 2bx = a^2 + 5ab + 6b^2. \\ \text{Dividing by } a + 2b, & \quad x = a + 3b. \end{aligned}$$

8.
$$\frac{a^2}{bx} + \frac{b^2}{ax} = \frac{a+b}{ab} - \frac{3(a+b)}{x}.$$

 Clearing,
 Dividing by $a+b$,
 Transposing, etc.,

$$\frac{a^3 + b^3}{a^2 - ab + b^2} = \frac{(a+b)x - 3ab(a+b)}{x - 3ab}.$$

$$x = a^2 + 2ab + b^2, \text{ or } (a+b)^2.$$

9.
$$\frac{a^2 + b^2}{2bx} - \frac{a-b}{2bx^2} = \frac{b}{x}.$$

 Clearing,
 Transposing, etc.,
 Dividing by $a^2 - b^2$,

$$\frac{a^2 + b^2}{2bx} - \frac{a-b}{2bx^2} = \frac{b}{x}.$$

$$(a^2 + b^2)x - (a-b) = 2b^2x.$$

$$(a^2 - b^2)x = a - b.$$

$$x = \frac{a-b}{a^2 - b^2} = \frac{1}{a+b}.$$

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10.
$$a + \frac{b(x+a)}{x-a} = \frac{2ab}{x-a}.$$

 Clearing,
 Transposing, etc.,
 Dividing by $a+b$,

$$ax - a^2 + bx + ab = 2ab.$$

$$ax + bx = a^2 + ab.$$

$$x = a.$$

11.
$$\frac{x-2a}{a} + \frac{x}{b} = \frac{a^2 + b^2}{ab}.$$

 Clearing,
 Transposing,
 Dividing by $a+b$,

$$bx - 2ab + ax = a^2 + b^2.$$

$$ax + bx = a^2 + 2ab + b^2.$$

$$x = a + b.$$

12.
$$6x + 18 \left(1 - \frac{a}{2}\right) = a(x-a).$$

 Expanding,
 Transposing, etc.,
 Dividing by $a-6$,

$$6x + 18 - 9a = ax - a^2.$$

$$ax - 6x = a^2 - 9a + 18.$$

$$x = a - 3.$$

13.
$$b(2x - 9c - 14b) = c(c-x).$$

 Expanding,
 Transposing,
 Dividing by $2b+c$,

$$2bx - 9bc - 14b^2 = c^2 - cx.$$

$$2bx + cx = 14b^2 + 9bc + c^2.$$

$$x = 7b + c.$$

14.
$$a(x-a-2b) + b(x-b) + c(x+c) = 0.$$

 Expanding,
 Transposing,
 Dividing by $a+b+c$,

$$ax - a^2 - 2ab + bx - b^2 + cx + c^2 = 0.$$

$$ax + bx + cx = a^2 + 2ab + b^2 - c^2.$$

$$x = a + b - c.$$

15.
$$(a-x)(x-b) + (a+x)(x-b) = (a-b)^2.$$

 Uniting terms,
 Expanding,
 Canceling $-2ab = -2ab$ and dividing by $2a$,

$$2a(x-b) = (a-b)^2.$$

$$2ax - 2ab = a^2 - 2ab + b^2.$$

$$x = \frac{a^2 + b^2}{2a}.$$

16.
$$\frac{a-b+c}{x+a} = \frac{b-a+c}{x-a}.$$

 Expanding, $ax - bx + cx - a^2 + ab - ac = bx - ax + cx + ab - a^2 + ac.$
 Canceling $cx - a^2 + ab = cx + ab - a^2$, transposing and uniting terms,

$$2ax - 2bx = 2ac.$$

 Dividing by $2(a-b)$,

$$x = \frac{ac}{a-b}.$$

$$17. \quad \frac{1}{a(b-x)} + \frac{1}{b(c-x)} - \frac{1}{a(c-x)} = 0.$$

$$\text{Clearing,} \quad b(c-x) + a(b-x) - b(b-x) = 0.$$

$$\text{Expanding,} \quad bc - bx + ab - ax - b^2 + bx = 0.$$

$$\text{Transposing, etc.,} \quad ax = ab - b^2 + bc = b(a - b + c).$$

$$\text{Dividing by } a, \quad x = \frac{b}{a}(a - b + c).$$

$$18. \quad \frac{x-1}{a-1} - \frac{a-1}{x-1} = \frac{x^2 - a^2}{(a-1)(x-1)}.$$

$$\text{Clearing,} \quad (x-1)^2 - (a-1)^2 = x^2 - a^2.$$

$$\text{Expanding, etc.,} \quad x^2 - 2x + 1 - a^2 + 2a - 1 = x^2 - a^2.$$

$$\text{Canceling } x^2 - a^2 = x^2 - a^2 \text{ and } 1 - 1 = 0, \text{ and transposing,}$$

$$-2x = -2a.$$

$$\therefore x = a.$$

$$19. \quad \frac{a+x}{a} - \frac{2x}{a+x} + \frac{x^2(x-a)}{a(a^2-x^2)} = \frac{1}{3}.$$

$$\frac{a+x}{a} - \frac{2x}{a+x} - \frac{x^2}{a(a+x)} = \frac{1}{3}.$$

Multiplying by $a(a+x)$,

$$a^2 + 2ax + x^2 - 2ax - x^2 = \frac{1}{3}a(a+x).$$

$$\text{Canceling and clearing,} \quad 3a^2 = a^2 + ax.$$

$$\therefore x = 2a.$$

$$20. \quad \frac{x+a}{b} + \frac{x+c}{a} + \frac{x+b}{c} = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1.$$

$$\frac{x}{b} + \frac{a}{b} + \frac{x}{a} + \frac{c}{a} + \frac{x}{c} + \frac{b}{c} = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1.$$

$$\text{Canceling,} \quad \frac{x}{b} + \frac{x}{a} + \frac{x}{c} = 1.$$

$$\text{Clearing,} \quad cax + bcx + abx = abc.$$

$$\text{Dividing by } ab + bc + ca, \quad x = \frac{abc}{ab + bc + ca}.$$

$$21. \quad \frac{x^2 - ax - bx + ab}{x-a} = \frac{x^2 - 2bx + 2b^2}{x-b} - \frac{c^2}{x-c}.$$

$$\text{Simplifying,} \quad x-b = x-b + \frac{b^2}{x-b} - \frac{c^2}{x-c}.$$

$$\text{Canceling and clearing,} \quad 0 = b^2x - b^2c - c^2x + bc^2.$$

$$\text{Transposing, etc.,} \quad b^2x - c^2x = b^2c - bc^2.$$

$$\text{Dividing by } b^2 - c^2, \quad x = \frac{bc(b+c)}{b^2 - c^2} = \frac{bc}{b+c}.$$

$$22. \quad \frac{1}{m+n} - \frac{2mn}{(m+n)^3} - \frac{m}{(m+n)^3} = \frac{x-n}{(m+n)^2}.$$

Clearing,

$$m^2 + 2mn + n^2 - 2mn - m^2 - mn = mx - mn + nx - n^2.$$

$$\text{Transposing, etc.,} \quad mx + nx = 2n^2.$$

$$\text{Dividing by } m+n, \quad x = \frac{2n^2}{m+n}.$$

$$23. \quad \frac{x}{a+b+c} + \frac{x}{a+b-c} = a^2 + b^2 + c^2 + 2ab.$$

Clearing,

$$(a+b-c+a+b+c)x = (a^2 + 2ab + b^2 + c^2)(a^2 + 2ab + b^2 - c^2).$$

$$2(a+b)x = (a+b)^4 - c^4.$$

Dividing by $2(a+b)$,

$$x = \frac{(a+b)^4 - c^4}{2(a+b)}.$$

$$24. \quad \frac{x+2b}{x-b} + \frac{x+3b}{x+b} = \frac{x+b}{x-3b} + \frac{x+2b}{x+5b}.$$

Uniting terms in each member,

$$\frac{x^2 + 3bx + 2b^2 + x^2 + 2bx - 3b^2}{(x-b)(x+b)} = \frac{x^2 + 6bx + 5b^2 + x^2 - bx - 6b^2}{(x-3b)(x+5b)},$$

or

$$\frac{2x^2 + 5bx - b^2}{x^2 - b^2} = \frac{2x^2 + 5bx - b^2}{x^2 + 2bx - 15b^2}.$$

Since the numerators are equal, the denominators are equal.

$$\therefore x^2 - b^2 = x^2 + 2bx - 15b^2;$$

whence,

$$x = 7b.$$

$$25. \quad \frac{2x+3a}{x+a} + \frac{3x+7a}{x+2a} = \frac{2a}{x+4a} + 5.$$

Reducing to mixed numbers,

$$2 + \frac{a}{x+a} + 3 + \frac{a}{x+2a} = \frac{2a}{x+4a} + 5.$$

Canceling $2+3=5$ and dividing by a ,

$$\frac{1}{x+a} + \frac{1}{x+2a} = \frac{2}{x+4a}.$$

Clearing, $x^2 + 6ax + 8a^2 + x^2 + 5ax + 4a^2 = 2x^2 + 6ax + 4a^2$.

Canceling and transposing, $5ax = -8a^2$.

$$\therefore x = -\frac{8a}{5}.$$

$$26. \quad \frac{x+7a}{x+6a} + \frac{x-a}{x-3a} = \frac{x+7a}{x+a} + \frac{x-a}{x+2a}.$$

Reducing to mixed numbers,

$$1 + \frac{a}{x+6a} + 1 + \frac{2a}{x-3a} = 1 + \frac{6a}{x+a} + 1 - \frac{3a}{x+2a}.$$

Canceling $1+1=1+1$ and dividing by a ,

$$\frac{1}{x+6a} + \frac{2}{x-3a} = \frac{6}{x+a} - \frac{3}{x+2a}.$$

Clearing, $(x-3a)(x+a)(x+2a) + 2(x+6a)(x+a)(x+2a)$
 $= 6(x+6a)(x-3a)(x+2a) - 3(x+6a)(x-3a)(x+a).$

Expanding, $x^3 - 7a^2x - 6a^3 + 2x^3 + 18ax^2 + 40a^2x + 24a^3$
 $= 6x^3 + 30ax^2 - 72a^2x - 216a^3 - 3x^3 - 12ax^2 + 45a^2x + 54a^3.$

Uniting terms, etc.,

$$60a^2x = -180a^3.$$

$$\therefore x = -3a.$$

$$27. \quad (a-b)(x-c) - (b-c)(x-a) = (c-a)(x-b).$$

Expanding, etc.,

$$ax - ac - bx + bc - bx + ab + cx - ac = cx - bc - ax + ab.$$

Canceling $ab + cx = cx + ab$, transposing, etc.,

$$2ax - 2bx = 2ac - 2bc.$$

Dividing by $2(a-b)$,

$$x = c.$$

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29. Let $x = \frac{1}{2}$ of first part, or $\frac{1}{3}$ of second, or $\frac{1}{5}$ of third.

Then, $2x =$ first part,

$3x =$ second part,

and $5x =$ third part ;

$$\therefore 2x + 3x + 5x = 40.$$

Solving, $x = 4$;

$2x = 8$, first part,

whence, $3x = 12$, second part,

and $5x = 20$, third part.

30. Let $x =$ first part divided by 5, or second multiplied by 2, or third increased by 5.

Then, $5x =$ first part,

$\frac{x}{2} =$ second part,

and $x - 5 =$ third part ;

$$\therefore 5x + \frac{x}{2} + x - 5 = 60.$$

Solving, $x = 10$;

whence, $5x = 50$, first part,

$\frac{x}{2} = 5$, second part,

and $x - 5 = 5$, third part.

31. Let $x =$ first part divided by 2, or second diminished by 2, or third multiplied by 2, or fourth increased by 2.

Then, $2x =$ first part,

$x + 2 =$ second part,

$\frac{x}{2} =$ third part,

and $x - 2 =$ fourth part ;

$$\therefore 2x + x + 2 + \frac{x}{2} + x - 2 = 72.$$

Solving, $x = 16$;

whence, $2x = 32$, first part,

$x + 2 = 18$, second part,

$\frac{x}{2} = 8$, third part,

and $x - 2 = 14$, fourth part.

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33. Let $x =$ number of days it will take both.

Then, $\frac{1}{x} =$ part of the work both can do in 1 day ;

$$\therefore \frac{1}{x} = \frac{1}{10} + \frac{1}{15}.$$

Solving, $x = 6$.

Hence, A and B together can do the work in 6 days.

34. Let x = number of hours it will take all.

Then, $\frac{1}{x}$ = part all can fill in 1 hour;

$$\therefore \frac{1}{x} = \frac{1}{5} + \frac{1}{6} + \frac{1}{10}.$$

Solving, $x = 2\frac{1}{2}$.

Hence, the three pipes together can fill the cistern in $2\frac{1}{2}$ hours.

35. Let x = number of days it will take all

Then, $\frac{1}{x}$ = part all can do in 1 day;

$$\therefore \frac{1}{x} = \frac{1}{10} + \frac{1}{12} + \frac{1}{8}.$$

Solving, $x = 3\frac{9}{7}$.

Hence, A, B, and C together can do the work in $3\frac{9}{7}$ days.

36. Let x = number of days in which A can finish the work after both have worked 3 days.

Then, $x + 3$ = number of days A works,

and $\frac{x+3}{6}$ = part of the work A does.

But since B does $\frac{3}{8}$ of the work, A must do $\frac{5}{8}$ of it;

$$\therefore \frac{x+3}{6} = \frac{5}{8}.$$

Solving, $x = \frac{3}{4}$.

Hence, after both have worked 3 days, A can finish the work in $\frac{3}{4}$ of a day.

37. Let $\frac{1}{x}$ = part of the wall C can build in 1 day.

Then, $\frac{2}{x}$ = part of the wall B can build in 1 day.

Since A can build $\frac{1}{15}$ of the wall in 1 day and all can build $\frac{1}{6}$ of it in 1 day, B and C can build $\frac{1}{6} - \frac{1}{15}$ of it in 1 day;

$$\therefore \frac{1}{x} + \frac{2}{x} = \frac{1}{6} - \frac{1}{15}.$$

Solving, $\frac{1}{x} = \frac{1}{30}$;

whence, $\frac{2}{x} = \frac{1}{15}$.

Hence, B can build $\frac{1}{15}$ of the wall in 1 day, or all of it in 15 days, and C can build $\frac{1}{30}$ of it in 1 day, or all of it in 30 days.

38. Let $\frac{1}{x}$ = part of the ditch all can dig in 1 day.

Then, in accordance with the suggestion in the text,

$$\frac{2}{x} = \frac{1}{10} + \frac{1}{6} + \frac{2}{15} = \frac{2}{5}.$$

$$\therefore \frac{1}{x} = \frac{1}{5}, \text{ part of the ditch all can dig in 1 day;}$$

whence, $\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$, part of the ditch A can dig in 1 day,

and $\frac{1}{5} - \frac{2}{10} = \frac{1}{10}$, part of the ditch B can dig in 1 day,

Hence, alone A can dig the ditch in 30 days, B in 15 days, and C in 10 days.

39. Let $\frac{1}{x}$ = part of the car all can load in 1 hour.

Then, since A and B can load $\frac{1}{30}$ of it in 1 hour, B and C $\frac{7}{24}$ of it in 1 hour, and A and C $\frac{13}{40}$ of it in 1 hour,

$$\frac{1}{30} + \frac{7}{24} + \frac{13}{40} = \text{twice the part all can load in 1 hour};$$

$$\therefore \frac{2}{x} = \frac{11}{30} + \frac{7}{24} + \frac{13}{40}.$$

Solving, $\frac{1}{x} = \frac{59}{120}$, part of the car all can load in 1 hour;

whence, $\frac{59}{120} - \frac{7}{24} = \frac{1}{5}$, part A can load in 1 hour,

$\frac{59}{120} - \frac{13}{40} = \frac{1}{6}$, part B can load in 1 hour,

and $\frac{59}{120} - \frac{11}{30} = \frac{1}{8}$, part C can load in 1 hour.

Hence, alone A can load the car in 5 hours, B in 6 hours, and C in 8 hours.

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40. Let

x = number of oranges.

Then,

$$\frac{3}{4}x \cdot 4 + \frac{1}{4}x \cdot 8 = x \cdot \frac{9}{2} + 90.$$

Solving,

$$x = 72.$$

Hence, the boy bought 72 oranges.

41. Let

x = numerator.

Then,

$x + 15$ = denominator;

$$\therefore \frac{x}{x+15} = \frac{4}{7}.$$

Solving,

$$x = 20;$$

whence,

$$x + 15 = 35.$$

Hence, the fraction is $\frac{20}{35}$.

42. Let

$2x$ = denominator.

Then,

$x + 3$ = numerator;

$$\therefore \frac{x+3}{2x} = \frac{2}{3}.$$

Solving,

$$x = 9;$$

whence, $2x = 18$ and $x + 3 = 12$.

Hence, the fraction is $\frac{12}{18}$.

43. Let

x = numerator.

Then,

$x + 8$ = denominator;

$$\therefore \frac{x-5}{x+8} = \frac{1}{3}.$$

Solving,

$$x = 9;$$

whence,

$$x + 8 = 17.$$

Hence, the fraction is $\frac{9}{17}$.

45. Let

x = digit in units' place.

Then,

$3x$ = digit in tens' place,

$30x + x$, or $31x$ = the number,

and

$2x$ = difference of digits;

$$\therefore \frac{31x - 33}{2x} = 10.$$

Solving,

$x = 3$, digit in units' place;

whence,

$3x = 9$, digit in tens' place.

Hence, the number is 93.

46. Let x = digit in tens' place.
 Then, $2x$ = digit in units' place,
 $10x + 2x$, or $12x$ = the number,
 and $3x$ = sum of the digits;
 $\therefore \frac{12x + 27}{3x} = 6\frac{1}{4}$.

Solving, $x = 4$, digit in tens' place;
 whence, $2x = 8$, digit in units' place.
 Hence, the number is 48.

47. Let $2x$ = number of 5-cent pieces.
 Then, x = number of quarters,
 and $\frac{4}{3}x$ = number of dimes;
 $\therefore 2x \cdot 5 + x \cdot 25 + \frac{4}{3}x \cdot 10 = 145$.

Solving, $x = 3$;
 whence, $2x = 6$ and $\frac{4}{3}x = 4$.
 Hence, there are 3 quarters, 6 5-cent pieces, and 4 dimes.

48. Let x = number of dollars she had at first.
 Then, $\frac{2}{3}x + 10$ = number of dollars she spent the first time,
 and $\frac{2}{3}x - 10$ = number of dollars left first time;
 also $\frac{2}{3}(\frac{2}{3}x - 10) + 10$ = number of dollars she spent the second time,
 and $\frac{2}{3}(\frac{2}{3}x - 10) - 10$ = number of dollars left second time;
 $\therefore \frac{2}{3}(\frac{2}{3}x - 10) - 10 = 2$.
 Solving, $x = 50$.
 Hence, the woman had \$50 at first.

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49. Let x = number of dollars he had at first.
 Then, $x - (\frac{2}{3}x - 1) = \frac{2}{3}x - 1$.
 Solving, $x = 30$.
 Hence, he had \$30 at first.

50. Let x = number of cents she had.
 Then, $\frac{x-5}{12}$ = number of cents 1 apple cost,
 and $\frac{x-6}{10}$ = number of cents 1 orange cost;
 $\therefore 6\left(\frac{x-5}{12} + \frac{x-6}{10}\right) = x - 2$.

Solving, $x = 41$.
 Hence, she had 41 cents.

51. Let x = number of cents he had at first.
 Then, $\frac{1}{2}x + \frac{1}{3}$ = number of cents spent first time.
 Subtracting, $\frac{1}{2}x - \frac{1}{3}$ = number of cents left first time.
 Then, $\frac{1}{4}x - \frac{1}{3} + \frac{1}{3}$ = number of cents spent second time.
 Subtracting, $\frac{1}{4}x - \frac{1}{4}$ = number of cents left second time.
 Then, $\frac{1}{8}x - \frac{3}{8} + \frac{1}{2}$ = number of cents spent third time.
 Subtracting, $\frac{1}{8}x - \frac{1}{8}$ = number of cents left third time;
 that is, $\frac{1}{8}x - \frac{1}{8} = 2$.
 Solving, $x = 23$.
 Hence, he had 23 cents at first.

52. Let $5x$ = number of dollars the boat cost.
 Then, x = number of dollars in each share,
 and $4x + 1$ = number of dollars four had to pay;
 $\therefore \frac{4x + 1}{4} = x + \frac{x}{12}.$

Canceling, $\frac{1}{4} = \frac{x}{12}.$

$\therefore x = 3;$

whence, $5x = 15$, the number of dollars the boat cost.

53. From the conditions of the problem it is seen that A received $\frac{1}{3}$ of the sum of money, B $\frac{1}{4}$ of it, C $\frac{1}{5}$ of it, and D \$2800 less than $\frac{1}{5}$ of it.

Consequently, let x = number of dollars in whole sum.

Then, $x = \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x + \frac{1}{5}x - 2800.$

Solving, $x = 24000;$

whence, $\frac{1}{3}x = 8000$, $\frac{1}{4}x = 6000$, $\frac{1}{5}x = 4800$ and $\frac{1}{5}x - 2800 = 5200.$

Hence, A received \$8000, B \$6000, C \$4800, and D \$5200.

54. Let x = number of ounces of copper to be added.

Then, $90 + x$ = number of ounces new alloy weighs.

Since the weight of the alloy must contain 10 ounces the same number of times that the total weight of silver in the new alloy, or 6 ounces, contains the weight of silver in each 10 ounces of the new alloy, or $\frac{2}{5}$ of an ounce,

$$\frac{90 + x}{10} = \frac{6}{\frac{2}{5}} = 15.$$

Solving, $x = 60.$

Hence, 60 ounces of copper must be added.

55. Let x = number of pounds of fresh water to be added.

Then, $80 + x$ = number of pounds of water after the addition of fresh water.

Since the number of times the total weight of water contains 45 pounds is the same as the number of times the total weight of salt contains $1\frac{2}{3}$ pounds,

$$\frac{80 + x}{45} = \frac{4}{1\frac{2}{3}} = \frac{12}{5}.$$

Solving, $x = 28.$

Hence, 28 pounds of fresh water must be added.

57. Let x = number of men on a side at first.

Then, x^2 = number of men before battle,

and $x^2 - 60$ = number of men after battle,

also $(x - 1)^2 + 1$ = number of men after battle;

$$\therefore x^2 - 60 = (x - 1)^2 + 1.$$

Solving, $x = 31;$

whence, $x^2 = 961.$

Hence, there were 961 men in the regiment at first.

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58. Let x = number of men on a side at first.

Then, x^2 = number of men at first,

and $x^2 + 240$ = number of men after arrival of reinforcements;

$$\therefore x^2 + 240 = (x + 4)^2.$$

Solving, $x = 28;$

whence, $x^2 = 784.$

Hence, there were 784 men in the regiment at first.

59. Let x = number of days he worked.
 Then, $40 - x$ = number of days he was idle;
 $\therefore 3x - 1.20(40 - x) = 57$.
 Expanding, $3x - 48 + 1.2x = 57$.
 $\therefore x = 25$.
 Hence, he worked 25 days.

61. Let x = number of dollars property is worth.
 Then, $\frac{4}{100} \cdot \frac{2}{3}x + \frac{3}{100} \cdot \frac{1}{4}x + \frac{2}{100} \cdot (x - \frac{2}{3}x - \frac{1}{4}x) = 860$.
 $\frac{2}{15}x + \frac{3}{400}x + \frac{1}{500}x = 860$.
 Clearing, $32x + 9x + 2x = 860 \cdot 1200$.
 $\therefore x = 20 \cdot 1200 = 24000$.
 Hence, his property is worth \$24000.

62. Let x = number of dollars in first investment.
 Then, $4330 - x$ = number of dollars in second investment;
 $\therefore .12x - .05(4330 - x) = 251$.
 Expanding, $.12x - 216.5 + .05x = 251$.
 $\therefore x = 2750$;
 whence, $4330 - x = 1580$.
 Hence, he invested \$2750 at a gain of 12% and \$1580 at a loss of 5%.

63. Let x = number of dollars each man received.
 Then, $7 - x$ = number of dollars each woman received;
 $\therefore 20x + 25(7 - x) = 160$.
 Solving, $x = 3$.
 Hence, the men received 20 times \$3, or \$60, and the women received \$160 - \$60, or \$100.

65. Let x = number of minute spaces the minute hand travels after 1 o'clock before they come together.
 Then, $\frac{x}{12}$ = number of minute spaces the hour hand travels in the same time.
 Since they are 5 minute spaces apart at 1 o'clock,
 $x - \frac{x}{12} = 5$.
 Solving, $x = 5\frac{5}{11}$, the number of minutes after 1 o'clock.

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66. Let x = number of minute spaces the minute hand travels after 6 o'clock before they come together.
 Then, $\frac{x}{12}$ = number of minute spaces the hour hand travels in the same time.
 Since the hour hand is 30 minute spaces in advance at 6 o'clock,
 $x - \frac{x}{12} = 30$.
 Solving, $x = 32\frac{8}{11}$, the number of minutes after 6 o'clock.

67. Let x = number of minute spaces the minute hand travels after 10 o'clock before the hands are opposite.

Then, $\frac{x}{12}$ = number of minute spaces the hour hand travels in the same time.

Since the hour hand is 50 minutes in advance at 10 o'clock, and 30 minutes in advance when the hands are opposite,

$$x - \frac{x}{12} = 50 - 30.$$

Solving, $x = 21\frac{3}{4}$, the number of minutes after 10 o'clock.

68. Let x = number of minute spaces the minute hand travels after 4 o'clock before it is 15 minute spaces *behind* the hour hand.

Then, $\frac{x}{12}$ = number of minute spaces the hour hand travels in the same time.

Since the minute hand gains $20 - 15$, or 5 minute spaces,

$$x - \frac{x}{12} = 5.$$

Solving, $x = 5\frac{5}{11}$, the number of minutes after 4 o'clock.

Again, let x = number of minute spaces the minute hand travels before it is 15 minute spaces *ahead* of the hour hand.

Then, $\frac{x}{12}$ = number of minute spaces the hour hand travels in the same time.

Since the minute hand gains $20 + 15$, or 35 minute spaces,

$$x - \frac{x}{12} = 35.$$

Solving, $x = 38\frac{2}{11}$, the number of minutes after 4 o'clock.

Hence, the required times are $4:05\frac{5}{11}$ o'clock and $4:38\frac{2}{11}$ o'clock.

69. Let x = number of minute spaces the minute hand travels after 9 o'clock before it is 15 minute spaces behind the hour hand.

Then, $\frac{x}{12}$ = number of minute spaces the hour hand travels in the same time.

Since the minute hand gains $45 - 15$, or 30 minutes,

$$x - \frac{x}{12} = 30.$$

Solving, $x = 32\frac{8}{11}$, the number of minutes after 9 o'clock.

70. Let x = number of miles downstream.

$\frac{x}{6}$ = number of hours downstream,

and $\frac{x}{3}$ = number of hours upstream;

$$\therefore \frac{x}{6} + \frac{x}{3} = 9.$$

Solving, $x = 18$.

Hence, he can row 18 miles downstream and return in 9 hours.

71. Let x = number of miles uphill.

Then, $\frac{x}{3}$ = number of hours uphill,

and $\frac{60 - x}{4}$ = number of hours downhill;

$$\therefore \frac{x}{3} + \frac{60 - x}{4} = 17.$$

Solving, $x = 24$.

Hence, 24 miles are uphill, and $60 - 24$, or 36 miles, downhill.

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73. Let $3x$ = number of leaps the hound takes.

Then, $5x$ = number of leaps the fox takes.

Suppose a = number of feet in 1 leap of the fox.

Then, $\frac{7a}{3}$ = number of feet in 1 leap of the hound,

$7ax$ = number of feet the hound runs,

and $5ax$ = number of feet the fox runs.

Since the fox has a start equal to $70a$ feet,

$$7ax - 5ax = 70a.$$

Solving, $x = 35$;

whence, $3x = 105$.

Hence, the hound must take 105 leaps to catch the fox.

74. Let $4x$ = number of leaps the dog takes.

Then, $5x$ = number of leaps the rabbit takes.

Suppose a = number of feet in 1 leap of the rabbit.

Then, $\frac{4a}{3}$ = number of feet in 1 leap of the dog,

$\frac{16ax}{3}$ = number of feet the dog runs,

and $5ax$ = number of feet the rabbit runs.

Since the rabbit has a start equal to $20a$ feet,

$$\frac{16ax}{3} - 5ax = 20a.$$

Solving, $x = 60$;

whence, $5x = 300$, and $4x = 240$.

Hence, the rabbit takes 300 leaps, and the dog takes 240 leaps

75. Let $8x$ = number of leaps the hound takes.

Then, $7x$ = number of leaps the rabbit takes.

Suppose a = number of feet in 1 leap of the hound.

Then, $\frac{5a}{6}$ = number of feet in 1 leap of the rabbit,

$8ax$ = number of feet the hound runs,

and $\frac{35ax}{6}$ = number of feet the rabbit runs.

Since the rabbit has a start equal to $39a$ feet,

$$8ax - \frac{35ax}{6} = 39a.$$

Solving, $x = 18$;

whence, $8x = 144$.

Hence, the hound must take 144 leaps to catch the rabbit.

76. Let x = number of miles per hour the pedestrian travels.

Then, $3x$ = number of miles per hour the wheelman travels.

Since each travels $6\frac{3}{4}$ hours and both together travel 108 miles,

$$\frac{27}{4}(x + 3x) = 108.$$

$$27x = 108.$$

$$\therefore x = 4;$$

whence, $3x = 12.$

Hence, the wheelman's rate is 12 miles an hour, the pedestrian's, 4 miles an hour.

77. Let x = number of pounds of lead.

Then, $159 - x$ = number of pounds of iron.

Since, when weighed in water, the lead loses $\frac{5}{7}x$ pounds, the iron $\frac{2}{15}(159 - x)$ pounds, and both together 16 pounds,

$$\frac{5x}{7} + \frac{2(159 - x)}{15} = 16.$$

Solving, $x = 114;$

whence, $159 - x = 45.$

Hence, there are 114 pounds of lead and 45 pounds of iron.

78. Let x = number of ounces of gold.

Then, $320 - x$ = number of ounces of silver.

Since in water 1 ounce of gold loses $\frac{5}{7}$ ounces in weight, and 1 ounce of silver $\frac{2}{11}$ ounces,

$$\frac{5}{7}x + \frac{2}{11}(320 - x) = 320 - 298.$$

Solving, $x = 194;$

whence, $320 - x = 126.$

Hence, there are 194 ounces of gold and 126 ounces of silver.

79. Let x = number of dollars in original capital.

Then, $\frac{5}{4}x - 800$ = number of dollars at end of 1st year,

$\frac{23}{10}x - 1000 - 800$ = number of dollars at end of 2d year,

$\frac{123}{64}x - 2250 - 800$ = number of dollars at end of 3d year;

$$\therefore \frac{123}{64}x - 2250 - 800 = 6325.$$

Solving, $x = 4800.$

Hence, his original capital was \$4800.

80. Let x = number of dollars in original capital.

Then, $\frac{6}{5}x - 1000$ = number of dollars at end of 1st year.

$\frac{36}{10}x - 1200 - 1000$ = number of dollars at end of 2d year,

and $\frac{216}{125}x - 2640 - 1000$ = number of dollars at end of 3d year;

$$\therefore \frac{216}{125}x - 2640 - 1000 = \frac{8}{5}x.$$

Solving, $x = 28437.50.$

Hence, his original capital was \$28437.50.

81. Let x = number of minutes it will take.

Then, $\frac{1}{x}$ = part all can fill in 1 minute.

Since the first pipe can fill $\frac{1}{20}$ of the cistern in 1 minute, the second $\frac{1}{15}$, and the third can empty $\frac{1}{10}$ of the cistern in 1 minute, the part all can fill in 1 minute is the algebraic sum of $\frac{1}{20}$, $\frac{1}{15}$ and $-\frac{1}{10}$;

$$\therefore \frac{1}{x} = \frac{1}{20} + \frac{1}{15} - \frac{1}{10}.$$

Solving, $x = 60.$

Hence, it will take 60 minutes, or 1 hour, to fill the cistern.

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82. Let x = number of miles from A to B.

Then, $\frac{x}{2}$ = number of hours occupied in going,

and $\frac{x}{3\frac{1}{5}}$ = number of hours occupied in returning;

$$\therefore \frac{x}{2} + \frac{5x}{16} = 13.$$

Solving, $x = 16$, number of miles from A to B.

83. Let x = number of miles an hour freight train runs.

Then, $\frac{8x}{3}$ = number of miles freight train runs,

and $\frac{3x}{60}$ of 40, or 64 = number of miles express train runs.

Since the trains run the same distance,

$$\frac{8x}{3} = 64.$$

$$\therefore x = 24.$$

Hence, the freight train runs 24 miles per hour.

84. Let x = number of miles from Albany.

Then, $\frac{x}{\frac{3}{2}}$ = number of hours it takes boat from Albany,

and $\frac{148 - x}{\frac{5}{4}}$ = number of hours it takes boat from Syracuse;

$$\frac{2x}{3} = \frac{4(148 - x)}{5}.$$

Solving, $x = 80\frac{8}{11}$.

Hence, the canal boats meet $80\frac{8}{11}$ miles from Albany.

85. Let $5x$ = number of miles an hour downstream.

Then, $3x$ = number of miles an hour upstream.

By the 2d condition,

$$5x - 4 = 2(3x - 4).$$

Solving, $x = 4$;

whence, $5x = 20$, number of miles an hour downstream,

and $3x = 12$, number of miles an hour upstream.

86. See next page.

87. Let x = number of days B worked.

Then, $10 - x$ = number of days A worked.

Since A could do $\frac{1}{6}$ of the work and B $\frac{1}{14}$ of it in 1 day, and since the whole work may be represented by $\frac{6}{6}$, or $\frac{14}{14}$, or 1,

$$\frac{10 - x}{6} + \frac{x}{14} = 1.$$

Solving, $x = 7$, number of days B worked.

88. Let x = number of pounds of gunpowder.

Then, $\frac{3}{4}x + 10$ = number of pounds of niter,

$\frac{1}{2}x + 3$ = number of pounds of sulphur,

and $\frac{3}{40}x + 1 - 3$ = number of pounds of charcoal;

$$\therefore x = \frac{3}{4}x + 10 + \frac{1}{2}x + 3 + \frac{3}{40}x + 1 - 3.$$

Solving, $x = 120$, number of pounds of gunpowder.

86. Let $\frac{1}{x}$ = part of the work A can do in 1 day.

Then, $\frac{1}{2}$ of $\frac{1}{x}$, or $\frac{1}{2x}$ = part of the work B can do in 1 day,

whence, $\frac{3}{4}$ of $\frac{1}{2x}$, or $\frac{3}{8x}$ = part of the work C can do in 1 day.

Since all together can do the work in $\frac{1}{5}$ days, or $\frac{5}{1}$ of it in 1 day,

$$\frac{5}{16} = \frac{1}{x} + \frac{1}{2x} + \frac{3}{8x}.$$

Since A can do $\frac{1}{x}$ of the work in 1 day, he can do $\frac{x}{x}$ of it, or all of it, in x days. Similarly, B can do it in $2x$ days, and C in $\frac{8}{3}x$ days.

Solving the equation, $x = 6$;

whence, $2x = 12$ and $\frac{8}{3}x = 16$.

Hence, alone A can do the work in 6 days, B in 12 days, and C in 16 days.

89. Let x = number of volumes of science.

Then, $2x$ = number of volumes of history,

whence, $3(2x - 500)$ = number of volumes of juvenile books;

also $\frac{1}{2}x$ = number of volumes of fiction,

whence, $\frac{\frac{1}{2}x + 500}{8}$ = number of volumes of reference;

$$\therefore 16000 = x + 2x + 3(2x - 500) + \frac{1}{2}x + \frac{\frac{1}{2}x + 500}{8}.$$

Simplifying, $279000 = 279x$;

$$\therefore x = 1000,$$

whence, $2x = 2000$, $3(2x - 500) = 4500$, $\frac{1}{2}x = 7500$ and $\frac{\frac{1}{2}x + 500}{8} = 1000$.

Hence, there were 1000 volumes in the department of science, 2000 in the department of history, 4500 in the juvenile department, 7500 in the department of fiction, and 1000 in the reference department.

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90. Let x = number of dollars the estate was worth.

Then, $\frac{x - 1000}{3}$ = number of dollars A received,

$\frac{x - 1000}{4}$ = number of dollars B received,

$\frac{\frac{2}{3}x}{5}$, or $\frac{6x}{25}$ = number of dollars C received,

and $\frac{\frac{1}{3}x}{6}$, or $\frac{x}{5}$ = number of dollars D received;

$$\therefore x = \frac{x - 1000}{3} + \frac{x - 1000}{4} + \frac{6x}{25} + \frac{x}{5}.$$

Solving, $x = 25000$;

whence, $\frac{x - 1000}{3} = 8000$, $\frac{x - 1000}{4} = 6000$, $\frac{6x}{25} = 6000$, and $\frac{x}{5} = 5000$.

Hence, A received \$8000, B \$6000, C \$6000, and D \$5000.

91. Let $3x$ = number of steps the father takes after son starts.
 Then, $5x$ = number of steps the son takes.
 Suppose a = number of feet in 1 step of father.
 Then, $\frac{2}{3}a$ = number of feet in 1 step of son.
 $3ax$ = number of feet the father goes after son starts,
 and $\frac{10}{3}ax$ = number of feet the son goes.
 Since the son goes $36a$ feet farther than the father,
 $\frac{10}{3}ax = 3ax + 36a$.
 Solving, $x = 108$;
 whence, $5x = 540$.

Hence, the son must take 540 steps to overtake the father.

92. Let x = number of dollars purse was worth.
 Then, $3x$ = number of dollars in money,
 and $8x$ = number of dollars ring was worth.
 Since the ring was worth \$10 more than the money,
 $8x - 3x = 10$.

Solving, $x = 2$;
 whence, $3x = 6$ and $8x = 16$.

Hence, the purse was worth \$2, and it contained \$6 in money and a ring worth \$16.

93. Since the whole mass of brass and iron occupies the same space as 7 pounds of water,

let x = number of pounds of water the brass displaces,
 whence, $7 - x$ = number of pounds of water the iron displaces.

Then, $\frac{4}{5}x + \frac{1}{2}(7 - x) = 57$.

Solving, $x = 5$;
 whence, $7 - x = 2$.

Hence, the mass contains $\frac{4}{5}$ times 5, or 42, pounds of brass, and $\frac{1}{2}$ times 2, or 15, pounds of iron.

94. Let x = number of dollars in annual expenses.

Then, $\frac{1}{3}(4725) - x$, or $6300 - x$ = number of dollars in his capital at the beginning of the 2d year,

$\frac{1}{3}(6300 - x) - x$, or $8400 - \frac{7}{3}x$ = number of dollars in his capital at the beginning of the 3d year,

and $\frac{1}{3}(8400 - \frac{7}{3}x) - x$, or $11200 - \frac{37}{9}x$ = number of dollars in his capital at the beginning of the 4th year, or at the end of the 3d year, expenses deducted;

$$\therefore 11200 - \frac{37}{9}x = 3800.$$

Solving, $x = 1800$.

Hence, his annual expenses were \$1800.

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96. See next page.

97. See next page.

98. Let x = one part.

Then, $b - x$ = the other part;

$$\therefore x = m(b - x).$$

Solving, $x = \frac{mb}{m + 1}$, the first part;

whence, $b - x = \frac{b}{m + 1}$, the other part.

Substituting 100 for b and 4 for m , the parts of 100 are 80 and 20.

96. Let x = number of cents first receives.

Then, $x - d$ = number of cents second receives;

$$\therefore x + x - d = c.$$

Solving, $x = \frac{c + d}{2}$, number of cents first receives;

whence, $x - d = \frac{c - d}{2}$, number of cents second receives.

If $c = 50$ and $d = 10$, the number of cents the first receives is $\frac{50 + 10}{2}$,
or 30, and the number of cents the second receives is $\frac{50 - 10}{2}$, or 20.

97. Let x = number of dollars saddle is worth.

Then, mx = number of dollars horse is worth;

$$\therefore mx + x = a.$$

Solving, $x = \frac{a}{m + 1}$, number of dollars saddle is worth;

whence, $mx = \frac{ma}{m + 1}$, number of dollars horse is worth.

If $a = 160$ and $m = 3$, $x = \frac{160}{3 + 1}$, or 40, and $mx = 40 \times 3 = 120$; that
is, the horse is worth \$120, and the saddle \$40.

99. Let mx = number of dollars the first receives.

Then, nx = number of dollars the second receives;

$$a = mx + nx.$$

Solving, $x = \frac{a}{m + n}$;

whence, $mx = \frac{ma}{m + n}$, number of dollars the first receives,

and $nx = \frac{na}{m + n}$, number of dollars the second receives.

If $a = 40000$, $m = 5$, and $n = 3$, the share of the first heir is $\frac{5 \times 40000}{5 + 3}$
dollars, or 25000 dollars, and the share of the second is $\frac{3 \times 40000}{5 + 3}$ dollars,
or 15000 dollars.

100. See next page.

101. Let mx = number of pounds of first.

Then, nx = number of pounds of second;

$$\therefore a = mx + nx.$$

Solving, $x = \frac{a}{m + n}$;

whence, $mx = \frac{ma}{m + n}$, number of pounds of first,

and $nx = \frac{na}{m + n}$, number of pounds of second.

Substituting 5 for m , 16 for n , and 4200 for a ,

$$\frac{ma}{m + n} = \frac{5 \times 4200}{21} = 1000, \text{ number of pounds of tin,}$$

$$\text{and } \frac{na}{m + n} = \frac{16 \times 4200}{21} = 3200, \text{ number of pounds of copper.}$$

100. Let x = number of days it will take both.

Then, $\frac{1}{x}$ = part both can do in 1 day;

$$\therefore \frac{1}{x} = \frac{1}{a} + \frac{1}{b}.$$

Solving, $x = \frac{ab}{a+b}$, number of days it will take both;

if $a=10$ and $b=15$, $x = \frac{1 \times 50}{25} = 6$.

102. Let x = number of hours the first rides.

Then, $x - a$ = number of hours the second rides,

rx = number of miles the first rides,

and $p(x - a)$ = number of miles the second rides.

Since the second wheelman must ride as far as the first,

$$p(x - a) = rx.$$

Solving, $x = \frac{ap}{p-r}$;

whence, $rx = \frac{ap}{p-r}$, number of miles each rides;

if $r = 10$, $p = 12$, and $a = 8$,

$$rx = \frac{8 \times 12 \times 10}{12 - 10} = 480, \text{ the number of miles.}$$

103. Let x = number of hours going.

Then, $h - x$ = number of hours returning,

ax = number of miles, going,

and $b(h - x)$ = number of miles, returning.

Since the distance was the same, going and returning,

$$ax = b(h - x).$$

Solving, $x = \frac{bh}{a+b}$; whence, $ax = \frac{abh}{a+b}$.

Hence, he went $\frac{abh}{a+b}$ miles from home; or, if $a = 4$, $b = 3\frac{1}{2}$, and $h = 15$.

he went $\frac{4 \times \frac{7}{2} \times 15}{4 + \frac{7}{2}}$ miles, or 28 miles from home.

SIMULTANEOUS SIMPLE EQUATIONS

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$$2. \quad \begin{cases} 7x - 5y = 52, & (1) \\ 2x + 5y = 47. & (2) \end{cases}$$

$$(1) + (2), \quad 9x = 99. \quad (2)$$

$$x = 11. \quad (3)$$

Substituting (3) in (2), $y = 5$.

$$3. \quad \begin{cases} 3x + 2y = 23, & (1) \\ x + y = 8. & (2) \end{cases}$$

$$(2) \times 2, \quad 2x + 2y = 16. \quad (3)$$

$$(1) - (3), \quad x = 7. \quad (4)$$

Substituting (4) in (2), $y = 1$.

$$4. \quad \begin{cases} 3x - 4y = 7, & (1) \\ x + 10y = 25. & (2) \end{cases}$$

$$(2) \times 3, \quad 3x + 30y = 75. \quad (3)$$

$$(3) - (1), \quad 34y = 68. \quad (4)$$

$$y = 2. \quad (4)$$

Substituting (4) in (1), $x = 5$.

$$5. \quad \begin{cases} 2x - 10y = 15, & (1) \\ 2x - 4y = 18. & (2) \end{cases}$$

$$(2) - (1), \quad 6y = 3. \quad (3)$$

$$y = \frac{1}{2}. \quad (3)$$

Substituting (3) in (1), $x = 10$.

$$\begin{aligned}
 6. \quad & \begin{cases} 3x - y = 4, & (1) \\ x + 3y = -2. & (2) \end{cases} \\
 (1) \times 3, & \quad 9x - 3y = 12. & (3) \\
 (2) + (3), & \quad 10x = 10. & (4) \\
 & \quad x = 1. & (4)
 \end{aligned}$$

Substituting (4) in (2), $y = -1$.

$$\begin{aligned}
 7. \quad & \begin{cases} 4x - y = 19, & (1) \\ x + 3y = 21. & (2) \end{cases} \\
 (1) \times 3, & \quad 12x - 3y = 57. & (3) \\
 (2) + (3), & \quad 13x = 78. & (4) \\
 & \quad x = 6. & (4)
 \end{aligned}$$

Substituting (4) in (2), $y = 5$.

$$\begin{aligned}
 8. \quad & \begin{cases} x + 2y = 5, & (1) \\ 2x + y = 1. & (2) \end{cases} \\
 \text{Adding, and dividing by 3,} & \quad x + y = 2. & (3) \\
 (2) - (3), & \quad x = -1. & (4) \\
 (1) - (3), & \quad y = 3. & (4)
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \begin{cases} 2x + 3y = 17, & (1) \\ 3x + 2y = 18. & (2) \end{cases} \\
 \text{Adding, and dividing by 5,} & \quad x + y = 7. & (3) \\
 (3) \times 2, & \quad 2x + 2y = 14. & (4) \\
 (2) - (4), & \quad x = 4. & (4) \\
 (1) - (4), & \quad y = 3. & (4)
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \begin{cases} 3x + 4y = 25, & (1) \\ 4x + 3y = 31. & (2) \end{cases} \\
 (1) + (2), & \quad 7x + 7y = 56. & (3) \\
 (3) \times \frac{3}{7}, & \quad 3x + 3y = 24. & (4) \\
 (2) - (4), & \quad x = 7. & (4) \\
 (1) - (4), & \quad y = 1. & (4)
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \begin{cases} 5x + 6y = 32, & (1) \\ 7x - 3y = 22. & (2) \end{cases} \\
 (2) \times 2, & \quad 14x - 6y = 44. & (3) \\
 (1) + (3), & \quad 19x = 76. & (4) \\
 & \quad x = 4. & (4) \\
 \text{Substituting (4) in (1),} & \quad y = 2. & (4)
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & \begin{cases} 3x + 6y = 39, & (1) \\ 9x - 4y = 51. & (2) \end{cases} \\
 (1) \times \frac{2}{3}, & \quad 2x + 4y = 26. & (3) \\
 (2) + (3), & \quad 11x = 77. & (4) \\
 & \quad x = 7. & (4) \\
 \text{Substituting (4) in (1),} & \quad y = 3. & (4)
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & \begin{cases} 7x - 9y = 6, & (1) \\ x + 2y = 14. & (2) \end{cases} \\
 (2) \times 7, & \quad 7x + 14y = 98. & (3) \\
 (3) - (1), & \quad 23y = 92. & (4) \\
 & \quad y = 4. & (4) \\
 \text{Substituting (4) in (2),} & \quad x = 6. & (4)
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \begin{cases} 13x - y = 20, & (1) \\ 4x + 2y = 20. & (2) \end{cases} \\
 (2) \div 2, & \quad 2x + y = 10. & (3) \\
 (1) + (3), & \quad 15x = 30. & (4) \\
 & \quad x = 2. & (4) \\
 \text{Substituting (4) in (3),} & \quad y = 6. & (4)
 \end{aligned}$$

15. See next page.

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$$\begin{aligned}
 17. \quad & \begin{cases} x + 14y = 38, & (1) \\ 14x + y = 142. & (2) \end{cases} \\
 (1) + (2), & \quad 15x + 15y = 180. & (3) \\
 (3) \times \frac{1}{15}, & \quad 14x + 14y = 168. & (4) \\
 (4) - (1), & \quad 13x = 130. & (4) \\
 & \quad x = 10. & (4) \\
 (4) - (2), & \quad 13y = 26. & (4) \\
 & \quad y = 2. & (4)
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & \begin{cases} 5x + y = 12, & (1) \\ x + 5y = 36. & (2) \end{cases} \\
 \text{Adding, and dividing by 6,} & \quad x + y = 8. & (3) \\
 (1) - (3), & \quad 4x = 4. & (4) \\
 & \quad x = 1. & (4) \\
 (2) - (3), & \quad 4y = 28. & (4) \\
 & \quad y = 7. & (4)
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & \begin{cases} 3x + 11y = 67, & (1) \\ 5x - 3y = 5. & (2) \end{cases} \\
 (1) \times 3, & \quad 9x + 33y = 201. & (3) \\
 (2) \times 11, & \quad 55x - 33y = 55. & (4) \\
 (3) + (4), & \quad 64x = 256. & (5) \\
 & \quad x = 4. & (5) \\
 \text{Substituting (5) in (1),} & \quad y = 5. & (5)
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & \begin{cases} \frac{x}{4} + \frac{y}{2} = 12, & (1) \\ \frac{x}{4} - \frac{y}{2} = -2. & (2) \end{cases} \\
 (1) + (2), & \quad \frac{x}{2} = 10. & (3) \\
 & \quad x = 20. & (3) \\
 (1) - (2), & \quad y = 14. & (3)
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & \begin{cases} \frac{x}{3} + \frac{y}{3} = 7, & (1) \\ \frac{x}{6} + \frac{y}{2} = 6\frac{1}{2}. & (2) \end{cases} \\
 (2) \times 2, & \quad \frac{x}{3} + y = 13. & (3) \\
 (3) - (1), & \quad \frac{2y}{3} = 6. & (4) \\
 & \quad y = 9. & (4) \\
 \text{Substituting (4) in (1),} & \quad x = 12. & (4)
 \end{aligned}$$

$$\begin{array}{ll}
 15. & \begin{cases} 8x - 3y = 44, & (1) \\ 7x - 5y = 29. & (2) \end{cases} \\
 (1) \times 5, & 40x - 15y = 220. & (3) \\
 (2) \times 3, & 21x - 15y = 87. & (4) \\
 (3) - (4), & 19x = 133. & \\
 & x = 7. & (5)
 \end{array}$$

Substituting (5) in (1), $y = 4$.

$$\begin{array}{ll}
 16. & \begin{cases} 6x - 5y = 33, & (1) \\ 4x + 4y = 44. & (2) \end{cases} \\
 (2) \div 4, & x + y = 11. & (3) \\
 (3) \times 5, & 5x + 5y = 55. & (4) \\
 (1) + (4), & 11x = 88. & \\
 & x = 8. & (5)
 \end{array}$$

Substituting (5) in (3), $y = 3$.

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2. See next column.

$$\begin{array}{ll}
 3. & \begin{cases} 5x + y = 22, & (1) \\ x + 5y = 14. & (2) \end{cases} \\
 \text{From (1),} & y = 22 - 5x. & (3) \\
 \text{From (2),} & y = \frac{14 - x}{5}. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } y, \\
 22 - 5x = \frac{14 - x}{5}.
 \end{array}$$

$$x = 4. \quad (5)$$

Substituting (5) in (3),
 $y = 2$.

$$\begin{array}{ll}
 4. & \begin{cases} 2x + 3y = 24, & (1) \\ 5x - 3y = 18. & (2) \end{cases} \\
 \text{From (1),} & 3y = 24 - 2x. & (3) \\
 \text{From (2),} & 3y = 5x - 18. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } 3y, \\
 24 - 2x = 5x - 18.
 \end{array}$$

$$x = 6. \quad (5)$$

Substituting (5) in (3),
 $y = 4$.

$$\begin{array}{ll}
 5. & \begin{cases} 3x + 5y = 14, & (1) \\ 2x - 3y = 3. & (2) \end{cases} \\
 \text{From (1),} & x = \frac{14 - 5y}{3}. & (3)
 \end{array}$$

$$\begin{array}{ll}
 \text{From (2),} & x = \frac{3y + 3}{2}. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } x, \\
 \frac{14 - 5y}{3} = \frac{3y + 3}{2}.
 \end{array}$$

$$y = 1. \quad (5)$$

Substituting (5) in (4),
 $x = 3$.

$$\begin{array}{ll}
 6. & \begin{cases} 3x + 2y = 36, & (1) \\ 5x - 9y = 23. & (2) \end{cases} \\
 \text{From (1),} & x = \frac{36 - 2y}{3}. & (3)
 \end{array}$$

$$\begin{array}{ll}
 \text{From (2),} & x = \frac{9y + 23}{5}. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } x, \\
 \frac{36 - 2y}{3} = \frac{9y + 23}{5}.
 \end{array}$$

$$y = 3. \quad (5)$$

Substituting (5) in (3),
 $x = 10$.

$$\begin{array}{ll}
 2. & \begin{cases} 3x - 2y = 10, & (1) \\ x + y = 70. & (2) \end{cases}
 \end{array}$$

$$\begin{array}{ll}
 \text{From (1),} & x = \frac{2y + 10}{3}. & (3)
 \end{array}$$

$$\begin{array}{ll}
 \text{From (2),} & x = 70 - y. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } x, \\
 \frac{2y + 10}{3} = 70 - y.
 \end{array}$$

$$y = 40. \quad (5)$$

Substituting (5) in (4),
 $x = 30$.

$$\begin{array}{ll}
 7. & \begin{cases} 2x + 7y = 8, & (1) \\ 3x + 9y = 9. & (2) \end{cases}
 \end{array}$$

$$\begin{array}{ll}
 \text{From (1),} & x = \frac{8 - 7y}{2}. & (3)
 \end{array}$$

$$\begin{array}{ll}
 \text{From (2),} & x = 3 - 3y. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } x, \\
 \frac{8 - 7y}{2} = 3 - 3y.
 \end{array}$$

$$y = 2. \quad (5)$$

Substituting (5) in (4),
 $x = -3$.

$$\begin{array}{ll}
 8. & \begin{cases} 4x + 6y = 19, & (1) \\ 3x - 2y = \frac{2}{3}. & (2) \end{cases}
 \end{array}$$

$$\begin{array}{ll}
 \text{From (1),} & y = \frac{19 - 4x}{6}. & (3)
 \end{array}$$

$$\begin{array}{ll}
 \text{From (2),} & y = \frac{6x - 9}{4}. & (4)
 \end{array}$$

$$\begin{array}{l}
 \text{Equating the values of } y, \\
 \frac{19 - 4x}{6} = \frac{6x - 9}{4}.
 \end{array}$$

$$x = \frac{5}{2}. \quad (5)$$

Substituting (5) in (3),
 $y = \frac{3}{2}$.

$$9. \quad \begin{cases} 4x + 3y = 34, & (1) \\ 11x + 5y = 87. & (2) \end{cases}$$

$$\text{From (1),} \quad y = \frac{34 - 4x}{3}. \quad (3)$$

$$\text{From (2),} \quad y = \frac{87 - 11x}{5}. \quad (4)$$

$$\text{Equating the values of } y, \\ \frac{34 - 4x}{3} = \frac{87 - 11x}{5}.$$

$$x = 7. \quad (5)$$

$$\text{Substituting (5) in (3),} \\ y = 2.$$

$$10. \quad \begin{cases} 4x - 13y = 5, & (1) \\ 3x + 11y = -17. & (2) \end{cases}$$

$$\text{From (1),} \quad x = \frac{13y + 5}{4}. \quad (3)$$

$$\text{From (2),} \quad x = \frac{-11y - 17}{3}. \quad (4)$$

$$\text{Equating the values of } x, \\ \frac{13y + 5}{4} = \frac{-11y - 17}{3}.$$

$$y = -1. \quad (5)$$

$$\text{Substituting (5) in (3),} \\ x = -2.$$

$$11. \quad \begin{cases} 18x - 3y = 4y, & (1) \\ 1 - 4x + 3y = 27. & (2) \end{cases}$$

$$\text{From (1),} \quad y = \frac{18x}{7}. \quad (3)$$

$$\text{From (2),} \quad y = \frac{4x + 26}{3}. \quad (4)$$

$$\text{Equating the values of } y, \\ \frac{18x}{7} = \frac{4x + 26}{3}.$$

$$x = 7. \quad (5)$$

$$\text{Substituting (5) in (3),} \\ y = 18.$$

$$12. \quad \begin{cases} 7y - x = 0, & (1) \\ x + 2y = 18. & (2) \end{cases}$$

$$\text{From (1),} \quad x = 7y. \quad (3)$$

$$\text{From (2),} \quad x = 18 - 2y. \quad (4)$$

Equating the values of x ,

$$7y = 18 - 2y. \quad (5)$$

$$\text{Substituting (5) in (3),} \\ x = 14.$$

$$13. \quad \begin{cases} 3y + 9 = 5x, & (1) \\ 16 - 2x = 5y. & (2) \end{cases}$$

$$\text{From (1),} \quad y = \frac{5x - 9}{3}. \quad (3)$$

$$\text{From (2),} \quad y = \frac{16 - 2x}{5}. \quad (4)$$

$$\text{Equating the values of } y, \\ \frac{5x - 9}{3} = \frac{16 - 2x}{5}.$$

$$x = 3. \quad (5)$$

$$\text{Substituting (5) in (4),} \\ y = 2.$$

$$14. \quad \begin{cases} 5x - 40 = y, & (1) \\ 5y - 60 = x. & (2) \end{cases}$$

$$\text{From (2),} \quad y = \frac{x}{5} + 12. \quad (3)$$

Equating the values of y ,

$$5x - 40 = \frac{x}{5} + 12. \quad (4)$$

$$x = 10\frac{2}{5}.$$

$$\text{Substituting (4) in (3),} \\ y = 14\frac{1}{5}.$$

$$15. \quad \begin{cases} 2y - 11x = 67, & (1) \\ 2x + 5y = 20. & (2) \end{cases}$$

$$\text{From (1),} \quad y = \frac{11x + 67}{2}. \quad (3)$$

$$\text{From (2),} \quad y = \frac{20 - 2x}{5}. \quad (4)$$

$$\text{Equating the values of } y, \\ \frac{11x + 67}{2} = \frac{20 - 2x}{5}.$$

$$x = -5. \quad (5)$$

$$\text{Substituting (5) in (4),} \\ y = 6.$$

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$$2. \quad \begin{cases} x - y = 4, & (1) \\ 4y - x = 14. & (2) \end{cases}$$

$$\text{From (1),} \quad x = y + 4. \quad (3)$$

$$\text{Substituting (3) in (2),} \\ 4y - (y + 4) = 14. \quad (4)$$

$$y = 6.$$

$$\text{Substituting (4) in (3),} \\ x = 10.$$

$$3. \quad \begin{cases} x + y = 10, & (1) \\ 6x - 7y = 34. & (2) \end{cases}$$

$$\text{From (1),} \quad x = 10 - y. \quad (3)$$

$$\text{Substituting (3) in (2),} \\ 6(10 - y) - 7y = 34. \quad (4)$$

$$y = 2.$$

$$\text{Substituting (4) in (3),} \\ x = 8.$$

$$\begin{cases} 3x - 4y = 26, & (1) \\ x - 8y = 22. & (2) \end{cases}$$

$$\text{From (2), } x = 8y + 22. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (1),} \\ 3(8y + 22) - 4y = 26. \end{aligned}$$

$$y = -2. \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ x = 6. \end{aligned}$$

$$\begin{cases} 6y - 10x = 14, & (1) \\ y - x = 3. & (2) \end{cases}$$

$$\text{From (2), } y = x + 3. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (1),} \\ 6(x + 3) - 10x = 14. \end{aligned}$$

$$x = 1. \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ y = 4. \end{aligned}$$

$$\begin{cases} y + 1 = 3x, & (1) \\ 5x + 9 = 3y. & (2) \end{cases}$$

$$\text{From (1), } y = 3x - 1. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (2),} \\ 5x + 9 = 9x - 3. \end{aligned}$$

$$x = 3. \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ y = 8. \end{aligned}$$

$$\begin{cases} 17 = 3x + z, & (1) \\ 7 = 3z - 2x. & (2) \end{cases}$$

$$\text{From (1), } z = 17 - 3x. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (2),} \\ 7 = 3(17 - 3x) - 2x. \end{aligned}$$

$$x = 4. \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ z = 5. \end{aligned}$$

$$\begin{cases} 4y = 10 - x, & (1) \\ y - x = 5. & (2) \end{cases}$$

$$\text{From (2), } y = x + 5. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (1),} \\ 4x + 20 = 10 - x. \end{aligned}$$

$$x = -2. \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ y = 3. \end{aligned}$$

$$\begin{cases} 7z - 3x = 18, & (1) \\ 2z - 5x = 1. & (2) \end{cases}$$

$$\text{From (2), } z = \frac{5x + 1}{2}. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (1),} \\ 7\left(\frac{5x + 1}{2}\right) - 3x = 18. \end{aligned}$$

$$\begin{aligned} 35x + 7 - 6x = 36. \\ x = 1. \end{aligned} \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ z = 3. \end{aligned}$$

$$\begin{cases} 3 - 15y = -x, & (1) \\ 3 + 15y = 4x. & (2) \end{cases}$$

$$\text{From (1), } 15y = x + 3. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (2),} \\ 3 + x + 3 = 4x. \end{aligned}$$

$$x = 2. \quad (4)$$

$$\begin{aligned} \text{Substituting (4) in (3),} \\ y = \frac{1}{3}. \end{aligned}$$

$$\begin{cases} 1 - x = 3y, & (1) \\ 3(1 - x) = 40 - y. & (2) \end{cases}$$

$$\begin{aligned} \text{Substituting } 3y \text{ for } 1 - x \text{ in (2),} \\ 9y = 40 - y. \end{aligned}$$

$$y = 4. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (1),} \\ 1 - x = 12. \\ x = -11. \end{aligned}$$

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$$\begin{cases} x + \frac{y}{3} = 11, & (1) \\ \frac{x}{3} + 3y = 21. & (2) \end{cases}$$

$$\text{From (2), } \frac{x}{3} = 21 - 3y. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) } \times 3 \text{ in (1),} \\ 63 - 9y + \frac{y}{3} = 11. \end{aligned}$$

$$\begin{aligned} -26y = -156. \\ y = 6. \end{aligned} \quad (4)$$

$$\text{Substituting (4) in (1), } x = 9.$$

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$$\begin{cases} \frac{x}{3} = 11 - \frac{y}{2}, & (1) \end{cases}$$

$$\begin{cases} \frac{x}{3} + \frac{2y}{7} = 8. & (2) \end{cases}$$

$$\begin{aligned} \text{Substituting (1) in (2),} \\ 11 - \frac{y}{2} + \frac{2y}{7} = 8. \end{aligned}$$

$$\begin{aligned} -\frac{3y}{14} = -3. \\ y = 14. \end{aligned} \quad (3)$$

$$\text{Substituting (3) in (1), } x = 12.$$

$$13. \quad \begin{cases} \frac{3x}{4} + \frac{4y}{5} = 21, & (1) \\ \frac{2x}{3} + \frac{3y}{5} = 17. & (2) \end{cases}$$

$$(1) \times 3, \quad \frac{9x}{4} + \frac{12y}{5} = 63. \quad (3)$$

$$(2) \times 4, \quad \frac{8x}{3} + \frac{12y}{5} = 68. \quad (4)$$

$$(4) - (3), \quad \frac{5x}{12} = 5. \quad (5)$$

$$\text{Substituting (5) in (2),} \quad x = 12. \quad (5)$$

$$\frac{3y}{5} = 9.$$

$$y = 15.$$

$$15. \quad \begin{cases} \frac{x}{8} + 4y = 15, & (1) \\ \frac{x}{6} + \frac{2y}{3} = 6. & (2) \end{cases}$$

$$(2) \times 6, \quad x + 4y = 36. \quad (3)$$

$$(3) - (1), \quad \frac{7x}{8} = 21. \quad (4)$$

$$x = 24.$$

$$\text{Substituting (4) in (3), } y = 3.$$

$$16. \quad \begin{cases} \frac{x-1}{4} + y = 3, & (1) \\ \frac{x-1}{4} + 4y = 9. & (2) \end{cases}$$

$$(2) - (1), \quad 3y = 6. \quad (3)$$

$$y = 2.$$

$$19. \quad \begin{cases} \frac{x+y}{2} - \frac{x-y}{3} = 8, & (1) \\ \frac{x+y}{3} + \frac{x-y}{4} = 11. & (2) \end{cases}$$

$$\text{Multiplying (1) by 2,} \quad x + y - \frac{2(x-y)}{3} = 16. \quad (3)$$

$$\text{Multiplying (2) by 3,} \quad x + y + \frac{3(x-y)}{4} = 33. \quad (4)$$

$$\text{Subtracting (3) from (4),} \quad \frac{17(x-y)}{12} = 17. \quad (5)$$

$$\text{Substituting (5) in (1),} \quad x - y = 12. \quad (5)$$

$$\text{Adding (5) and (6),} \quad \frac{x+y}{2} - 4 = 8. \quad (6)$$

$$\text{Subtracting (5) from (6),} \quad x + y = 24. \quad (6)$$

$$2x = 36.$$

$$x = 18.$$

$$2y = 12.$$

$$y = 6.$$

$$\text{Substituting (3) in (1),} \quad \frac{x-1}{4} + 2 = 3. \quad (1)$$

$$x = 5.$$

$$17. \quad \begin{cases} \frac{3x}{4} + \frac{2y}{3} = 20, & (1) \\ \frac{x}{2} + \frac{3y}{4} = 17. & (2) \end{cases}$$

$$(1) \times 2, \quad \frac{3x}{2} + \frac{4y}{3} = 40. \quad (3)$$

$$(2) \times 3, \quad \frac{3x}{2} + \frac{9y}{4} = 51. \quad (4)$$

$$(4) - (3), \quad \frac{11y}{12} = 11. \quad (5)$$

$$y = 12.$$

$$\text{Substituting (5) in (2),} \quad \frac{x}{2} + 9 = 17. \quad (5)$$

$$x = 16.$$

$$18. \quad \begin{cases} \frac{x}{3} = \frac{y}{2}, & (1) \\ \frac{x}{3} - \frac{y}{3} = 1. & (2) \end{cases}$$

$$\text{Substituting (1) in (2),} \quad \frac{y}{2} - \frac{y}{3} = 1. \quad (3)$$

$$\frac{y}{6} = 1.$$

$$y = 6. \quad (3)$$

$$\text{Substituting (3) in (1), } \frac{x}{3} = 3. \quad (4)$$

$$x = 9.$$

20.

$$\begin{cases} \frac{x}{2} - \frac{y}{3} - 1 = 0, & (1) \\ \frac{2x-1}{2} - \frac{3y-1}{3} = \frac{5}{6}. & (2) \end{cases}$$

Clearing (1) of fractions, etc.,
Simplifying (2),

$$3x - 2y = 6. \quad (3)$$

$$x - \frac{1}{2} - y + \frac{1}{3} = \frac{5}{6}. \quad (4)$$

Multiplying (4) by 2,
Subtracting (5) from (3),
Substituting (6) in (4),

$$2x - 2y = 2. \quad (5)$$

$$x = 4. \quad (6)$$

$$y = 3.$$

21.

$$\begin{cases} \frac{7+x}{5} - \frac{2x-y}{4} = 3y-5, & (1) \\ \frac{5y-7}{2} + \frac{4x-3}{6} = 18-5x. & (2) \end{cases}$$

Reducing (1),
Reducing (2),
Subtracting (3) from (4),
Multiplying (5) by $\frac{3}{4}$,
Multiplying (4) by 2,
Adding (6) and (7),

$$6x + 55y = 128. \quad (3)$$

$$34x + 15y = 132. \quad (4)$$

$$28x - 40y = 4. \quad (5)$$

$$21x - 30y = 3. \quad (6)$$

$$68x + 30y = 264. \quad (7)$$

$$89x = 267. \quad (8)$$

$$x = 3. \quad (9)$$

$$y = 2.$$

Substituting (8) in (4),

22.

$$\begin{cases} \frac{x}{2} - 12 = \frac{y}{4} + 8, & (1) \\ \frac{x+y}{5} + \frac{x}{3} = \frac{2y-x}{4} + 35. & (2) \end{cases}$$

Reducing (1),
Reducing (2),
Multiplying (3) by 18,
Subtracting (5) from (4),

$$2x - y = 80. \quad (3)$$

$$47x - 18y = 2100. \quad (4)$$

$$36x - 18y = 1440. \quad (5)$$

$$11x = 660. \quad (6)$$

$$x = 60. \quad (7)$$

$$y = 40.$$

Substituting (6) in (3),

23.

$$\begin{cases} \frac{1}{x-1} - \frac{3}{x+y} = 0, & (1) \\ \frac{3}{x-y} + 3 = 0. & (2) \end{cases}$$

Solving (1) for y ,
Solving (2) for y ,
Equating the values of y ,

$$y = 2x - 3. \quad (3)$$

$$y = x + 1. \quad (4)$$

$$2x - 3 = x + 1. \quad (5)$$

$$x = 4. \quad (6)$$

$$y = 5.$$

Substituting (5) in (4),

24.

$$\begin{cases} \frac{x}{2} - 12 = \frac{y+32}{4}, & (1) \\ \frac{y}{8} + \frac{3x-2y}{5} = 25. & (2) \end{cases}$$

Reducing (1),
Reducing (2),
Multiplying (3) by 12,
Subtracting (5) from (4),
Substituting (6) in (3),

$$2x - y = 80. \quad (3)$$

$$24x - 11y = 1000. \quad (4)$$

$$24x - 12y = 960. \quad (5)$$

$$y = 40. \quad (6)$$

$$x = 60.$$

$$25. \quad \begin{cases} \frac{.2y + .5}{1.5} = \frac{.49x - .7}{4.2}, & (1) \\ \frac{.5x - .2}{1.6} = \frac{.41}{16} - \frac{1.5y - 11}{8}. & (2) \end{cases}$$

Reducing the second member of (1) by dividing both terms by .7,

$$\frac{.2y + .5}{1.5} = \frac{.7x - 1}{6}. \quad (3)$$

Multiplying (3) by 6,

$$.8y + 2 = .7x - 1. \quad (4)$$

Multiplying (2) by 16,

$$7x - 8y = 30. \quad (5)$$

$$5x - 2 = 41 - 3y + 22.$$

$$5x + 3y = 65. \quad (6)$$

Multiplying (5) by 3,

$$21x - 24y = 90. \quad (7)$$

Multiplying (6) by 8,

$$40x + 24y = 520. \quad (8)$$

Adding (7) and (8),

$$61x = 610. \quad (9)$$

Substituting (9) in (6),

$$\begin{aligned} x &= 10. \\ y &= 5. \end{aligned}$$

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$$26. \quad \begin{cases} \frac{x+y}{5} + \frac{x-y}{5} = \frac{2x-y}{8} + \frac{4y-x}{5} + 1, & (1) \\ \frac{3x}{4} - \frac{y}{5} = y + 12. & (2) \end{cases}$$

Reducing (1),

$$14x - 27y = 40. \quad (3)$$

Reducing (2),

$$15x - 24y = 240. \quad (4)$$

Subtracting (3) from (4),

$$x + 3y = 200. \quad (5)$$

Multiplying (5) by 8,

$$8x + 24y = 1600. \quad (6)$$

Adding (4) and (6),

$$23x = 1840. \quad (7)$$

$$x = 80.$$

Substituting (7) in (5),

$$y = 40.$$

27.

$$\begin{cases} x + \frac{1}{2}(3x - y - 1) = \frac{1}{4} + \frac{3}{4}(y - 1), & (1) \\ \frac{1}{5}(4x + 3y) = \frac{1}{10}(7y + 24). & (2) \end{cases}$$

Reducing (1),

$$10x - 5y = 0. \quad (3)$$

Reducing (2),

$$8x - y = 24. \quad (4)$$

Substituting (3) in (4),

$$x = 4. \quad (5)$$

Substituting (5) in (3),

$$y = 8.$$

28. See next page.

$$29. \quad \begin{cases} \frac{3x - 5y}{3} - \frac{2x - 8y - 9}{12} = \frac{31}{12}, & (1) \\ \left(\frac{x}{7} + \frac{y}{4} + 1\frac{1}{3}\right) - \left(4x - \frac{y}{8} - 25\right) = \frac{5}{6}. & (2) \end{cases}$$

Reducing (1),

$$5x - 6y = 11. \quad (3)$$

Clearing (2) of fractions,

$$24x + 42y + 224 - 672x + 21y + 4200 = 140. \quad (4)$$

$$-648x + 63y = -4284. \quad (5)$$

Multiplying (4) by 2,

$$-1296x + 126y = -8568. \quad (6)$$

Multiplying (5) by 21,

$$105x - 126y = 231. \quad (7)$$

Adding (6) and (7),

$$-1191x = -8337. \quad (8)$$

$$x = 7.$$

Substituting (7) in (3),

$$y = 4.$$

28.

$$\begin{cases} \frac{6x+9}{4} + \frac{3x+5y}{4x-6} = 3\frac{1}{4} + \frac{3x+4}{2}, & (1) \end{cases}$$

$$\begin{cases} \frac{8y+7}{10} + \frac{6x-3y}{2(y-4)} = 4 + \frac{4y-9}{5}. & (2) \end{cases}$$

From (1),

$$\frac{6x}{4} + 2\frac{1}{4} + \frac{3x+5y}{4x-6} = 3\frac{1}{4} + \frac{3x}{2} + 2.$$

Canceling, etc.,

$$\frac{3x+5y}{4x-6} = 3. \quad (3)$$

Reducing (3),

$$9x-5y = 18. \quad (4)$$

From (2),

$$\frac{8y}{10} + \frac{7}{10} + \frac{6x-3y}{2(y-4)} = 4 + \frac{4y}{5} - \frac{9}{5}.$$

Canceling, etc.,

$$\frac{6x-3y}{2(y-4)} = \frac{3}{2}. \quad (5)$$

Reducing (5),

$$x = y - 2. \quad (6)$$

Substituting (6) in (4),

$$9y - 18 - 5y = 18. \quad (7)$$

Substituting (7) in (6),

$$\begin{aligned} y &= 9. \\ x &= 7. \end{aligned}$$

30.

$$\begin{cases} x - 20 - \frac{2y-x}{23-x} = \frac{2x-59}{2}, & (1) \end{cases}$$

$$\begin{cases} y - \frac{3-y}{x-18} - 30 = \frac{3y-73}{3}. & (2) \end{cases}$$

From (1), canceling $x - 20 = \frac{2x-40}{2}$,

$$-\frac{2y-x}{23-x} = -\frac{19}{2}.$$

$$17x + 4y = 437. \quad (3)$$

From (2), canceling $y - 30 = \frac{3y-90}{3}$,

$$-\frac{3-y}{x-18} = \frac{17}{3}.$$

$$17x - 3y = 297. \quad (4)$$

Subtracting (4) from (3),

$$7y = 140.$$

Substituting (5) in (4),

$$\begin{aligned} y &= 20. \\ x &= 21. \end{aligned} \quad (5)$$

31.

$$\begin{cases} \frac{3x+6}{7} - \frac{x+5}{3y-5} = \frac{6x-2}{14}, & (1) \end{cases}$$

$$\begin{cases} \frac{2y-3}{6} + \frac{3y+4}{5x-7} = \frac{3y+5}{9}. & (2) \end{cases}$$

Subtracting $\frac{3x+6}{7}$ from both members of (1), and $\frac{2y-3}{6}$ from both members of (2),

$$-\frac{x+5}{3y-5} = -1, \quad (3)$$

and

$$\frac{3y+4}{5x-7} = \frac{19}{18}. \quad (4)$$

Reducing (3),

$$x - 3y = -10. \quad (5)$$

Reducing (4),

$$95x - 54y = 205. \quad (6)$$

Multiplying (5) by 18,

$$18x - 54y = -180. \quad (7)$$

Subtracting (7) from (6),

$$77x = 385.$$

Substituting (8) in (5),

$$\begin{aligned} x &= 5. \\ y &= 5. \end{aligned} \quad (8)$$

$$32. \quad \begin{cases} \frac{x}{2} + \frac{16-x}{2} = 30 + \frac{5y+2x}{40-x}, & (1) \\ \frac{2y-3}{y+8} + \frac{83-8y}{8} = 10-y. & (2) \end{cases}$$

$$\text{From (1),} \quad \begin{aligned} \frac{x}{2} + 8 - \frac{x}{2} &= 30 + \frac{5y+2x}{40-x}, \\ \frac{5y+2x}{40-x} &= -22. \\ 4x - y &= 176. \end{aligned} \quad (3)$$

$$\text{From (2),} \quad \begin{aligned} \frac{2y-3}{y+8} + 10\frac{3}{8} - y &= 10-y, \\ \frac{2y-3}{y+8} &= -\frac{3}{8}, \\ 19y &= 0, \\ y &= 0, \\ x &= 44. \end{aligned} \quad (4)$$

Substituting (4) in (3),

$$33. \quad \begin{cases} 4x + \frac{2y-x}{17-3x} = 5 + \frac{16x-1}{4}, & (1) \\ 50 - \frac{y-1}{5x-10} = 8y + \frac{147-24y}{3}. & (2) \end{cases}$$

From (1), canceling and simplifying,

$$\frac{4y-x}{2(17-3x)} = \frac{19}{4}. \quad (3)$$

$$\text{Reducing (3),} \quad 55x + 8y = 323. \quad (4)$$

From (2), canceling $49 = \frac{147}{3}$ and $8y - \frac{24y}{3} = 0$, etc.,

$$1 - \frac{3y-3}{5x-10} = 0. \quad (5)$$

$$\text{Reducing (5),} \quad 5x - 3y = 7. \quad (6)$$

$$\text{Multiplying (6) by 11,} \quad 55x - 33y = 77. \quad (7)$$

$$\text{Subtracting (7) from (4),} \quad 41y = 246. \quad (8)$$

$$\text{Substituting (8) in (6),} \quad \begin{aligned} y &= 6, \\ x &= 5. \end{aligned}$$

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$$35. \quad \begin{cases} \frac{x}{2} - \frac{2y}{3} = -2, & (1) \\ \frac{5x}{2} + \frac{y}{3} = 12. & (2) \end{cases} \quad 36. \quad \begin{cases} \frac{x}{3} + \frac{y}{3} = 13, & (1) \\ \frac{4y}{3} - \frac{x}{3} = -8. & (2) \end{cases}$$

$$(2) \times 2, \quad 5x + \frac{2y}{3} = 24. \quad (3) \quad (1) + (2), \quad \frac{5y}{3} = 5.$$

$$(1) + (3), \quad \frac{11x}{2} = 22. \quad (4) \quad y = 3. \quad (3)$$

$$x = 4. \quad (4) \quad \text{Substituting (3) in (1),}$$

$$\text{Substituting (4) in (2),} \quad \frac{y}{3} = 12.$$

$$y = 6. \quad x = 36.$$

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| <p>37. $\begin{cases} \frac{5}{x} + \frac{6}{y} = 64, & (1) \\ \frac{6}{x} + \frac{5}{y} = 73\frac{1}{2}. & (2) \end{cases}$</p> <p>(1) + (2), $\frac{11}{x} + \frac{11}{y} = \frac{275}{2}. \quad (3)$</p> <p>(3) $\times \frac{5}{11}$, $\frac{5}{x} + \frac{5}{y} = \frac{125}{2}. \quad (4)$</p> <p>(2) - (4), $\frac{1}{x} = 11.$</p> <p>(1) - (4), $\frac{1}{y} = \frac{3}{2}.$</p> | <p>38. $\begin{cases} \frac{10}{x} + \frac{5}{y} = 20, & (1) \\ \frac{5}{x} + \frac{10}{y} = 57\frac{1}{2}. & (2) \end{cases}$</p> <p>(1) + (2), $\frac{15}{x} + \frac{15}{y} = \frac{155}{2}. \quad (3)$</p> <p>(3) $\div 3$, $\frac{5}{x} + \frac{5}{y} = \frac{155}{6}. \quad (4)$</p> <p>(1) - (4), $\frac{5}{x} = -\frac{35}{6}.$</p> <p>(2) - (4), $\frac{5}{y} = \frac{95}{3}.$</p> |
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|---|--|
| <p>39. $\begin{cases} \frac{5}{x} - \frac{3}{y} = -2, & (1) \\ \frac{25}{x} + \frac{1}{y} = 6. & (2) \end{cases}$</p> <p>(2) $\times 3$, $\frac{75}{x} + \frac{3}{y} = 18. \quad (3)$</p> <p>(1) + (3), $\frac{80}{x} = 16.$</p> <p>Substituting (4) in (2), $x = 5. \quad (4)$</p> | <p>Substituting (3) in (1), $\frac{15}{x} = 30.$</p> <p>$x = \frac{1}{2}. \quad (4)$</p> <p>Substituting (4) in (3), $y = \frac{1}{2}.$</p> |
| <p>40. $\begin{cases} \frac{2}{x} - \frac{3}{y} = 5, & (1) \\ \frac{5}{x} - \frac{2}{y} = 7. & (2) \end{cases}$</p> <p>(1) $\times 2$, $\frac{4}{x} - \frac{6}{y} = 10. \quad (3)$</p> <p>(2) $\times 3$, $\frac{15}{x} - \frac{6}{y} = 21. \quad (4)$</p> <p>(4) - (3), $\frac{11}{x} = 11.$</p> <p>$x = 1. \quad (5)$</p> <p>Substituting (5) in (1), $y = -1.$</p> | <p>41. $\begin{cases} \frac{4}{x} + \frac{3}{y} = \frac{9}{8}, & (1) \\ \frac{3}{x} + \frac{4}{y} = \frac{11}{12}. & (2) \end{cases}$</p> <p>(1) + (2), $\frac{7}{x} + \frac{7}{y} = \frac{49}{24}. \quad (3)$</p> <p>(3) $\times \frac{3}{7}$, $\frac{3}{x} + \frac{3}{y} = \frac{7}{8}. \quad (4)$</p> <p>(1) - (4), $\frac{1}{x} = \frac{1}{4}.$</p> <p>$x = 4.$</p> <p>(2) - (4), $\frac{1}{y} = \frac{1}{24}.$</p> <p>$y = 24.$</p> |
| <p>42. $\begin{cases} \frac{7}{x} + \frac{8}{y} = 30, & (1) \\ \frac{7}{x} + \frac{8}{y} = 30. & (2) \end{cases}$</p> <p>(2) - (1), $\frac{1}{x} - \frac{1}{y} = 0.$</p> <p>$y = x. \quad (3)$</p> | <p>43. $\begin{cases} \frac{3}{2x} - \frac{1}{y} = -3, & (1) \\ \frac{5}{2x} + \frac{3}{y} = 23. & (2) \end{cases}$</p> <p>(1) $\times 3$, $\frac{9}{2x} - \frac{3}{y} = -9. \quad (3)$</p> <p>(2) + (3), $\frac{7}{2x} = 14.$</p> <p>$x = \frac{1}{2}.$</p> <p>Substituting (4) in (1), $y = \frac{1}{8}.$</p> |

$$\begin{aligned}
 44. \quad & \begin{cases} \frac{7}{8}x - \frac{2}{3}y = 10, & (1) \\ \frac{5}{6}x - \frac{2}{11}y = 17. & (2) \end{cases} \\
 & (1) \times 3, \quad \frac{21}{8}x - \frac{2}{y} = 30. \quad (3) \\
 & (2) \times 11, \quad \frac{55}{6}x - \frac{2}{y} = 187. \quad (4) \\
 & (4) - (3), \quad \frac{55}{6}x - \frac{21}{8}x = 157.
 \end{aligned}$$

$$\begin{aligned}
 & \frac{157}{24}x = 157. \\
 & \frac{1}{x} = 24. \quad (5) \\
 & x = \frac{1}{24}. \\
 & \text{Substituting (5) in (3),} \\
 & -\frac{2}{y} = -33. \\
 & y = \frac{2}{33}.
 \end{aligned}$$

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$$\begin{aligned}
 2. \quad & \begin{cases} ax + by = m, & (1) \\ bx - ay = c. & (2) \end{cases} \\
 & (1) \times a, \quad a^2x + aby = am. \quad (3) \\
 & (2) \times b, \quad b^2x - aby = bc. \quad (4) \\
 & (3) + (4), \quad (a^2 + b^2)x = am + bc. \\
 & \quad \quad \quad x = \frac{am + bc}{a^2 + b^2}. \\
 & (1) \times b, \quad abx + b^2y = bm. \quad (5) \\
 & (2) \times a, \quad abx - a^2y = ac. \quad (6) \\
 & (5) - (6), \quad (a^2 + b^2)y = bm - ac. \\
 & \quad \quad \quad y = \frac{bm - ac}{a^2 + b^2}.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \begin{cases} ax - by = m, & (1) \\ cx - dy = r. & (2) \end{cases} \\
 & (1) \times d, \quad adx - bdy = dm. \quad (3) \\
 & (2) \times b, \quad bcx - bdy = br. \quad (4) \\
 & (3) - (4), \quad (ad - bc)x = dm - br. \\
 & \quad \quad \quad x = \frac{dm - br}{ad - bc}. \\
 & (1) \times c, \quad acx - bcy = cm. \quad (5) \\
 & (2) \times a, \quad acx - ady = ar. \quad (6) \\
 & (5) - (6), \quad (ad - bc)y = cm - ar. \\
 & \quad \quad \quad y = \frac{cm - ar}{ad - bc}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \begin{cases} ax = by, & (1) \\ x + y = ab. & (2) \end{cases} \\
 & (2) \times a, \quad ax + ay = a^2b. \quad (3) \\
 & \text{Substituting (1) in (3),} \\
 & \quad \quad \quad by + ay = a^2b. \\
 & \quad \quad \quad y = \frac{a^2b}{a + b}. \quad (4) \\
 & (4) \times b, \quad by = \frac{a^2b^2}{a + b}. \quad (5) \\
 & \text{Substituting (5) in (1),} \\
 & \quad \quad \quad ax = \frac{a^2b^2}{a + b}. \\
 & \quad \quad \quad x = \frac{ab^2}{a + b}.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \begin{cases} x - ay = n, & (1) \\ bx + y = p. & (2) \end{cases} \\
 & \text{From (1),} \quad x = ay + n. \quad (3) \\
 & \text{Substituting (3) in (2),} \\
 & \quad \quad \quad aby + bn + y = p. \\
 & \quad \quad \quad y = \frac{p - bn}{ab + 1}. \\
 & \text{From (2),} \quad y = p - bx. \quad (4) \\
 & \text{Substituting (4) in (1),} \\
 & \quad \quad \quad x - ap + abx = n. \\
 & \quad \quad \quad x = \frac{ap + n}{ab + 1}.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \begin{cases} a(x - y) = 5, & (1) \\ bx - cy = n. & (2) \end{cases} \\
 & (1) \times c, \quad acx - acy = 5c. \quad (3) \\
 & (2) \times a, \quad abx - acy = an. \quad (4) \\
 & (4) - (3), \quad (ab - ac)x = an - 5c. \\
 & \quad \quad \quad x = \frac{an - 5c}{ab - ac}. \\
 & (1) \times b, \quad abx - aby = 5b. \quad (5) \\
 & (2) \times a, \quad abx - acy = an. \quad (6) \\
 & (6) - (5), \quad (ab - ac)y = an - 5b. \\
 & \quad \quad \quad y = \frac{an - 5b}{ab - ac}.
 \end{aligned}$$

7. See next page.

$$\begin{aligned}
 8. \quad & \begin{cases} x + y = b - a, & (1) \\ bx - ay + 2ab = 0. & (2) \end{cases} \\
 & \text{From (1),} \quad y = b - a - x. \quad (3) \\
 & \text{Substituting (3) in (2),} \\
 & \quad \quad \quad bx - ab + a^2 + ax + 2ab = 0. \\
 & \quad \quad \quad x = \frac{-a^2 - ab}{a + b} = \frac{-a(a + b)}{a + b}, \\
 & \text{that is,} \quad x = -a. \quad (4) \\
 & \text{Substituting (4) in (1),} \\
 & \quad \quad \quad y = b.
 \end{aligned}$$

$$\begin{aligned} 7. \quad & \begin{cases} a(a-x) = b(y-b), & (1) \\ ax = by. & (2) \end{cases} \end{aligned}$$

$$\begin{aligned} (2)-(1), \quad & -a^2 + 2ax = b^2. \\ & x = \frac{a^2 + b^2}{2a}. \end{aligned}$$

$$\begin{aligned} (1) + (2), \quad & a^2 = 2by - b^2. \\ & y = \frac{a^2 + b^2}{2b}. \end{aligned}$$

$$9. \quad \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{1}{a}, & (1) \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{b}. & (2) \end{cases}$$

$$\begin{aligned} (1) + (2), \quad & \frac{2}{x} = \frac{1}{a} + \frac{1}{b} \\ & = \frac{a+b}{ab} \\ & x = \frac{2ab}{a+b}. \end{aligned}$$

$$\begin{aligned} (1) - (2), \quad & \frac{2}{y} = \frac{1}{a} - \frac{1}{b} \\ & = \frac{b-a}{ab} \\ & y = \frac{2ab}{b-a}. \end{aligned}$$

$$10. \quad \begin{cases} \frac{a}{x} - \frac{b}{y} = -1, & (1) \\ \frac{b}{x} - \frac{a}{y} = -1. & (2) \end{cases}$$

$$\begin{aligned} (1)-(2), \quad & \frac{a-b}{x} + \frac{a-b}{y} = 0. \\ & \frac{a-b}{y} = -\frac{a-b}{x}. \end{aligned}$$

$$\frac{y}{y} = -\frac{x}{x}. \quad (3)$$

Substituting (3) in (1),

$$\frac{a}{x} + \frac{b}{x} = -1.$$

$$\text{Substituting (4) in (3),}$$

$$y = a + b.$$

$$11. \quad \begin{cases} \frac{x}{a} + \frac{y}{b} = 2ab, & (1) \\ \frac{x}{ab} + \frac{y}{ab} = a + b. & (2) \end{cases}$$

$$(1) \div a, \quad \frac{x}{a^2} + \frac{y}{ab} = 2b. \quad (3)$$

$$\begin{aligned} (2)-(3), \quad & \left(\frac{1}{ab} - \frac{1}{a^2}\right)x = a - b. \\ & \left(\frac{a-b}{a^2b}\right)x = a - b. \end{aligned}$$

$$\begin{aligned} x &= a^2b. & (4) \\ \text{Substituting (4) in (1),} \end{aligned}$$

$$ab + \frac{y}{b} = 2ab.$$

$$y = ab^2.$$

12. See next page.

$$13. \quad \begin{cases} \frac{x}{a} + \frac{y}{b} = 1, & (1) \end{cases}$$

$$\begin{cases} \frac{x}{b} - \frac{y}{a} = \frac{1}{2} & (2) \end{cases}$$

$$(1) \div a, \quad \frac{x}{a^2} + \frac{y}{ab} = \frac{1}{a}. \quad (3)$$

$$(2) \div b, \quad \frac{x}{b^2} - \frac{y}{ab} = \frac{1}{2b}. \quad (4)$$

$$\begin{aligned} (3) + (4), \quad & \left(\frac{1}{b^2} + \frac{1}{a^2}\right)x = \frac{1}{2b} + \frac{1}{a}. \\ & \frac{a^2 + b^2}{a^2b^2}x = \frac{a + 2b}{2ab}. \\ & x = \frac{ab(a+2b)}{2(a^2 + b^2)}. \quad (5) \end{aligned}$$

Substituting (5) in (1),

$$\begin{aligned} & \frac{b(a+2b)}{2(a^2 + b^2)} + \frac{y}{b} = 1. \\ ab^2 + 2b^3 + 2(a^2 + b^2)y &= 2a^2b + 2b^3. \\ y = \frac{2a^2b - ab^2}{2(a^2 + b^2)} &= \frac{ab(2a - b)}{2(a^2 + b^2)}. \end{aligned}$$

$$14. \quad \begin{cases} \frac{1}{ax} + \frac{1}{by} = c, & (1) \end{cases}$$

$$\begin{cases} \frac{1}{bx} - \frac{1}{ay} = d. & (2) \end{cases}$$

$$(1) \times b, \quad \frac{b}{ax} + \frac{1}{y} = bc. \quad (3)$$

$$(2) \times a, \quad \frac{a}{bx} - \frac{1}{y} = ad. \quad (4)$$

$$(3) + (4), \quad \frac{a}{bx} + \frac{b}{ax} = ad + bc. \quad (5)$$

$$\text{From (5),} \quad \frac{a^2 + b^2}{a^2b} = \frac{ad + bc}{x}.$$

$$x = \frac{a^2b(ad + bc)}{a^2 + b^2}.$$

$$(1) \times a, \quad \frac{1}{x} + \frac{a}{by} = ac. \quad (6)$$

$$(2) \times b, \quad \frac{1}{x} - \frac{b}{ay} = bd. \quad (7)$$

$$(6) - (7), \quad \frac{a}{by} + \frac{b}{ay} = ac - bd. \quad (8)$$

$$\begin{aligned} \text{From (8),} \quad & \frac{a^2 + b^2}{a^2b} = \frac{aby(ac - bd)}{a^2 + b^2}. \\ & y = \frac{ab(ac - bd)}{a^2b(ac - bd)}. \end{aligned}$$

$$12. \quad \begin{cases} \frac{x}{a} + \frac{y}{b} - 2 = 0, & (1) \\ bx - ay = 0. & (2) \end{cases}$$

$$\text{From (1), } bx + ay = 2ab. \quad (3)$$

$$(3) + (2), \quad 2bx = 2ab. \quad (3)$$

$$(3) - (2), \quad 2ay = 2ab.$$

$$x = a.$$

$$y = b.$$

$$15. \quad \begin{cases} \frac{a}{x} + \frac{b}{y} = c, & (1) \\ \frac{m}{x} + \frac{n}{y} = e. & (2) \end{cases}$$

$$(1) \times n, \quad \frac{an}{x} + \frac{bn}{y} = cn. \quad (3)$$

$$(2) \times b, \quad \frac{bm}{x} + \frac{bn}{y} = be. \quad (4)$$

$$(3) - (4), \quad (an - bm) \frac{1}{x} = cn - be.$$

$$x = \frac{an - bm}{cn - be}.$$

$$(1) \times m, \quad \frac{am}{x} + \frac{bm}{y} = cm. \quad (5)$$

$$(2) \times a, \quad \frac{am}{x} + \frac{an}{y} = ae. \quad (6)$$

$$(6) - (5), \quad (an - bm) \frac{1}{y} = ae - cm.$$

$$y = \frac{an - bm}{ae - cm}.$$

$$16. \quad \begin{cases} \frac{x+1}{y+1} = \frac{a+b+1}{a-b+1}, & (1) \\ \frac{y+1}{x-y} = 2b. & (2) \end{cases}$$

$$\text{From (2), } x = y + 2b. \quad (3)$$

$$\text{Substituting (3) in (1),}$$

$$\frac{y+2b+1}{y+1} = \frac{a+b+1}{a-b+1}.$$

$$\text{Reducing to mixed numbers and canceling 1 from each member,}$$

$$\frac{2b}{y+1} = \frac{2b}{a-b+1}.$$

$$y+1 = a-b+1. \quad (4)$$

$$\text{Substituting (4) in (3),}$$

$$x = a + b.$$

$$17. \quad \begin{cases} \frac{x+y}{a} = \frac{x-y}{b}, & (1) \\ \frac{x-y}{a} = \frac{y-a}{b}. & (2) \end{cases}$$

$$(1) + (2), \quad \frac{2x}{a} = \frac{x-a}{b}.$$

$$2bx = ax - a^2.$$

$$x = \frac{a^2}{a-2b}. \quad (3)$$

$$\text{From (1), } bx + by = ax - ay.$$

$$y = \frac{a-b}{a+b} x. \quad (4)$$

$$\text{Substituting (3) in (4),}$$

$$y = \frac{a^2(a-b)}{(a+b)(a-2b)}.$$

$$18. \quad \begin{cases} \frac{1}{x-a} = \frac{1}{a-y}, & (1) \\ \frac{x+y}{x-y} = a. & (2) \end{cases}$$

$$\text{From (1), } x = 2a - y. \quad (3)$$

$$\text{Substituting (3) in (2),}$$

$$\frac{2a}{2a-2y} = a.$$

$$\frac{1}{a-y} = 1. \quad (4)$$

$$\text{Substituting (4) in (3),}$$

$$x = a + 1.$$

$$19. \quad \begin{cases} \frac{x}{a} + \frac{y}{b} = c, & (1) \\ \frac{x}{b} + \frac{y}{c} = d. & (2) \end{cases}$$

$$(1) \times b, \quad \frac{bx}{a} + y = bc. \quad (3)$$

$$(2) \times c, \quad \frac{cx}{b} + y = cd. \quad (4)$$

$$(3) - (4), \quad \frac{bx}{a} - \frac{cx}{b} = bc - cd.$$

$$(b^2 - ac)x = ab(bc - cd).$$

$$x = \frac{abc(b-d)}{b^2 - ac}.$$

$$(1) \times a, \quad x + \frac{ay}{b} = ac. \quad (5)$$

$$(2) \times b, \quad x + \frac{by}{c} = bd. \quad (6)$$

$$(6) - (5), \quad \frac{by}{c} - \frac{ay}{b} = bd - ac.$$

$$(b^2 - ac)y = bc(bd - ac).$$

$$y = \frac{bc(bd - ac)}{b^2 - ac}.$$

$$20. \begin{cases} \frac{a}{a+x} + \frac{b}{b-y} = \frac{a}{b}, & (1) \\ \frac{b}{a+x} + \frac{a}{b-y} = \frac{b}{a}. & (2) \end{cases}$$

Subtracting (2) from (1),

$$\frac{a-b}{a+x} + \frac{b-a}{b-y} = \frac{a^2-b^2}{ab}. \quad (3)$$

Dividing (3) by $a-b$,

$$\frac{1}{a+x} - \frac{1}{b-y} = \frac{a+b}{ab}. \quad (4)$$

Multiplying (4) by b ,

$$\frac{b}{a+x} - \frac{b}{b-y} = \frac{a+b}{a}. \quad (5)$$

$$(1) + (5), \quad \frac{a+b}{a+x} = \frac{a^2+ab+b^2}{ab}.$$

$$\begin{aligned} a^2b + ab^2 \\ = a^3 + a^2b + ab^2 + x(a^2 + ab + b^2). \\ x = \frac{-a^3}{a^2 + ab + b^2}. \end{aligned}$$

$$(2) - (5), \quad \frac{a+b}{b-y} = -1. \\ y = a + 2b.$$

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2. Let x = number of dollars received for each cow,
 and y = number of dollars received for each horse.
 Then, $3x + 7y = 600$,
 and $3x + 3y = 300$.
 Solving, $x = 25$, and $y = 75$.

Hence, he received \$25 each for cows and \$75 each for horses.

3. Let x = number of cents per yard for first kind,
 and y = number of cents per yard for second kind.
 Then, $20x + 50y = 3000$,
 and $30x + 20y = 2300$.
 Solving, $x = 50$, and $y = 40$.

Hence, the price of the first is 50 cents per yard, and the price of the second is 40 cents per yard.

4. Let x = number of cents each bushel of wheat costs
 and y = number of cents each bushel of rye costs.
 Then, $45x + 37y = 6270$,
 and $37x + 25y = 4830$.
 Solving, $x = 90$, and $y = 60$.

Hence, wheat costs 90 cents per bushel, and rye 60 cents per bushel.

5. Let x = number of apples,
 and y = number of oranges.
 Then, $4x + 5y = 95$,
 and $x + y = 22$.
 Solving, $x = 15$, number of apples,
 and $y = 7$, number of oranges.

6. Let x = number of years in A's age,
 and y = number of years in B's age.
 Then, $x - 5 = \frac{1}{3}(y - 5)$,
 and $x + 10 = \frac{1}{2}(y + 10)$.
 Solving, $x = 20$, and $y = 50$.

Hence, A's age is 20 years, and B's is 50 years.

7. Let x = number of years in A's age,
 and y = number of years in B's age.
 Then, $\frac{1}{2}x + 2y = 70$,
 and $\frac{3}{8}x + \frac{1}{3}y = 70$.
 Solving, $x = 20$, and $y = 30$.
 Hence, A's age was 20 years, and B's was 30 years.

8. Let x = number of cakes at 2 cents each,
 and y = number of cakes at 3 cents each.
 Then, $2x + 3y = 28$,
 and $x + y = 12$.
 Solving, $x = 8$, and $y = 4$.
 Hence, he can buy 8 cakes at 2 cents each and 4 at 3 cents each.

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10. Let x = number of dollars A has,
 and y = number of dollars B has.
 Then, $4(x - 100) = y + 100$,
 and $x + 200 = 4(y - 200)$.
 Solving, $x = 200$, and $y = 300$.
 Hence, A has \$200 and B has \$300.

11. Let x = number of cents A had,
 and y = number of cents B had.
 Then, $2(x + 20) = y - 20$,
 and $5(x - 25) = y + 25$.
 Solving, $x = 70$, and $y = 200$.
 Hence, A had \$.70 and B had \$2.

12. Let x = number of dollars A had,
 and y = number of dollars B had.
 Then, $x + 300 = 2y$,
 and $y - 300 = \frac{1}{3}x$.
 Solving, $x = 500$, and $y = 400$.
 Hence, A had \$500 and B had \$400.

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14. Let x = the numerator of the fraction,
 and y = the denominator.
 Then, $\frac{x+1}{y} = \frac{3}{4}$,
 and $\frac{x}{y+2} = \frac{1}{2}$.
 Solving, $x = 5$, and $y = 8$.
 Hence, the fraction is $\frac{5}{8}$.

15. Let x = the numerator of the fraction,
 and y = the denominator.
 Then, $\frac{x-2}{y-2} = \frac{1}{3}$,
 and $\frac{x+2}{y+2} = \frac{3}{5}$.
 Solving, $x = 4$, and $y = 8$.
 Hence, the fraction is $\frac{4}{8}$.

17. Let $x =$ digit in tens' place,
 and $y =$ digit in units' place,
 whence, $10x + y =$ the number,
 and $10y + x =$ the number with its digits reversed.
 Then, $x + y = 12$,
 and $10x + y - 18 = 10y + x$.
 Solving, $x = 7$,
 and $y = 5$.

Hence, the number is $70 + 5$, or 75.

18. Let $x =$ digit in tens' place,
 and $y =$ digit in units' place,
 whence, $10x + y =$ the number.
 Then, $\frac{10x + y}{x + y} = 8$,
 and $x - 3y = 1$.
 Solving, $x = 7$, and $y = 2$.

Hence, the number is $70 + 2$, or 72.

19. Let $x =$ digit in tens' place,
 and $y =$ digit in units' place,
 whence, $10x + y =$ the number,
 and $10y + x =$ the number with its digits reversed.
 Then, $x + y = 12$,
 and $10y + x = 2(10x + y) - 12$.
 Solving, $x = 4$, and $y = 8$.

Hence, the number is 48.

20. Let $x =$ number of acres at \$40 an acre,
 and $y =$ number of acres at \$15 an acre.
 Then, $x + y = 100$.
 and $40x + 15y = 3250$.
 Solving, $x = 70$, and $y = 30$.

Hence, there were 70 acres bought for \$40 an acre, and 30 acres bought for \$15 an acre.

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21. Let $x =$ number of 50-cent tickets,
 and $y =$ number of 35-cent tickets.
 Then, $x + y = 100$,
 and $50x + 35y = 3950$.
 Solving, $x = 30$, and $y = 70$.

Hence, 30 50-cent tickets and 70 35-cent tickets were sold.

22. Let $x =$ number of 25-cent pieces,
 and $y =$ number of 5-cent pieces.
 Then, $x + y = 80$,
 and $25x + 5y = 1600$.
 Solving, $x = 60$, and $y = 20$.

Hence, there were 60 25-cent pieces and 20 5-cent pieces.

23. Let x = number of apples,
 and y = number of oranges.
 Then, $3x + 5y = 100$,
 and $3 \cdot \frac{2}{5}x + 5 \cdot \frac{1}{4}y = 34$.
 Solving, $x = 20$, and $y = 8$.
 Hence, he bought 20 apples and 8 oranges.

24. Let x = number of days it will take A,
 and y = number of days it will take B.
 Then, $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$, (1)
 and $\frac{5}{x} + \frac{26}{y} = 1$. (2)

Multiplying (1) by 5 and subtracting the result from (2),

$$\frac{21}{y} = \frac{7}{12};$$

$$\therefore y = 36;$$

whence, substituting in (1), $x = 18$.

Hence, A can do the work in 18 days, and B in 36 days.

25. Let x = number of days in which the blacksmith could have
 done the work,
 and y = number of days in which his son could have done it.
 Then, $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$,
 and $\frac{8}{x} + \frac{3}{y} = 1$.
 Solving, $x = 10$, and $y = 15$.

Hence, the blacksmith could have done the work alone in 10 days, and
 his son could have done it in 15 days.

26. Let x = number of days in which the man can dig the ditch,
 and y = number of days in which either son can dig it.

Then, $\frac{1}{x} + \frac{2}{y} = \frac{1}{6}$,
 and $\frac{7}{x} + \frac{7}{y} + \frac{2}{y} = 1$.

Solving, $x = 10$, and $y = 30$.

Hence, the father can dig the ditch in 10 days, and either son can dig it
 in 30 days.

27. Let x = number of persons,
 and y = number of cents in share of each.
 Then, $75x = xy + 125$, (1)
 and $50x = xy - 250$. (2)
 Subtracting (2) from (1), $25x = 375$,
 whence, $x = 15$. (3)

Substituting (3) in the first member of (2),

$$xy = 1000.$$

Hence, there were 15 persons, and the whole expense of hiring the coach
 was 1000 cents, or \$10.

28. Let x = number of miles per hour the train ran,
 and y = number of hours it ran.
 Then, $(x + 5)(y - 2) = xy$, (1)
 and $(x - 5)(y + \frac{5}{2}) = xy$. (2)
 Reducing (1), $2x - 5y = -10$. (3)
 Reducing (2), $x - 2y = 5$. (4)
 Solving (3) and (4), $x = 45$, and $y = 20$.

Hence, the whole distance was 20 times 45 miles, or 900 miles, and the rate of the train was 45 miles an hour.

29. Let x = number of persons,
 and y = number of dollars in share of each.
 Then, $(x + 4)(y - 3) = xy$, (1)
 and $(x - 2)(y + 2) = xy$. (2)
 Reducing (1), $3x - 4y = -12$. (3)
 Reducing (2), $x - y = 2$. (4)
 Solving (3) and (4), $x = 20$, and $y = 18$.

Hence, there were 20 persons, and the share of each was \$18.

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30. Let x = number of cents eggs were worth per dozen,
 and y = number of cents potatoes were worth per bushel.
 Then, $6x = 2y + 10$,
 and $10x = 4y - 10$.
 Solving, $x = 15$, and $y = 40$.

Hence, eggs were worth 15 cents, and potatoes 40 cents.

31. Let x = number of feet in length,
 and y = number of feet in width.
 Then, $(x + 5)(y + 2) = xy + 140$, (1)
 and $(x + 10)(y + 7) = xy + 390$. (2)
 Reducing (1), $2x + 5y = 130$. (3)
 Reducing (2), $7x + 10y = 320$. (4)
 Solving (3) and (4), $x = 20$, and $y = 18$.

Hence, the floor is 20 feet long and 18 feet wide.

32. Let x = digit in tens' place,
 and y = digit in units' place.
 Then, $10x + y$ = the number,
 and $10y + x$ = the number with digits in reverse order;
 $\therefore 10x + y + 54 = 10y + x$, (1)
 and $x + y = 8$. (2)
 Reducing (1), $x - y = -6$. (3)
 From (2) and (3), $x = 1$, and $y = 7$. (4)

Hence, the number is 17.

33. Let $x =$ digit in tens' place,
 and $y =$ digit in units' place.
 Then, $10x + y =$ the number,
 and $10y + x =$ the number with digits in reverse order;
 $\therefore 10x + y + 13 = 5(x + y),$ (1)
 $10x + y + 36 = 10y + x.$ (2)
 Reducing (2), $x - y = -4.$ (3)
 Reducing (1), $5x - 4y, \text{ or } x + 4(x - y) = -13.$ (4)
 Substituting (3) in (4), $x = 3,$
 whence, from (3), $y = 7.$
 Hence, the number is 37.

34. Let $x =$ number of days it takes A,
 and $y =$ number of days it takes B.
 Then, $\frac{1}{x} + \frac{1}{y} = \frac{1}{a},$ (1)
 and $\frac{m}{x} + \frac{n}{y} = 1.$ (2)
 Multiplying (1) by $n,$ $\frac{n}{x} + \frac{n}{y} = \frac{n}{a}.$ (3)
 Subtracting (3) from (2), $\frac{m-n}{x} = \frac{a-n}{a},$
 whence, $x = \frac{a(m-n)}{a-n}.$
 Similarly, subtracting (2) from (1) $\times m,$
 $y = \frac{a(m-n)}{m-a}.$
 Hence, A can do the work in $\frac{a(m-n)}{a-n}$ days, and B in $\frac{a(m-n)}{m-a}$ days.

35. Let $x =$ number of days A must work,
 and $y =$ number of days B must work.
 Then, $\frac{x}{c} + \frac{y}{d} = 1,$ (1)
 and $x + y = a.$ (2)
 Clearing (1) of fractions, $dx + cy = cd.$ (3)
 Subtracting (3) from (2) $\times c,$ $(c-d)x = ac - cd.$
 $\therefore x = \frac{c(a-d)}{c-d}.$
 Subtracting (2) $\times d$ from (3), $(c-d)y = cd - ad.$
 $\therefore y = \frac{d(c-a)}{c-d}.$
 Hence, A must work $\frac{c(a-d)}{c-d}$ days, and B $\frac{d(c-a)}{c-d}$ days.

36. Let $x =$ number of gallons from 1st cask,
 and $y =$ number of gallons from 2d cask.
 Then, $\frac{2}{3}x + \frac{4}{9}y = 8,$
 and $x + y = 8 + 11.$
 Solving, $x = 10,$ and $y = 9.$
 Hence, 10 gallons must be drawn from the first cask, and 9 gallons from the second.

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37. Let x = number of oranges A had bought,
 and y = number of cents each had cost.
 Then, $(x + 3)(y - 1) = xy$, (1)
 and $(x - 2)(y + 1) = xy$. (2)
 Reducing (1), $3y - x = 3$. (3)
 Reducing (2), $x - 2y = 2$. (4)
 Adding (3) and (4), $y = 5$. (5)
 Substituting (5) in (4), $x = 12$.
 Hence, he had bought 12 oranges at 5 cents each.

38. Let x = number of cents 1 pound of first kind is worth,
 and y = number of cents 1 pound of second kind is worth.
 Then, considering two mixtures of 12 pounds,
 and $7x + 5y = 46 \times 12$, (1)
 $5x + 7y = 50 \times 12$. (2)
 $\frac{5}{12}$ of the sum of (1) and (2), $5x + 5y = 40 \times 12$. (3)
 Subtracting (3) from (1), $2x = 6 \times 12$.
 $\therefore x = 36$.
 Subtracting (3) from (2), $2y = 10 \times 12$.
 $\therefore y = 60$.

Hence, the first kind is worth 36 cents a pound, and the second kind 60 cents a pound.

39. Let x = number of dollars invested at 5%,
 and y = number of dollars invested at 4%.
 Then, $x + y = 4000$, (1)
 and $.05x + .04y = 175$. (2)
 Multiplying (2) by 25, $1.25x + y = 4375$. (3)
 Subtracting (1) from (3), $.25x = 375$.
 $\therefore x = 1500$. (4)
 Substituting (4) in (1), $y = 2500$.
 Hence, he invested \$1500 at 5% and \$2500 at 4%.

40. Let x = number of dollars invested at $r\%$,
 and y = number of dollars invested at $s\%$.
 Then, $x + y = a$, (1)
 and $\frac{rx}{100} + \frac{sy}{100} = b$. (2)
 Clearing (2) of fractions, $rx + sy = 100b$. (3)
 Multiplying (1) by s , $sx + sy = sa$. (4)
 Subtracting (4) from (3), $(r - s)x = 100b - sa$.
 $\therefore x = \frac{100b - sa}{r - s}$.
 Multiplying (1) by r , $rx + ry = ra$. (5)
 Subtracting (3) from (5), $(r - s)y = ra - 100b$.
 $\therefore y = \frac{ra - 100b}{r - s}$.

Hence, he invested $\frac{100b - sa}{r - s}$ dollars at $r\%$, and $\frac{ra - 100b}{r - s}$ dollars at $s\%$.

41. Let x = number of dollars in principal,
and y = number of hundredths indicated by rate.

Then, since the interest for 1 year is equal to the difference between the amount and the principal, divided by the number of years,

$$\frac{xy}{100} = \frac{b - x}{t}, \quad (1)$$

and

$$\frac{xy}{100} = \frac{a - x}{s}. \quad (2)$$

Eliminating y by comparison,

$$x = \frac{bs - at}{s - t}. \quad (3)$$

Substituting (3) in (1) or in (2),

$$y = \frac{100(a - b)}{bs - at}.$$

Hence, the principal was $\frac{bs - at}{s - t}$ dollars and the rate $\frac{100(a - b)}{bs - at}\%$.

42. Let x = number of dollars invested by A,
and y = number of hundredths indicated by A's rate.

Then, $(x - 100) \frac{y + 2}{100} = \frac{xy}{100} + 56, \quad (1)$

and $(x - 100) \frac{y + 1}{100} = \frac{xy}{100} + 25. \quad (2)$

Subtracting (2) from (1),
Clearing of fractions,

$$\frac{1}{100}(x - 100) = 31, \\ x - 100 = 3100. \quad (3)$$

Reducing (2),

$$x - 100 y = 2600. \quad (4)$$

Subtracting (4) from (3),

$$100 y = 600. \\ \therefore y = 6.$$

Hence, A invested \$3200 at 6%, and since B invested \$100 less at a rate 2% higher, B invested \$3100 at 8%.

43. Let x = number of miles an hour the crew can row in still water,
and y = number of miles an hour the stream runs.

Then, since the rate of the crew in still water is increased by that of the current when they row downstream, and decreased by that of the current when they row upstream, and since the time is equal to the distance divided by the rate,

$$\frac{8}{x + y} + \frac{8}{x - y} = \frac{3}{2}, \quad (1)$$

and

$$\frac{12}{x + y} + \frac{6}{x - y} = \frac{3}{2}. \quad (2)$$

Subtracting (1) from (2),

$$\frac{4}{x + y} - \frac{2}{x - y} = 0. \quad (3)$$

Clearing of fractions, $4x - 4y - 2x - 2y = 0.$

$$x = 3y. \quad (4)$$

Substituting (4) in (1),

$$\frac{6}{y} = \frac{3}{2}.$$

$$\therefore y = 4. \quad (5)$$

Substituting (5) in (4),

$$x = 12.$$

Hence, the rate of the crew in still water is 12 miles an hour, and the velocity of the stream is 4 miles an hour.

44. Let x = number of miles he can row in still water,
and y = number of miles an hour the stream runs.

Then, since his rate of rowing is increased when he rows downstream, and diminished when he rows upstream, by the velocity of the current, and since the time is equal to the distance divided by the rate,

$$\frac{15}{x+y} + \frac{15}{x-y} = 11, \quad (1)$$

and

$$\frac{8}{x+y} = \frac{3}{x-y}. \quad (2)$$

From (2),

Substituting (3) in (1),

$$y = \frac{5}{11}x. \quad (3)$$

whence, by (3),

$$x = \frac{55}{16}, \quad (4)$$

$$y = \frac{5}{48}.$$

Hence, the man's rate of rowing in still water is $3\frac{7}{16}$ miles an hour, and the velocity of the stream is $1\frac{5}{16}$ miles an hour.

45. Let x = number of quires of paper,
and y = number of bunches of envelopes.

Then, $\frac{1}{x}$ = part of box occupied by 1 quire of paper,

and $\frac{1}{y}$ = part of box occupied by 1 bunch of envelopes ;

$$\therefore \frac{18}{x} + \frac{18}{y} = 1,$$

and $\frac{20}{x} + \frac{15}{y} = 1.$

Solving, $x = 30$, and $y = 45$.

Hence, the box will hold 30 quires of paper or 45 bunches of envelopes.

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46. Let x = number of arithmetics it will hold,
and y = number of algebras it will hold.

Then, $\frac{1}{x}$ = part of space occupied by 1 arithmetic,

and $\frac{1}{y}$ = part of space occupied by 1 algebra ;

$$\therefore \frac{20}{x} + \frac{24}{y} = 1,$$

and $\frac{15}{x} + \frac{36}{y} = 1.$

Solving, $x = 30$, and $y = 72$.

Hence, the shelf will hold 30 arithmetics or 72 algebras.

47. Let x = number of hours the first rowed,
and y = number of hours the second rowed.

Then, since the first rowed $\frac{1}{10}$ of the distance for every hour he rowed, and the second $\frac{1}{14}$ of the distance for every hour he rowed,

$$\frac{x}{10} + \frac{y}{14} = 1,$$

and $x + y = 12.$

Solving, $x = 5$, and $y = 7$.

Hence, the first rowed 5 hours, the second 7 hours.

48. Let $5x$ = number of miles per hour the train ran at first.
 Then, $6x$ = number of miles per hour it ran after the delay.
 Also, let y = whole number of miles.

Then, since the train ran $\frac{2}{3}$ hours at $5x$ miles per hour,

$$y - 8x = \text{number of miles traveled after the delay};$$

$$\therefore \frac{y - 8x}{6x} = \text{number of hours required after the delay},$$

and $\frac{y - 8x}{5x}$ = number of hours that would have been required at the original rate to complete the journey.

Since the train was only 16 minutes late after being detained 40 minutes, it made up $40 - 16$, or 24, minutes, or $\frac{2}{5}$ of an hour, by increasing its rate;

$$\therefore \frac{y - 8x}{6x} = \frac{y - 8x}{5x} - \frac{2}{5}. \quad (1)$$

$$\text{Similarly,} \quad \frac{y - 8x - 10}{6x} = \frac{y - 8x - 10}{5x} - \frac{1}{3}. \quad (2)$$

$$\text{Subtracting (2) from (1),} \quad \frac{10}{6x} = \frac{10}{5x} - \frac{1}{15}. \quad (3)$$

$$\therefore x = 5; \text{ whence } 5x = 25. \quad (3)$$

Substituting (3) in (1),

$$y = 100$$

Hence, the train set out at the rate of 25 miles an hour, and traveled in all 100 miles.

49. Let x = number of persons,
 and y = number of cents each should pay.

$$\text{Then,} \quad ax = xy - b, \quad (1)$$

$$\text{and} \quad cx = xy + d. \quad (2)$$

$$\text{Subtracting (1) from (2),} \quad (c - a)x = b + d. \quad (3)$$

$$\therefore x = \frac{b + d}{c - a}. \quad (3)$$

$$\text{Multiplying (1) by } c, \quad acx = cxy - bc. \quad (4)$$

$$\text{Multiplying (2) by } a, \quad acx = axy + ad. \quad (5)$$

$$\text{By comparison,} \quad cxy - bc = axy + ad. \quad (6)$$

$$\text{Transposing, etc.,} \quad (c - a)xy = ad + bc. \quad (6)$$

$$\therefore y = \frac{ad + bc}{(c - a)x} = \frac{ad + bc}{b + d}. \quad (6)$$

Hence, there are $\frac{b + d}{c - a}$ persons, and each should pay $\frac{ad + bc}{b + d}$ cents.

50. Let x = number of persons expected,
 and y = number of dollars to be paid to each.

$$\text{Then,} \quad x + a = \text{actual number of persons,} \quad (1)$$

$$\text{and} \quad y - b = \text{actual number of dollars each received;} \quad (2)$$

further, under the last supposition, had there been $x - c$ persons, the share of each would have been $y + d$ dollars;

$$\therefore xy = (x + a)(y - b), \quad (1)$$

$$\text{and} \quad xy = (x - c)(y + d). \quad (2)$$

$$\text{Reducing (1),} \quad bx - ay = -ab. \quad (3)$$

$$\text{Reducing (2),} \quad dx - cy = cd. \quad (4)$$

$$\text{From (3),} \quad x + a = \frac{a}{b}y. \quad (5)$$

Solving (3) and (4) for y ,
$$y = \frac{bd(a+c)}{ad-bc}. \quad (6)$$

Substituting (6) in (5),
$$x + a = \frac{ad(a+c)}{ad-bc}. \quad (7)$$

Subtracting b from each member of (6),
$$y - b = \frac{bc(b+d)}{ad-bc}. \quad (8)$$

Hence, there were $\frac{ad(a+c)}{ad-bc}$ persons and each received $\frac{bc(b+d)}{ad-bc}$ dollars.

Substituting 5 for a , 100 for b , 4 for c , and 125 for d ,
by (7), number of persons = 25,
and by (8), number of dollars each received = 400.

51. Let x = number of days in which A can do the work,
and y = number of days in which B can do it.

Then,
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a}, \quad (1)$$

and
$$\frac{b}{x} + \frac{c}{y} = 1. \quad (2)$$

Subtracting (1) $\times c$ from (2),
$$\frac{b}{x} - \frac{c}{x} = 1 - \frac{c}{a}. \quad (3)$$

Dividing both members by $b - c$,
$$\frac{1}{x} = \frac{a - c}{a(b - c)}. \quad (4)$$

$$\therefore x = \frac{a(b - c)}{a - c}. \quad (5)$$

Substituting (4) in (1),
$$\frac{1}{y} = \frac{b - a}{a(b - c)}. \quad (6)$$

$$\therefore y = \frac{a(b - c)}{b - a}. \quad (7)$$

Substituting $5\frac{5}{11}$ for a , 5 for b , and 6 for c in (5) and (7),
 $x = 10$, and $y = 12$.

Hence, A can do the work in $\frac{a(b - c)}{a - c}$, or 10, days, and B can do it in $\frac{a(b - c)}{b - a}$, or 12, days.

52. Let x = number of crowns,
and y = number of guineas.

Then, $\frac{1}{x}$ = part of purse occupied by 1 crown,

and $\frac{1}{y}$ = part of purse occupied by 1 guinea;

$$\therefore \frac{c}{x} + \frac{d}{y} = 1, \quad (1)$$

and
$$\frac{a}{x} + \frac{b}{y} = \frac{m}{n}. \quad (2)$$

Multiplying (1) by b ,
$$\frac{bc}{x} + \frac{bd}{y} = b. \quad (3)$$

Multiplying (2) by d ,
$$\frac{ad}{x} + \frac{bd}{y} = \frac{dm}{n}. \quad (4)$$

Subtracting (3) from (4), $(ad - bc) \frac{1}{x} = \frac{dm - bn}{n}.$

$$\therefore x = \frac{n(ad - bc)}{dm - bn}. \quad (5)$$

Multiplying (1) by a , $\frac{ac}{x} + \frac{ad}{y} = a.$ (6)

Multiplying (2) by c , $\frac{ac}{x} + \frac{bc}{y} = \frac{cm}{n}.$ (7)

Subtracting (7) from (6), $(ad - bc) \frac{1}{y} = \frac{an - cm}{n}.$

$$\therefore y = \frac{n(ad - bc)}{an - cm}. \quad (8)$$

Substituting 12 for c , 6 for d , 4 for a , 6 for b , 1 for m , and 2 for n ,
 (5) becomes $x = 16$,
 and (8) becomes $y = 24$.

Hence, the purse will hold $\frac{n(ad - bc)}{dm - bn}$, or 16, crowns; or $\frac{n(ad - bc)}{an - cm}$, or 24, guineas.

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53. See next page.

54. Let x = number of miles per hour 1st train runs,
 and y = number of miles per hour 2d train runs.

Since they are m miles apart, and approach each other at the rate of $x + y$ miles per hour, and meet in b hours,

$$\frac{m}{x + y} = b. \quad (1)$$

Also, since the first train has traveled ax miles when the second starts, by the 2d condition,

$$\frac{m - ax}{x + y} = c. \quad (2)$$

Reducing (1), $bx + by = m.$ (3)

Reducing (2), $(a + c)x + cy = m.$ (4)

Subtracting (3) from (4), and transposing the terms containing y ,

$$(a - b + c)x = (b - c)y.$$

$$\therefore x = \frac{b - c}{a - b + c}y. \quad (5)$$

Substituting (5) in (3), $\frac{b^2 - bc}{a - b + c}y + \frac{ab - b^2 + bc}{a - b + c}y = m.$

$$\therefore y = \frac{m(a - b + c)}{ab}. \quad (6)$$

Substituting (6) in (5), $x = \frac{m(b - c)}{ab}. \quad (7)$

Substituting 800 for m , 9 for c , $1\frac{3}{5}$ for a , and 10 for b , in (7) and (6),
 $x = 50$, and $y = 30$.

Hence, the rate of the train from A is $\frac{m(b - c)}{ab}$, or 50, miles per hour.

and the rate of the train from B is $\frac{m(a - b + c)}{ab}$, or 30, miles per hour.

53. Let x = number of gallons per hour discharged by larger pump,
and y = number of gallons per hour discharged by smaller pump.

Then, $x + y = m,$ (1)

and $x = \frac{bm}{a}.$ (2)

Substituting (2) in (1), $y = \frac{m(a-b)}{a}.$ (3)

Substituting 5 for a , 4 for b , and 1250 for m ,
(2) becomes $x = 1000$,
and (3) becomes $y = 250$.

Hence, the larger pump discharges $\frac{bm}{a}$, or 1000, gallons per hour, and
the smaller discharges $\frac{m(a-b)}{a}$, or 250, gallons per hour.

55. Let x = number of cents better wine is worth per quart,
and y = number of cents poorer wine is worth per quart.

Then, $ax + by = (a+b)c,$ (1)

and $bx + ay = (a+b)d.$ (2)

Adding and dividing by $a+b$, $x + y = c + d.$ (3)

Subtracting (3) $\times b$ from (1), $(a-b)x = ac - bd.$ (3)
 $\therefore x = \frac{ac - bd}{a - b}.$ (4)

Subtracting (3) $\times b$ from (2), $(a-b)y = ad - bc.$
 $\therefore y = \frac{ad - bc}{a - b}.$ (5)

Substituting 40 for a , 20 for b , 100 for c , and 80 for d , in (4) and (5),
 $x = 120$, and $y = 60$.

Hence, the better wine is worth $\frac{ac - bd}{a - b}$, or 120, cents per quart, and the
poorer is worth $\frac{ad - bc}{a - b}$, or 60, cents per quart.

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| | | | | | |
|----------------------------------|--|-----|-----------------------------------|---|------|
| 2. | $\begin{cases} 2x - y + 2z = 12, \\ x + 3y + z = 41, \\ 2x + y + 4z = 22. \end{cases}$ | (1) | Substituting (5) in (6), $x = 5.$ | (7) | |
| | | (2) | Substituting (7) and (5) in (1), | | |
| | | (3) | $z = 12.$ | | |
| (2) $\times 2$, | $2x + 6y + 2z = 82.$ | (4) | | | |
| (4) - (1), | $7y = 70.$ | | | | |
| | $y = 10.$ | (5) | | | |
| (3) - (1), | $2y + 2z = 10.$ | (6) | | | |
| Substituting (5) in (6), | $z = -5.$ | (7) | | | |
| Substituting (5) and (7) in (2), | $x = 16.$ | | | | |
| | $\begin{cases} 3x + 5y - z = 8, \\ 4x + 3y + 2z = 47, \\ 6x + 5y - 2z = 11. \end{cases}$ | (1) | | | |
| | | (2) | | | |
| | | (3) | | | |
| (1) $\times 2$, | $6x + 10y - 2z = 16.$ | (4) | | | |
| (4) - (3), | $5y = 5.$ | | | | |
| | $y = 1.$ | (5) | | | |
| (2) + (3), | $10x + 8y = 58.$ | (6) | | | |
| | | | 4. | $\begin{cases} x + 3y - z = 10, \\ 2x + 5y + 4z = 57, \\ 3x - y + 2z = 15. \end{cases}$ | (1) |
| | | | | (2) | |
| | | | | (3) | |
| | | | (3) $\times 3$, | $9x - 3y + 6z = 45.$ | (4) |
| | | | (1) + (4), | $10x + 5z = 55.$ | |
| | | | | $2x + z = 11.$ | (5) |
| | | | (3) $\times 5$, | $15x - 5y + 10z = 75.$ | (6) |
| | | | (2) + (6), | $17x + 14z = 132.$ | (7) |
| | | | (5) $\times 14$, | $28x + 14z = 154.$ | (8) |
| | | | (8) - (7), | $11x = 22.$ | |
| | | | | $x = 2.$ | (9) |
| | | | Substituting (9) in (5), | $z = 7.$ | (10) |
| | | | Substituting (9) and (10) in (1), | $y = 5.$ | |

$$\begin{aligned}
 5. \quad & \begin{cases} x + y + z = 53, & (1) \\ x + 2y + 3z = 105, & (2) \\ x + 3y + 4z = 134. & (3) \end{cases} \\
 (2) - (1), & \quad y + 2z = 52. & (4) \\
 (3) - (2), & \quad y + z = 29. & (5) \\
 (4) - (5), & \quad z = 23. & (6) \\
 \text{Substituting (6) in (5), } & y = 6. & (7) \\
 \text{Substituting (7) and (6) in (1),} & x = 24.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \begin{cases} x - y + z = 30, & (1) \\ 3y - x - z = 12, & (2) \\ 7z - y + 2x = 141. & (3) \end{cases} \\
 (1) + (2), & \quad 2y = 42. & (4) \\
 & \quad y = 21.
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (4) in (1),} & \quad x + z = 51. & (5) \\
 (3) - (1), & \quad x + 6z = 111. & (6) \\
 (6) - (5), & \quad 5z = 60. & (7) \\
 & \quad z = 12. \\
 (5) - (7), & \quad x = 39.
 \end{aligned}$$

7. See next page.

$$\begin{aligned}
 8. \quad & \begin{cases} x + 3y + 4z = 83, & (1) \\ x + y + z = 29, & (2) \\ 6x + 8y + 3z = 156. & (3) \end{cases} \\
 (1) - (2), & \quad 2y + 3z = 54. & (4) \\
 (2) \times 6, & 6x + 6y + 6z = 174. & (5) \\
 (3) - (5), & \quad 2y - 3z = -18. & (6) \\
 (4) + (6), & \quad 4y = 36. & (7) \\
 & \quad y = 9. \\
 (4) - (6), & \quad 6z = 72. & (8) \\
 & \quad z = 12. \\
 (2) - (7) - (8), & \quad x = 8.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \begin{cases} 2x + 3y + 4z = 29, & (1) \\ 3x + 2y + 5z = 32, & (2) \\ 4x + 3y + 2z = 25. & (3) \end{cases} \\
 (3) - (2), & \quad x + y - 3z = -7. & (4) \\
 (1) + (3), & 6x + 6y + 6z = 54. & (5) \\
 & \quad x + y + z = 9. \\
 (5) - (4), & \quad 4z = 16. & (6) \\
 & \quad z = 4. \\
 (2) - (1), & \quad x - y + z = 3. & (7) \\
 (5) - (7), & \quad 2y = 6. & (8) \\
 & \quad y = 3. \\
 (5) - (6) - (8), & \quad x = 2.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \begin{cases} 2x - 3y + 4z - v = 4, & (1) \\ 4x + 2y - z + 2v = 13, & (2) \\ x - y + 2z + 3v = 17, & (3) \\ 3x + 2y - z + 4v = 20. & (4) \end{cases}
 \end{aligned}$$

To eliminate z from the given equations,

$$(2) - (4), \quad x - 2v = -7; \quad (5)$$

$$\begin{aligned}
 (3) \times 2, & \quad 2x - 2y + 4z + 6v = 34; & (6) \\
 (6) - (1), & \quad y + 7v = 30; & (7) \\
 (2) \times 2, & \quad 8x + 4y - 2z + 4v = 26; & (8) \\
 (3) + (8), & \quad 9x + 3y + 7v = 43. & (9)
 \end{aligned}$$

To eliminate y from the resulting equations, (5), (7), and (9),

$$\begin{aligned}
 (7) \times 3, & \quad 3y + 21v = 90; & (10) \\
 (9) - (10), & \quad 9x - 14v = -47. & (11) \\
 \text{To eliminate } x \text{ from the resulting} & \text{equations, (5) and (11),} \\
 (5) \times 9, & \quad 9x - 18v = -63. & (12) \\
 (11) - (12), & \quad 4v = 16. & (13) \\
 & \quad v = 4.
 \end{aligned}$$

$$\text{Substituting (13) in (5),} \quad x = 1. \quad (14)$$

$$\text{Substituting (13) in (7),} \quad y = 2. \quad (15)$$

$$\text{Substituting (13), (14), and (15) in (3),} \quad z = 3.$$

$$\begin{aligned}
 11. \quad & \begin{cases} 3x - 2y + z = 2, & (1) \\ 2x + 5y + 2z = 27, & (2) \\ x + 3y + 3z = 25. & (3) \end{cases} \\
 (1) + (2), & \quad 5x + 3y + 3z = 29. & (4) \\
 (4) - (3), & \quad 4x = 4. & (5) \\
 & \quad x = 1.
 \end{aligned}$$

$$\text{Substituting (5) in (1),} \quad -2y + z = -1. \quad (6)$$

$$\text{Substituting (5) in (3),} \quad 3y + 3z = 24.$$

$$\begin{aligned}
 (7) - (6), & \quad y + z = 8. & (7) \\
 & \quad 3y = 9. & (8) \\
 & \quad y = 3. \\
 (7) - (8), & \quad z = 5.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & \begin{cases} 4x - 5y + 3z = 14, & (1) \\ x + 7y - z = 13, & (2) \\ 2x + 5y + 5z = 36. & (3) \end{cases} \\
 (2) \times 2, & 2x + 14y - 2z = 26. & (4) \\
 (4) - (3), & \quad 9y - 7z = -10. & (5) \\
 (3) \times 2, & 4x + 10y + 10z = 72. & (6) \\
 (6) - (1), & \quad 15y + 7z = 58. & (7) \\
 (5) + (7), & \quad 24y = 48. & (8) \\
 & \quad y = 2.
 \end{aligned}$$

$$\text{Substituting (8) in (7),} \quad 7z = 28. \quad (9)$$

$$\text{Substituting (8) and (9) in (2),} \quad z = 4. \quad (9)$$

$$x = 3.$$

$$\begin{aligned}
 7. \quad & \begin{cases} 8x - 5y + 2z = 53, & (1) \\ x + y - z = 9, & (2) \\ 13x - 9y + 3z = 71. & (3) \end{cases} \\
 (2) \times 2, & 2x + 2y - 2z = 18. & (4) \\
 (1) + (4), & 10x - 3y = 71. & (5) \\
 (2) \times 3, & 3x + 3y - 3z = 27. & (6) \\
 (3) + (6), & 16x - 6y = 98. & (7) \\
 & 8x - 3y = 49. & (7) \\
 (5) - (7), & 2x = 22. & (8) \\
 & x = 11. & (9)
 \end{aligned}$$

Substituting (8) in (5), $y = 13$.
 Substituting (8) and (9) in (2),
 $z = 15$.

$$\begin{aligned}
 13. \quad & \begin{cases} 2x + y - 3z + 4w = 44, & (1) \\ 3x - 2y + z - w = -1, & (2) \\ 4x - y + 2z + w = 55, & (3) \\ 5x - 3y + 4z - w = 39. & (4) \end{cases}
 \end{aligned}$$

To eliminate w ,

$$\begin{aligned}
 (2) \times 4, & 12x - 8y + 4z - 4w = -4; & (5) \\
 (1) + (5), & 14x - 7y + z = 40; & (6) \\
 (2) + (3), & 7x - 3y + 3z = 54; & (7) \\
 (4) - (2), & 2x - y + 3z = 40. & (8)
 \end{aligned}$$

To eliminate x from (6), (7), and (8),

$$\begin{aligned}
 (7) \times 2, & 14x - 6y + 6z = 108; & (9) \\
 (9) - (6), & y + 5z = 68; & (10) \\
 (8) \times 7, & 14x - 7y + 21z = 280; & (11) \\
 (11) - (6), & 20z = 240. & (12) \\
 & z = 12. & (12)
 \end{aligned}$$

To eliminate z from (10) and (12),

$$\begin{aligned}
 \text{Substituting (12) in (10),} & y = 8. & (13)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (12) and (13) in (8),} & x = 6. & (14)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (14), (13), and (12) in (3),} & w = 15.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \begin{cases} 7x - 1 = 3y, & (1) \\ 11z - 1 = 7v, & (2) \\ 4z - 1 = 7y, & (3) \\ 19x - 1 = 3v. & (4) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (2) - (3), & 7z = 7(v - y). & (5) \\
 & z = v - y. & (5)
 \end{aligned}$$

$$\begin{aligned}
 (4) - (1), & 12x = 3v - 3y. & (6) \\
 & 4x = v - y. & (6)
 \end{aligned}$$

From (5) and (6), $z = 4x$.

$$\begin{aligned}
 (1) \times 7, & 49x - 7 = 21y. & (8)
 \end{aligned}$$

$$\begin{aligned}
 (2) \times 3, & 33z - 3 = 21v. & (9)
 \end{aligned}$$

Substituting (7) in (9),

$$\begin{aligned}
 132x - 3 = 21v. & (10)
 \end{aligned}$$

$$\begin{aligned}
 (10) - (8), & 83x + 4 = 21(v - y). & (11)
 \end{aligned}$$

Substituting (6) in (11),

$$\begin{aligned}
 83x + 4 = 21(4x). & (12) \\
 x = 4. & (12)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (12) in (7),} & z = 16.
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (12) in (1),} & y = 9.
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (12) in (4),} & v = 25.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \begin{cases} x + \frac{1}{2}y + \frac{1}{3}z = 32, & (1) \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15, & (2) \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12. & (3) \end{cases}
 \end{aligned}$$

Clearing of fractions,

$$\begin{aligned}
 6x + 3y + 2z = 192, & (4)
 \end{aligned}$$

$$\begin{aligned}
 20x + 15y + 12z = 900, & (5)
 \end{aligned}$$

$$\begin{aligned}
 15x + 12y + 10z = 720. & (6)
 \end{aligned}$$

$$\begin{aligned}
 (5) - (4), & 14x + 12y + 10z = 708. & (7)
 \end{aligned}$$

$$\begin{aligned}
 (6) - (7), & x = 12. & (8)
 \end{aligned}$$

Substituting (8) in (4) and (6),

$$\begin{aligned}
 3y + 2z = 120, & (9)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } 12y + 10z = 540. & (10)
 \end{aligned}$$

$$\begin{aligned}
 (9) \times 5, & 15y + 10z = 600. & (11)
 \end{aligned}$$

$$\begin{aligned}
 (11) - (10), & 3y = 60. & (12) \\
 & y = 20. & (12)
 \end{aligned}$$

Substituting (12) in (9), $z = 30$.

$$\begin{aligned}
 16. \quad & \begin{cases} \frac{1}{6}x - \frac{1}{5}y + \frac{1}{4}z = 3, & (1) \\ \frac{1}{5}x - \frac{1}{4}y + \frac{1}{3}z = 1, & (2) \\ \frac{1}{4}x - \frac{1}{3}y + \frac{1}{2}z = 5. & (3) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (1) \times 2, & \frac{1}{3}x - \frac{2}{5}y + \frac{1}{2}z = 6. & (4)
 \end{aligned}$$

$$\begin{aligned}
 (4) - (3), & \frac{1}{12}x - \frac{1}{15}y = 1. & (5)
 \end{aligned}$$

$$\begin{aligned}
 (2) \times \frac{5}{2}, & \frac{1}{2}x - \frac{5}{8}y + \frac{1}{2}z = \frac{5}{2}. & (6)
 \end{aligned}$$

$$\begin{aligned}
 (6) - (3), & \frac{1}{4}x - \frac{7}{24}y = \frac{5}{2}. & (7)
 \end{aligned}$$

$$\begin{aligned}
 (5) \times 3, & \frac{1}{4}x - \frac{1}{5}y = 3. & (8)
 \end{aligned}$$

$$\begin{aligned}
 (8) - (7), & \frac{1}{20}y = \frac{11}{2}. & (9) \\
 & y = 60. & (9)
 \end{aligned}$$

Substituting (9) in (8), $x = 60$.

Substituting (9) and (10) in (1),

$$\begin{aligned}
 z = 20.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & \begin{cases} \frac{x+y}{3} + 3z = 29, & (1) \\ \frac{2x-y}{2} + 2z = 22, & (2) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 & 3x - y = 3(z - 1). & (3)
 \end{aligned}$$

Reducing the given equations,

$$\begin{aligned}
 x + y + 9z = 87, & (4)
 \end{aligned}$$

$$\begin{aligned}
 2x - y + 4z = 44, & (5)
 \end{aligned}$$

$$\begin{aligned}
 3x - y - 3z = -3. & (6)
 \end{aligned}$$

$$\begin{aligned}
 (4) + (5), & 3x + 13z = 131. & (7)
 \end{aligned}$$

$$\begin{aligned}
 (6) - (5), & x - 7z = -47. & (8)
 \end{aligned}$$

$$\begin{aligned}
 (8) \times 3, & 3x - 21z = -141. & (9)
 \end{aligned}$$

$$\begin{aligned}
 (7) - (9), & 34z = 272. & (10) \\
 & z = 8. & (10)
 \end{aligned}$$

Substituting (10) in (8),

$$\begin{aligned}
 x = 9. & (11)
 \end{aligned}$$

Substituting (10) and (11) in (3),

$$\begin{aligned}
 y = 6.
 \end{aligned}$$

$$18. \begin{cases} 3x + y - z + 2v = 0, & (1) \\ 3y - 2x + z - 4v = 21, & (2) \\ x - y + 2z - 3v = 6, & (3) \\ 4x + 2y - 3z + v = 12. & (4) \end{cases}$$

To eliminate z ,

$$(1) + (2), \quad x + 4y - 2v = 21; \quad (5)$$

adding (2), (3), and (4),

$$3x + 4y - 6v = 39; \quad (6)$$

adding (1) $\times 2$ to (3),

$$7x + y + v = 6. \quad (7)$$

To eliminate y from (5), (6), and (7),

$$(6) - (5), \quad 2x - 4v = 18, \quad (8)$$

$$\text{or} \quad x - 2v = 9; \quad (8)$$

Subtracting (6) from (7) $\times 4$,

$$25x + 10v = -15, \quad (9)$$

$$\text{or} \quad 5x + 2v = -3. \quad (9)$$

$$(8) + (9), \quad 6x = 6. \quad (10)$$

$$x = 1. \quad (10)$$

Substituting (10) in (9),

$$v = -4. \quad (11)$$

Substituting (10) and (11) in (7),

$$y = 3. \quad (12)$$

Substituting (10), (11), and (12) in (2),

$$z = -2.$$

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$$20. \begin{cases} x + y = 9, & (1) \\ y + z = 7, & (2) \\ z + x = 5. & (3) \end{cases}$$

Adding the given equations,

$$2x + 2y + 2z = 21. \quad (4)$$

Subtracting (2), (3), and (1), successively, from (4),

$$x = \frac{1}{2}, y = \frac{1}{2}, \text{ and } z = \frac{3}{2}.$$

$$21. \begin{cases} v + x + y = 15, & (1) \\ x + y + z = 18, & (2) \\ y + z + v = 17, & (3) \\ z + v + x = 16. & (4) \end{cases}$$

Adding the given equations,

$$3v + 3x + 3y + 3z = 66. \quad (5)$$

$$v + x + y + z = 22. \quad (5)$$

Subtracting (2), (3), (4) and (1), successively, from (5),

$$v = 4, x = 5, y = 6, \text{ and } z = 7.$$

$$22. \begin{cases} \frac{1}{x} + \frac{1}{y} = 6, & (1) \\ \frac{1}{x} + \frac{1}{z} = 10, & (2) \\ \frac{1}{y} + \frac{1}{z} = 8. & (3) \end{cases}$$

Adding the given equations,

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 24. \quad (4)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 12. \quad (4)$$

Subtracting (2), (3), and (1), successively, from (4),

$$\frac{1}{x} = 2, \frac{1}{y} = 4, \text{ and } \frac{1}{z} = 6.$$

$$\therefore x = \frac{1}{2}, y = \frac{1}{4}, \text{ and } z = \frac{1}{6}.$$

$$23. \begin{cases} \frac{xy}{x+y} = \frac{1}{5}, & (1) \\ \frac{yz}{y+z} = \frac{1}{6}, & (2) \\ \frac{zx}{z+x} = \frac{1}{7}. & (3) \end{cases}$$

Taking the reciprocals of both members of each equation, Ax. 5,

$$\frac{1}{x} + \frac{1}{y} = 5, \quad (4)$$

$$\frac{1}{y} + \frac{1}{z} = 6, \quad (5)$$

$$\frac{1}{z} + \frac{1}{x} = 7. \quad (6)$$

Adding (4), (5), and (6),

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 18.$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9. \quad (7)$$

Subtracting (5), (6), and (4), successively from (7),

$$\frac{1}{x} = 3, \frac{1}{y} = 2, \text{ and } \frac{1}{z} = 4.$$

$$\therefore x = \frac{1}{3}, y = \frac{1}{2}, \text{ and } z = \frac{1}{4}.$$

$$24. \begin{cases} x + 3y + z = 14, & (1) \\ x + y + 3z = 16, & (2) \\ 3x + y + z = 20. & (3) \end{cases}$$

Adding the given equations,

$$5x + 5y + 5z = 50. \quad (4)$$

$$x + y + z = 10. \quad (4)$$

Subtracting (4) from (3), (1), and (2) in succession,

$$2x = 10, 2y = 4, \text{ and } 2z = 6.$$

$$\therefore x = 5, y = 2, \text{ and } z = 3.$$

$$25. \begin{cases} y + z + v - x = 22, & (1) \\ z + v + x - y = 18, & (2) \\ v + x + y - z = 14, & (3) \\ x + y + z - v = 10. & (4) \end{cases}$$

Adding the given equations,

$$2v + 2x + 2y + 2z = 64.$$

$$v + x + y + z = 32. \quad (5)$$

Subtracting (4), (1), (2), and (3), in succession, from (5),

$$2v = 22, 2x = 10, 2y = 14 \text{ and } 2z = 18.$$

$$\therefore v = 11, x = 5, y = 7, \text{ and } z = 9.$$

$$26. \begin{cases} \frac{1}{x} + \frac{1}{y} - 1 = 0, & (1) \\ \frac{1}{x} + \frac{1}{y} + 3 = 0, & (2) \\ \frac{y}{z} + \frac{1}{x} - 2 = 0. & (3) \end{cases}$$

Adding the given equations,

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 0.$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0. \quad (4)$$

Subtracting (2), (3), and (1), in succession, from (4),

$$\frac{1}{x} - 3 = 0, \quad (5)$$

$$\frac{1}{y} + 2 = 0, \quad (6)$$

$$\text{and} \quad \frac{1}{z} + 1 = 0. \quad (7)$$

From (5), (6), and (7),

$$x = \frac{1}{3}, y = -\frac{1}{2}, \text{ and } z = -1.$$

$$27. \begin{cases} \frac{xy}{x+y} = \frac{1}{a}, & (1) \\ \frac{yz}{y+z} = \frac{1}{b}, & (2) \\ \frac{zx}{z+x} = \frac{1}{c}. & (3) \end{cases}$$

Taking the reciprocals of both members of each equation, Ax. 5,

$$\frac{1}{y} + \frac{1}{x} = a, \quad (4)$$

$$\frac{1}{z} + \frac{1}{y} = b, \quad (5)$$

$$\frac{1}{x} + \frac{1}{z} = c \quad (6)$$

Adding (4), (5), and (6),

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = a + b + c. \quad (7)$$

Subtracting (5), (6), and (4), each multiplied by 2, in succession from (7),

$$\frac{2}{x} = a - b + c,$$

$$\text{whence,} \quad x = \frac{2}{a - b + c};$$

$$\frac{2}{y} = a + b - c,$$

$$\text{whence,} \quad y = \frac{2}{a + b - c};$$

$$\text{and} \quad \frac{2}{z} = -a + b + c,$$

$$\text{whence,} \quad z = \frac{2}{b + c - a}.$$

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29. See next page.

30.

$$\begin{cases} x + y - z = 0, & (1) \\ x - y = 2b, & (2) \\ x + z = 3a + b. & (3) \end{cases}$$

Adding the given equations,

$$3x = 3a + 3b. \quad (4)$$

Subtracting (4) from (3),

$$x = a + b. \quad (5)$$

Subtracting (2) from (4),

$$z = 2a. \quad (6)$$

$$y = a - b. \quad (7)$$

31.

$$\begin{cases} v + x = 2a, & (1) \\ x + y = 2a - z, & (2) \\ y + z = a + b, & (3) \\ v - z = a + c. & (4) \end{cases}$$

Subtracting (3) from (2),

$$x - z = a - b - z. \quad (5)$$

Subtracting (5) from (1),

$$x = a - b. \quad (6)$$

Subtracting (4) from (6),

$$z = b - c. \quad (7)$$

Subtracting (7) from (3),

$$y = a + c. \quad (8)$$

$$29. \quad \begin{cases} axy - x - y = 0, & (1) \\ bzx - z - x = 0, & (2) \\ cyz - y - z = 0. & (3) \end{cases}$$

Dividing (1) by xy , (2) by zx , and (3) by yz ,

$$a - \frac{1}{y} - \frac{1}{x} = 0, \quad (4)$$

$$b - \frac{1}{x} - \frac{1}{z} = 0, \quad (5)$$

$$c - \frac{1}{z} - \frac{1}{y} = 0. \quad (6)$$

Upon comparison it will be seen that (4), (5), and (6) differ from (4), (5), and (6) in Ex. 27 only in having b instead of c and c instead of b . Hence, the solution of (4), (5), and (6) is given in Ex. 27, with b and c interchanged.

Hence,

$$x = \frac{2}{a - c + b}, \text{ or } \frac{2}{a + b - c},$$

$$y = \frac{2}{a + c - b}, \text{ or } \frac{2}{a - b + c},$$

and

$$z = \frac{2}{c + b - a}, \text{ or } \frac{2}{b + c - a}.$$

$$32. \quad \begin{cases} y + z - 3x = 2a, & (1) \\ z + x - 3y = 2b, & (2) \\ x + y - 3z = 2c, & (3) \\ 2x + 2y + v = 0. & (4) \end{cases}$$

Adding (1), (2), and (3),
Adding (1) and (5),

$$\begin{aligned} -x - y - z &= 2a + 2b + 2c. \\ -4x &= 4a + 2b + 2c. \end{aligned} \quad (5)$$

Adding (2) and (5),

$$x = -\frac{1}{2}(2a + b + c). \quad (6)$$

Adding (3) and (5),

$$\begin{aligned} -4y &= 2a + 4b + 2c. \\ y &= -\frac{1}{2}(2b + c + a). \end{aligned} \quad (7)$$

Substituting (6) and (7) in (4),

$$\begin{aligned} -4z &= 2a + 2b + 4c. \\ z &= -\frac{1}{2}(2c + a + b). \\ v &= 3a + 3b + 2c. \end{aligned}$$

33. See next page.

$$34. \quad \begin{cases} x + y + z = a + b + c, & (1) \\ x + 2y + 3z = b + 2c, & (2) \\ x + 3y + 4z = b + 3c. & (3) \end{cases}$$

Subtracting (1) from (2),
Subtracting (2) from (3),
Subtracting (5) from (4),
Subtracting (6) from (5),
Subtracting (5) from (1),

$$\begin{aligned} y + 2z &= c - a. & (4) \\ y + z &= c. & (5) \\ z &= -a. & (6) \\ y &= a + c. & (7) \\ x &= a + b. & (8) \end{aligned}$$

35.

$$\begin{cases} v + x + y = a + 2b + c, & (1) \\ x + y + z = 3b, & (2) \\ y + z + v = a + b, & (3) \\ z + v + x = a + 3b - c. & (4) \end{cases}$$

Adding the given equations,

$$\begin{aligned} 3v + 3x + 3y + 3z &= 3a + 9b. \\ v + x + y + z &= a + 3b. \end{aligned} \quad (5)$$

Subtracting (2), (3), (4), and (1) successively from (5),

$$v = a, x = 2b, y = c, \text{ and } z = b - c.$$

$$33. \quad \begin{cases} abxyz + cxy - ayz - bzx = 0, & (1) \\ bcxyz + ayz - bzx - cxy = 0, & (2) \\ caxyz + bzx - cxy - ayz = 0. & (3) \end{cases}$$

Dividing each equation by xyz , transposing, etc.,

$$\frac{a}{x} + \frac{b}{y} - \frac{c}{z} = ab, \quad (4)$$

$$-\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = bc, \quad (5)$$

and

$$\frac{a}{x} - \frac{b}{y} + \frac{c}{z} = ca. \quad (6)$$

Adding (4), (5), and (6),

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = ab + bc + ca. \quad (7)$$

Subtracting (5) from (7),

$$\frac{2a}{x} = ab + ca.$$

$$x = \frac{2}{b+c}.$$

Subtracting (6) from (7),

$$\frac{2b}{y} = ab + bc.$$

$$y = \frac{2}{a+c}.$$

Subtracting (4) from (7),

$$\frac{2c}{z} = bc + ca.$$

$$z = \frac{2}{a+b}.$$

$$36. \quad \begin{cases} ax + by + cz = 3, & (1) \\ x + y = \frac{a+b}{ab}, & (2) \\ y + z = \frac{b+c}{bc}. & (3) \end{cases}$$

Multiplying (3) by c ,

$$cy + cz = \frac{b+c}{b} = 1 + \frac{c}{b}. \quad (4)$$

Subtracting (4) from (1),

$$ax + (b-c)y = 2 - \frac{c}{b}. \quad (5)$$

Multiplying (2) by a ,

$$ax + ay = 1 + \frac{a}{b}. \quad (6)$$

Subtracting (6) from (5),

$$(b-c-a)y = 1 - \frac{c}{b} - \frac{a}{b} = \frac{b-c-a}{b}.$$

$$y = \frac{1}{b}. \quad (7)$$

Substituting (7) in (2) and in (3),

$$x = \frac{1}{a} \text{ and } z = \frac{1}{c}.$$

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1. Let x = number of cents in price of rye,
 and y = number of cents in price of wheat,
 Then, z = number of cents in price of oats.
 and $2x + 3y + 4z = 480,$ (1)
 $3x + 5y + 2z = 640,$ (2)
 $2x + 4y + 3z = 530.$ (3)

Subtracting (3) from (2), $x + y - z = 110.$ (4)

Subtracting (1) from (3), $y - z = 50.$ (5)

Subtracting (5) from (4), $x = 60.$ (6)

Substituting (6) in one of the given equations and combining the result with (5), $y = 80$ and $z = 30.$

Hence, the price of rye was 60 cents, of wheat 80 cents, and of oats 30 cents, per bushel.

2. Let $x =$ number of cents in price of wheat,
and $y =$ number of cents in price of corn,
 $z =$ number of cents in price of rye.

Then, $5x + 2y + 3z = 660,$ (1)

$2x + 3y + 5z = 580,$ (2)

$3x + 5y + 2z = 560.$ (3)

Adding the given equations and dividing the result by 10,

$x + y + z = 180.$ (4)

Subtracting (4) $\times 3$ from (1), and (4) $\times 2$ from (3),

$2x - y = 120,$ (5)

and $x + 3y = 200.$ (6)

Solving (5) and (6), $x = 80,$ and $y = 40,$
whence, from (4), $z = 60.$

Hence, the price of wheat was 80 cents, of corn 40 cents, and of rye 60 cents, per bushel.

3. Let $x =$ first part,
and $y =$ second part,
 $z =$ third part.

Then, $x + y + z = 90,$ (1)

$\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 30,$ (2)

and $x = 2z - 2y.$ (3)

Substituting (3) in (1), $3z - y = 90.$ (4)

Substituting (3) in (2), $\frac{5}{4}z - \frac{2}{3}y = 30.$ (5)

Solving (4) and (5), $z = 40,$ and $y = 30,$
whence, from (1), $x = 20.$

Hence, the parts of 90 are 20, 30, and 40.

4. Let $x =$ first number,
and $y =$ second number,
 $z =$ third number.

Then, $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 12,$ (1)

$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 9,$ (2)

and $x + y + z = 38.$ (3)

Subtracting (3) successively from (1) $\times 4$ and (2) $\times 5$,

$x + \frac{1}{3}y = 10,$ (4)

and $\frac{2}{3}x + \frac{1}{4}y = 7.$ (5)

Subtracting (4) from (5) $\times \frac{3}{2}$,

$\frac{1}{24}y = \frac{1}{2}.$

whence, from (4), $y = 12,$ second number,
and from (3), $x = 6,$ first number,
 $z = 20,$ third number.

5. Let x = first number,
 y = second number,
 and z = third number.
 Then,
$$x + y + z = 72, \quad (1)$$

$$\frac{x + y}{z} = \frac{7}{5}, \quad (2)$$
 and
$$2x - z = \frac{1}{4}y. \quad (3)$$
 Multiplying (2) by z ,

$$x + y = \frac{7}{5}z. \quad (4)$$
 Subtracting (4) from (1),

$$z = 72 - \frac{7}{5}z.$$

$$\therefore z = 30, \text{ third number.} \quad (5)$$
 Substituting (5) in (3) and (4),

$$2x - \frac{1}{4}y = 30, \quad (6)$$
 and
$$x + y = 42. \quad (7)$$
 Solving (6) and (7),

$$y = 24, \text{ second number,}$$
 and
$$x = 18, \text{ first number.}$$

6. Let x = number of days it will take A,
 y = number of days it will take B,
 and z = number of days it will take C.
 Then,
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{10}, \quad (1)$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{8}, \quad (2)$$
 and
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{12}. \quad (3)$$
 Adding the given equations and dividing the result by 2,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{37}{240}. \quad (4)$$
 Subtracting (3), (2), and (1), successively, from (4), and solving,

$$x = 14\frac{2}{7},$$

$$y = 34\frac{2}{7},$$

$$z = 18\frac{6}{7}.$$

Hence, it will take A $14\frac{2}{7}$ days, B $34\frac{2}{7}$ days, and C $18\frac{6}{7}$ days.

7. See next page.

8. Let x = digit in hundreds' place,
 y = digit in tens' place,
 and z = digit in units' place.
 Then,
$$z - y = y - x, \quad (1)$$

$$\frac{100x + 10y + z}{x + y + z} = 15, \quad (2)$$
 and
$$100x + 10y + z + 396 = 100z + 10y + x. \quad (3)$$
 Reducing the given equations,

$$x - 2y + z = 0, \quad (4)$$

$$85x - 5y - 14z = 0, \quad (5)$$
 and
$$-x + z = 4. \quad (6)$$
 Subtracting (4) from (6),

$$2y - 2x = 4. \quad (7)$$
 Adding (5) and (4) $\times 14$,

$$99x - 33y = 0, \quad (8)$$
 whence,
$$y = 3x. \quad (9)$$
 Substituting (8) in (7),

$$x = 1. \quad (10)$$
 Substituting (9) in (8),

$$y = 3.$$
 Adding (9) and (6),

$$z = 5.$$
 Hence, the number is 135.

7. Let x = digit in hundreds' place,
 y = digit in tens' place,
 and z = digit in units' place.
- Then, $x + y + z = 14,$ (1)
 $100x + 10y + z + 693 = 100z + 10y + x,$ (2)
 and $z = y + 6.$ (3)
 Reducing (2), $x - z = -7.$ (4)
 Subtracting (4) from (1), $y + 2z = 21.$ (5)
 Adding (3) and (5), $3z = 27.$
 $\therefore z = 9,$ (6)
 whence, in (3), $y = 3.$ (7)
 Substituting (6) and (7) in (1),
 Hence, the number is 239.

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9. Let x = first number,
 y = second number,
 and z = third number.
- Then, $x + \frac{1}{3}y + \frac{1}{3}z = 36,$ (1)
 $\frac{1}{3}x + y + \frac{1}{3}z = 40,$ (2)
 and $\frac{1}{3}x + \frac{1}{3}y + z = 44.$ (3)
 Adding the given equations and dividing the result by 5,
 $\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z = 24.$ (4)
 Subtracting (4) from (1), (2), and (3), successively,
 $\frac{2}{3}x = 12, \frac{2}{3}y = 16, \text{ and } \frac{2}{3}z = 20.$
 Solving, $x = 18, y = 24, \text{ and } z = 30, \text{ the numbers.}$
10. Let x = first part,
 y = second part,
 and z = third part.
- Then, $x + y + z = 800,$ (1)
 $x + \frac{1}{2}y + \frac{2}{5}z = 400,$ (2)
 and $\frac{3}{4}x + y + \frac{1}{4}z = 400.$ (3)
 Subtracting (1) from (2) $\times 2$, and (3) from (1),
 $x - \frac{1}{5}z = 0,$ (4)
 and $\frac{1}{4}x + \frac{3}{4}z = 400.$ (5)
 From (4), $z = 5x.$ (6)
 Substituting (6) in (5), $\frac{1}{4}x + \frac{15}{4}x = 400.$
 $\therefore x = 100, \text{ first part,}$
 $z = 500, \text{ third part,}$
 whence, from (6), $y = 200, \text{ second part.}$
 and from (1),
11. Let x = number of miles from A to B,
 y = number of miles from B to C,
 and z = number of miles from C to A.
- Then, $z + y = 130,$ (1)
 $x + z = 110,$ (2)
 and $y + x = 140.$ (3)
 Adding the given equations and dividing by 2,
 $x + y + z = 190.$ (4)
 Subtracting (1), (2), and (3) successively from (4),
 $x = 60, y = 80, z = 50.$
 Hence, B is 60 miles from A, C 80 miles from B, and A 50 miles from C.

12. Let x = first number,
 y = second number,
 and z = third number.
- Then,
- $$x + \frac{1}{5}y + \frac{1}{5}z = 340, \quad (1)$$
- $$\frac{1}{5}x + y + \frac{1}{5}z = 600, \quad (2)$$
- and
- $$-\frac{1}{2}x + \frac{1}{2}y + z = 450. \quad (3)$$
- Adding (2) and (3),
- $$\frac{3}{5}y + \frac{3}{5}z = 1050. \quad (4)$$
- Multiplying (4) by $\frac{2}{3}$,
- $$\frac{2}{5}y + \frac{2}{5}z = 700. \quad (5)$$
- Subtracting (5) from (1),
- $$x = 200. \quad (6)$$
- From (5),
- $$y + z = 700. \quad (7)$$
- Substituting (6) in (2),
- $$y + \frac{1}{5}z = 500. \quad (8)$$
- Solving (7) and (8),
- $$z = 400, \text{ and } y = 300.$$
- Hence, the numbers are 200, 300, and 400.

13. Let x = number of cents 1 pound of first kind is worth,
 y = number of cents 1 pound of second kind is worth,
 and z = number of cents 1 pound of third kind is worth.
- Then,
- $$2x + 3y + 4z = 470, \quad (1)$$
- $$4x + 3y + 2z = 430, \quad (2)$$
- and
- $$z - \frac{3}{4}x - \frac{1}{5}y = 5. \quad (3)$$
- Multiplying (3) by -6 ,
- $$\frac{9}{2}x + \frac{3}{5}y - 6z = -30. \quad (4)$$
- Eliminating y between (1) and (2), and between (2) and (4),
- $$z - x = 20, \quad (5)$$
- and
- $$16z - x = 920. \quad (6)$$
- Solving (5) and (6),
- $$z = 60, \text{ and } x = 40,$$
- whence, in (1),
- $$y = 50.$$

Hence, the three kinds of tea were worth 40, 50, and 60 cents per pound, respectively.

14. Let x = number of dollars A has,
 y = number of dollars B has,
 and z = number of dollars C has.
- Then,
- $$x - 100 = y + 100, \quad (1)$$
- $$2(x - 100) = z + 100, \quad (2)$$
- and
- $$4(y - 100) = z + 100. \quad (3)$$
- Eliminating z between (2) and (3) by comparison,
- $$x - 100 = 2(y - 100). \quad (4)$$
- Eliminating x between (4) and (1) by comparison,
- $$2(y - 100) = y + 100.$$
- $$\therefore y = 300,$$
- whence, by substitution,
- $$x = 500, \text{ and } z = 700.$$
- Hence, A has \$500, B \$300, and C \$700.

15. Let x = number of gallons first jar holds,
 y = number of gallons second jar holds,
 and z = number of gallons third jar holds.
- Then,
- $$x + y + z = 4z, \quad (1)$$
- $$x + y + z = 2x + 4, \quad (2)$$
- $$x + y + z = 3y + 2. \quad (3)$$
- From (1),
- $$x = 3z - y. \quad (4)$$
- From (2),
- $$x = y + z - 4. \quad (5)$$
- Eliminating x between (4) and (5), transposing, etc.,
- $$y - z = 2. \quad (6)$$
- From (3),
- $$x = 2y - z + 2. \quad (7)$$

Eliminating x between (5) and (7), and transposing,

$$y - 2z = -6. \quad (8)$$

Subtracting (8) from (6),

$$z = 8,$$

whence, by substitution,

$$y = 10, \text{ and } x = 14.$$

Hence, the capacity of the largest jar is 14 gallons, of the second, 10 gallons, and of the third, 8 gallons.

16. Let

x = number of dollars A had at first,

y = number of dollars B had at first,

and

z = number of dollars C had at first.

Then, after the first giving, since all together had $x + y + z$ dollars,

$2y$ = number of dollars B had,

and

$2z$ = number of dollars C had,

whence,

$x - y - z$ = number of dollars A had;

similarly, after the second giving,

$2x - 2y - 2z$ = number of dollars A had,

$4z$ = number of dollars C had,

whence,

$-x + 3y - z$ = number of dollars B had;

finally, after the third giving, since each had \$8,

$4x - 4y - 4z = 8$, number of dollars A had,

$-2x + 6y - 2z = 8$, number of dollars B had,

and

$-x - y + 7z = 8$, number of dollars C had,

also,

$x + y + z = 24$, number of dollars all had.

Solving the first three equations, or any two of them together with the fourth, which is the sum of the first three equations,

$$x = 13, y = 7, \text{ and } z = 4.$$

Hence, at first A had \$13, B \$7, and C \$4.

17. Let

$5x$ = number of nuts in A's bag,

$5y$ = number of nuts in B's bag,

and

$5z$ = number of nuts in C's bag.

Then,

$$3x + y + z = 740, \quad (1)$$

$$3y + x + z = 580, \quad (2)$$

and

$$3z + x + y = 380, \quad (3)$$

also,

$$5x + 5y + 5z = 1700, \quad (4)$$

derived from the conditions or from (1), (2), and (3) by addition.

Solving, $x = 200, y = 120, \text{ and } z = 20.$

Hence, A had 1000 nuts, B 600, and C 100.

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$$20. (x + 2)^3 = x^3 + 3x^2(2) + 3x(2)^2 + (2)^3 = x^3 + 6x^2 + 12x + 8.$$

$$21. (a + 3)^3 = a^3 + 3a^2(3) + 3a(3)^2 + (3)^3 = a^3 + 9a^2 + 27a + 27.$$

$$22. (x + 4)^3 = x^3 + 3x^2(4) + 3x(4)^2 + (4)^3 = x^3 + 12x^2 + 48x + 64.$$

$$23. (x + 5)^3 = x^3 + 3x^2(5) + 3x(5)^2 + (5)^3 = x^3 + 15x^2 + 75x + 125.$$

$$24. (x - 2)^3 = x^3 - 3x^2(2) + 3x(2)^2 - (2)^3 = x^3 - 6x^2 + 12x - 8.$$

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37. $(x + 2y)^4 = x^4 + 4x^3(2y) + 6x^2(2y)^2 + 4x(2y)^3 + (2y)^4$
 $= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4.$
38. $(2x - y)^3 = (2x)^3 - 3(2x)^2y + 3(2x)y^2 - y^3$
 $= 8x^3 - 12x^2y + 6xy^2 - y^3.$
39. $(2x - 5)^3 = (2x)^3 - 3(2x)^2(5) + 3(2x)(5)^2 - (5)^3$
 $= 8x^3 - 60x^2 + 150x - 125.$
40. $(x^2 - 10)^4 = (x^2)^4 - 4(x^2)^3(10) + 6(x^2)^2(10)^2 - 4(x^2)(10)^3 + (10)^4$
 $= x^8 - 40x^6 + 600x^4 - 4000x^2 + 10000.$
41. $(1 - 3x^2)^4 = (1)^4 - 4(1)^3(3x^2) + 6(1)^2(3x^2)^2 - 4(1)(3x^2)^3 + (3x^2)^4$
 $= 1 - 12x^2 + 54x^4 - 108x^6 + 81x^8.$
42. $(5x^2 - ab)^3 = (5x^2)^3 - 3(5x^2)^2(ab) + 3(5x^2)(ab)^2 - (ab)^3$
 $= 125x^6 - 75abx^4 + 15a^2b^2x^2 - a^3b^3.$
43. $(1 + a^2b^2)^4 = (1)^4 + 4(1)^3(a^2b^2) + 6(1)^2(a^2b^2)^2 + 4(1)(a^2b^2)^3 + (a^2b^2)^4$
 $= 1 + 4a^2b^2 + 6a^4b^4 + 4a^6b^6 + a^8b^8.$
44. $(2ax - b)^5 = (2ax)^5 - 5(2ax)^4b + 10(2ax)^3b^2 - 10(2ax)^2b^3 + 5(2ax)b^4 - b^5$
 $= 32a^5x^5 - 80a^4bx^4 + 80a^3b^2x^3 - 40a^2b^3x^2 + 10ab^4x - b^5.$
45. $(1 - x)^7 = (1)^7 - 7(1)^6x + 21(1)^5x^2 - 35(1)^4x^3 + 35(1)^3x^4 - 21(1)^2x^5 + 7(1)x^6 - x^7$
 $= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7.$
46. $(1 - 2x)^6 = (1)^6 - 6(1)^5(2x) + 15(1)^4(2x)^2 - 20(1)^3(2x)^3 + 15(1)^2(2x)^4 - 6(1)(2x)^5 + (2x)^6$
 $= 1 - 12x + 60x^2 - 160x^3 + 240x^4 - 192x^5 + 64x^6.$
47. $(x - \frac{1}{2})^6 = x^6 - 6x^5(\frac{1}{2}) + 15x^4(\frac{1}{2})^2 - 20x^3(\frac{1}{2})^3 + 15x^2(\frac{1}{2})^4 - 6x(\frac{1}{2})^5 + (\frac{1}{2})^6$
 $= x^6 - 3x^5 + \frac{15}{4}x^4 - \frac{5}{2}x^3 + \frac{15}{8}x^2 - \frac{3}{16}x + \frac{1}{64}.$
48. $(\frac{1}{2}x - \frac{1}{3}y)^4 = (\frac{1}{2}x)^4 - 4(\frac{1}{2}x)^3(\frac{1}{3}y) + 6(\frac{1}{2}x)^2(\frac{1}{3}y)^2 - 4(\frac{1}{2}x)(\frac{1}{3}y)^3 + (\frac{1}{3}y)^4$
 $= \frac{1}{16}x^4 - \frac{1}{6}x^3y + \frac{1}{8}x^2y^2 - \frac{2}{27}xy^3 + \frac{1}{81}y^4.$
49. $(3a + \frac{1}{2})^5 = (2a)^5 + 5(2a)^4(\frac{1}{2}) + 10(2a)^3(\frac{1}{2})^2 + 10(2a)^2(\frac{1}{2})^3 + 5(2a)(\frac{1}{2})^4 + (\frac{1}{2})^5$
 $= 32a^5 + 40a^4 + 20a^3 + 5a^2 + \frac{5}{8}a + \frac{1}{32}.$
50. $(\frac{x}{y} - \frac{y}{x})^4 = (\frac{x}{y})^4 - 4(\frac{x}{y})^3(\frac{y}{x}) + 6(\frac{x}{y})^2(\frac{y}{x})^2 - 4(\frac{x}{y})(\frac{y}{x})^3 + (\frac{y}{x})^4$
 $= \frac{x^4}{y^4} - 4\frac{x^2}{y^2} + 6 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4}.$
51. $(\frac{x}{y} - \frac{y}{x})^6 = (\frac{x}{y})^6 - 6(\frac{x}{y})^5(\frac{y}{x}) + 15(\frac{x}{y})^4(\frac{y}{x})^2 - 20(\frac{x}{y})^3(\frac{y}{x})^3 + 15(\frac{x}{y})^2(\frac{y}{x})^4 - 6(\frac{x}{y})(\frac{y}{x})^5 + (\frac{y}{x})^6$
 $= \frac{x^6}{y^6} - 6\frac{x^4}{y^4} + 15\frac{x^2}{y^2} - 20 + 15\frac{y^2}{x^2} - 6\frac{y^4}{x^4} + \frac{y^6}{x^6}.$
52. $(3a^2 + \frac{b}{6})^3 = (3a^2)^3 + 3(3a^2)^2(\frac{b}{6}) + 3(3a^2)(\frac{b}{6})^2 + (\frac{b}{6})^3$
 $= 27a^6 + \frac{9}{2}a^4b + \frac{1}{4}a^2b^2 + \frac{1}{216}b^3.$

53. $\left(1 + \frac{3x}{2}\right)^6$
 $= (1)^6 + 5(1)^4 \left(\frac{3x}{2}\right) + 10(1)^3 \left(\frac{3x}{2}\right)^2 + 10(1)^2 \left(\frac{3x}{2}\right)^3 + 5(1) \left(\frac{3x}{2}\right)^4 + \left(\frac{3x}{2}\right)^6$
 $= 1 + \frac{15}{2}x + \frac{45}{2}x^2 + \frac{135}{8}x^3 + \frac{405}{16}x^4 + \frac{243}{32}x^5.$
54. $\left(\frac{3}{5} + \frac{5x}{3}\right)^4 = \left(\frac{3}{5}\right)^4 + 4\left(\frac{3}{5}\right)^3 \left(\frac{5x}{3}\right) + 6\left(\frac{3}{5}\right)^2 \left(\frac{5x}{3}\right)^2 + 4\left(\frac{3}{5}\right) \left(\frac{5x}{3}\right)^3 + \left(\frac{5x}{3}\right)^4$
 $= \frac{81}{625} + \frac{36}{25}x + 6x^2 + \frac{100}{9}x^3 + \frac{625}{81}x^4.$
55. $\left(\frac{1}{2x} - 2x\right)^5$
 $= \left(\frac{1}{2x}\right)^5 - 5\left(\frac{1}{2x}\right)^4 (2x) + 10\left(\frac{1}{2x}\right)^3 (2x)^2 - 10\left(\frac{1}{2x}\right)^2 (2x)^3 + 5\left(\frac{1}{2x}\right) (2x)^4 - (2x)^5$
 $= \frac{1}{32x^5} - \frac{5}{8x^3} + \frac{5}{x} - 20x + 40x^3 - 32x^5.$
56. $\left(\frac{1}{a} - a\right)^6 = \left(\frac{1}{a}\right)^6 - 6\left(\frac{1}{a}\right)^5 a + 15\left(\frac{1}{a}\right)^4 a^2 - 20\left(\frac{1}{a}\right)^3 a^3 + 15\left(\frac{1}{a}\right)^2 a^4 - 6\left(\frac{1}{a}\right) a^5 + a^6$
 $= \frac{1}{a^6} - \frac{6}{a^4} + \frac{15}{a^2} - 20 + 15a^2 - 6a^4 + a^6.$
57. $\left(x + \frac{1}{x}\right)^7$
 $= x^7 + 7x^6 \left(\frac{1}{x}\right) + 21x^5 \left(\frac{1}{x}\right)^2 + 35x^4 \left(\frac{1}{x}\right)^3 + 35x^3 \left(\frac{1}{x}\right)^4 + 21x^2 \left(\frac{1}{x}\right)^5 + 7x \left(\frac{1}{x}\right)^6 + \left(\frac{1}{x}\right)^7$
 $= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}.$
59. $(a + b - c - d)^3 = (\overline{a + b - c + d})^3$
 $= (a + b)^3 - 3(a + b)^2(c + d) + 3(a + b)(c + d)^2 - (c + d)^3$
 $= a^3 + 3a^2b + 3ab^2 + b^3 - 3(a^2c + 2abc + b^2c + a^2d + 2abd + b^2d)$
 $+ 3(ac^2 + 2acd + ad^2 + bc^2 + 2bcd + bd^2) - (c^3 + 3c^2d + 3cd^2 + d^3)$
 $= a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c - 3a^2d - 6abd - 3b^2d$
 $+ 3ac^2 + 6acd + 3ad^2 + 3bc^2 + 6bcd + 3bd^2 - c^3 - 3c^2d - 3cd^2 - d^3.$
60. $(a + x - y)^3 = (\overline{a + x - y})^3$
 $= (a + x)^3 - 3(a + x)^2y + 3(a + x)y^2 - y^3$
 $= a^3 + 3a^2x + 3ax^2 + x^3 - 3y(a^2 + 2ax + x^2) + 3ay^2 + 3xy^2 - y^3$
 $= a^3 + 3a^2x + 3ax^2 + x^3 - 3a^2y - 6axy - 3x^2y + 3ay^2 + 3xy^2 - y^3.$
61. $(a - m - n)^3 = (\overline{a - m - n})^3$
 $= (a - m)^3 - 3(a - m)^2n + 3(a - m)n^2 - n^3$
 $= a^3 - 3a^2m + 3am^2 - m^3 - 3n(a^2 - 2am + m^2) + 3an^2 - 3mn^2 - n^3$
 $= a^3 - 3a^2m + 3am^2 - m^3 - 3a^2n + 6amn - 3m^2n + 3an^2 - 3mn^2 - n^3.$
62. $(a - x + y)^3 = (\overline{a - x + y})^3$
 $= (a - x)^3 + 3(a - x)^2y + 3(a - x)y^2 + y^3$
 $= a^3 - 3a^2x + 3ax^2 - x^3 + 3y(a^2 - 2ax + x^2) + 3ay^2 - 3xy^2 + y^3$
 $= a^3 - 3a^2x + 3ax^2 - x^3 + 3a^2y - 6axy + 3x^2y + 3ay^2 - 3xy^2 + y^3.$

63. $(a - x - y)^3 = (\overline{a - x} - y)^3$
 $= (a - x)^3 - 3(a - x)^2 y + 3(a - x) y^2 - y^3$
 $= a^3 - 3a^2 x + 3ax^2 - x^3 - 3y(a^2 - 2ax + x^2) + 3ay^2 - 3xy^2 - y^3$
 $= a^3 - 3a^2 x + 3ax^2 - x^3 - 3a^2 y + 6axy - 3x^2 y + 3ay^2 - 3xy^2 - y^3.$
64. $(a + x + 2)^3 = (\overline{a + x} + 2)^3$
 $= (a + x)^3 + 3(a + x)^2(2) + 3(a + x)(2)^2 + (2)^3$
 $= a^3 + 3a^2 x + 3ax^2 + x^3 + 6(a^2 + 2ax + x^2) + 12a + 12x + 8$
 $= a^3 + 3a^2 x + 3ax^2 + x^3 + 6a^2 + 12ax + 6x^2 + 12a + 12x + 8.$
65. $(a - x - 2)^3 = (\overline{a - x} - 2)^3$
 $= (a - x)^3 - 3(a - x)^2(2) + 3(a - x)(2)^2 - (2)^3$
 $= a^3 - 3a^2 x + 3ax^2 - x^3 - 6(a^2 - 2ax + x^2) + 12a - 12x - 8$
 $= a^3 - 3a^2 x + 3ax^2 - x^3 - 6a^2 + 12ax - 6x^2 + 12a - 12x - 8.$
66. $(a + 2b - 3c)^3 = (\overline{a + 2b} - 3c)^3$
 $= (a + 2b)^3 - 3(a + 2b)^2(3c) + 3(a + 2b)(3c)^2 - (3c)^3$
 $= a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3 - 9c(a^2 + 4ab + 4b^2) + 27ac^2 + 54bc^2 - 27c^3$
 $= a^3 + 6a^2 b + 12ab^2 + 8b^3 - 9a^2 c - 36abc - 36b^2 c + 27ac^2 + 54bc^2 - 27c^3.$
67. $(a + b + x + y)^3 = (\overline{a + b} + \overline{x + y})^3$
 $= (a + b)^3 + 3(a + b)^2(x + y) + 3(a + b)(x + y)^2 + (x + y)^3$
 $= a^3 + 3a^2 b + 3ab^2 + b^3 + 3(a^2 x + 2abx + b^2 x + a^2 y + 2aby + b^2 y)$
 $+ 3(ax^2 + 2axy + ay^2 + bx^2 + 2bxy + by^2) + x^3 + 3x^2 y + 3xy^2 + y^3$
 $= a^3 + 3a^2 b + 3ab^2 + b^3 + 3a^2 x + 6abx + 3b^2 x + 3a^2 y + 6aby + 3b^2 y$
 $+ 3ax^2 + 6axy + 3ay^2 + 3bx^2 + 6bxy + 3by^2 + x^3 + 3x^2 y + 3xy^2 + y^3.$
68. $(a + b - x - y)^3 = (\overline{a + b} - \overline{x + y})^3$
 $= (a + b)^3 - 3(a + b)^2(x + y) + 3(a + b)(x + y)^2 - (x + y)^3$
 $= a^3 + 3a^2 b + 3ab^2 + b^3 - 3(a^2 x + 2abx + b^2 x + a^2 y + 2aby + b^2 y)$
 $+ 3(ax^2 + 2axy + ay^2 + bx^2 + 2bxy + by^2) - (x^3 + 3x^2 y + 3xy^2 + y^3)$
 $= a^3 + 3a^2 b + 3ab^2 + b^3 - 3a^2 x - 6abx - 3b^2 x - 3a^2 y - 6aby - 3b^2 y$
 $+ 3ax^2 + 6axy + 3ay^2 + 3bx^2 + 6bxy + 3by^2 - x^3 - 3x^2 y - 3xy^2 - y^3.$
69. $(a - b + x - y)^3 = (\overline{a - b} + \overline{x - y})^3$
 $= (a - b)^3 + 3(a - b)^2(x - y) + 3(a - b)(x - y)^2 + (x - y)^3$
 $= a^3 - 3a^2 b + 3ab^2 - b^3 + 3(a^2 x - 2abx + b^2 x - a^2 y + 2aby - b^2 y)$
 $+ 3(ax^2 - 2axy + ay^2 - bx^2 + 2bxy - by^2) + x^3 - 3x^2 y + 3xy^2 - y^3$
 $= a^3 - 3a^2 b + 3ab^2 - b^3 + 3a^2 x - 6abx + 3b^2 x - 3a^2 y + 6aby - 3b^2 y$
 $+ 3ax^2 - 6axy + 3ay^2 - 3bx^2 + 6bxy - 3by^2 + x^3 - 3x^2 y + 3xy^2 - y^3.$
70. $(a - b - x + y)^3 = (\overline{a - b} - \overline{x - y})^3$
 $= (a - b)^3 - 3(a - b)^2(x - y) + 3(a - b)(x - y)^2 - (x - y)^3$
 $= a^3 - 3a^2 b + 3ab^2 - b^3 - 3(a^2 x - 2abx + b^2 x - a^2 y + 2aby - b^2 y)$
 $+ 3(ax^2 - 2axy + ay^2 - bx^2 + 2bxy - by^2) - (x^3 - 3x^2 y + 3xy^2 - y^3)$
 $= a^3 - 3a^2 b + 3ab^2 - b^3 - 3a^2 x + 6abx - 3b^2 x + 3a^2 y - 6aby + 3b^2 y$
 $+ 3ax^2 - 6axy + 3ay^2 - 3bx^2 + 6bxy - 3by^2 - x^3 + 3x^2 y - 3xy^2 + y^3.$
71. $(a - b - x - y)^3 = (\overline{a - b} - \overline{x + y})^3$
 $= (a - b)^3 - 3(a - b)^2(x + y) + 3(a - b)(x + y)^2 - (x + y)^3$
 $= a^3 - 3a^2 b + 3ab^2 - b^3 - 3(a^2 x + 2abx + b^2 x + a^2 y + 2aby + b^2 y)$
 $+ 3(ax^2 + 2axy + ay^2 - bx^2 - 2bxy - by^2) - (x^3 + 3x^2 y + 3xy^2 + y^3)$
 $= a^3 - 3a^2 b + 3ab^2 - b^3 - 3a^2 x + 6abx - 3b^2 x - 3a^2 y + 6aby - 3b^2 y$
 $+ 3ax^2 + 6axy + 3ay^2 - 3bx^2 - 6bxy - 3by^2 - x^3 - 3x^2 y - 3xy^2 - y^3.$

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$$9. \quad \frac{(a+b)^2 - 4(a+b) + 4}{(a+b)^2} \mid \underline{a+b-2}$$

$$\frac{2(a+b)}{2(a+b)-2} \mid \frac{-4(a+b)+4}{-4(a+b)+4}$$

$$10. \quad \frac{x^6 + 4x^5 + 2x^4 + 9x^2 - 4x + 4}{x^6} \mid \underline{x^8 + 2x^2 - x + 2}$$

$$\begin{array}{r|l} 2x^3 & 4x^5 + 2x^4 \\ 2x^3 + 2x^2 & 4x^5 + 4x^4 \\ \hline 2x^3 + 4x^2 & -2x^4 + 9x^2 \\ 2x^3 + 4x^2 - x & -2x^4 - 4x^3 + x^2 \\ \hline 2x^3 + 4x^2 - 2x & 4x^3 + 8x^2 - 4x + 4 \\ 2x^3 + 4x^2 - 2x + 2 & 4x^3 + 8x^2 - 4x + 4 \end{array}$$

$$11. \quad \frac{9x^4 - 12x^3 + 10x^2 - 4x + 1}{9x^4} \mid \underline{3x^2 - 2x + 1}$$

$$\begin{array}{r|l} 6x^2 & -12x^3 + 10x^2 \\ 6x^2 - 2x & -12x^3 + 4x^2 \\ \hline 6x^2 - 4x & 6x^2 - 4x + 1 \\ 6x^2 - 4x + 1 & 6x^2 - 4x + 1 \end{array}$$

$$12. \quad \frac{x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4}{x^4} \mid \underline{x^2 - 3xy + 2y^2}$$

$$\begin{array}{r|l} 2x^2 & -6x^3y + 13x^2y^2 \\ 2x^2 - 3xy & -6x^3y + 9x^2y^2 \\ \hline 2x^2 - 6xy & 4x^2y^2 - 12xy^3 + 4y^4 \\ 2x^2 - 6xy + 2y^2 & 4x^2y^2 - 12xy^3 + 4y^4 \end{array}$$

$$13. \quad \frac{x^8 - 2a^2x^6 - a^4x^4 + 2a^6x^2 + a^8}{x^8} \mid \underline{x^4 - a^2x^2 - a^4}$$

$$\begin{array}{r|l} 2x^4 & -2a^2x^6 - a^4x^4 \\ 2x^4 - a^2x^2 & -2a^2x^6 + a^4x^4 \\ \hline 2x^4 - 2a^2x^2 & -2a^4x^4 + 2a^6x^2 + a^8 \\ 2x^4 - 2a^2x^2 - a^4 & -2a^4x^4 + 2a^6x^2 + a^8 \end{array}$$

$$14. \quad \frac{25x^4 - 30x^3 + 29x^2 - 12x + 4}{25x^4} \mid \underline{5x^2 - 3x + 2}$$

$$\begin{array}{r|l} 10x^2 & -30x^3 + 29x^2 \\ 10x^2 - 3x & -30x^3 + 9x^2 \\ \hline 10x^2 - 6x & 20x^2 - 12x + 4 \\ 10x^2 - 6x + 2 & 20x^2 - 12x + 4 \end{array}$$

$$15. \quad \frac{1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6}{1} \left| \frac{1 + 3x + 3x^2 + x^3}{1} \right.$$

$$\begin{array}{r|l} 2 & 6x + 15x^2 \\ 2 + 3x & 6x + 9x^2 \\ \hline 2 + 6x & 6x^2 + 20x^3 + 15x^4 \\ 2 + 6x + 3x^2 & 6x^2 + 18x^3 + 9x^4 \\ \hline 2 + 6x + 6x^2 & 2x^3 + 6x^4 + 6x^5 + x^6 \\ 2 + 6x + 6x^2 + x^3 & 2x^3 + 6x^4 + 6x^5 + x^6 \end{array}$$

$$16. \quad \frac{1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6}{1} \left| \frac{1 - x + x^2 - x^3}{1} \right.$$

$$\begin{array}{r|l} 2 & -2x + 3x^2 \\ 2 - x & -2x + x^2 \\ \hline 2 - 2x & 2x^2 - 4x^3 + 3x^4 \\ 2 - 2x + x^2 & 2x^2 - 2x^3 + x^4 \\ \hline 2 - 2x + 2x^2 & -2x^3 + 2x^4 - 2x^5 + x^6 \\ 2 - 2x + 2x^2 - x^3 & -2x^3 + 2x^4 - 2x^5 + x^6 \end{array}$$

$$17. \quad \frac{a^4 - 2a^2b + 2a^2c^2 - 2bc^2 + b^2 + c^4}{a^4} \left| \frac{a^2 - b + c^2}{a^2} \right.$$

$$\begin{array}{r|l} 2a^2 & -2a^2b + 2a^2c^2 - 2bc^2 + b^2 + c^4 \\ 2a^2 - b & -2a^2b + b^2 \\ \hline 2a^2 - 2b & 2a^2c^2 - 2bc^2 + c^4 \\ 2a^2 - 2b + c^2 & 2a^2c^2 - 2bc^2 + c^4 \end{array}$$

$$18. \quad \frac{4a^2 - 12ab + 9b^2 + 16ac - 24bc + 16c^2}{4a^2} \left| \frac{2a - 3b + 4c}{4a^2} \right.$$

$$\begin{array}{r|l} 4a & -12ab + 9b^2 \\ 4a - 3b & -12ab + 9b^2 \\ \hline 4a - 6b & 16ac - 24bc + 16c^2 \\ 4a - 6b + 4c & 16ac - 24bc + 16c^2 \end{array}$$

$$19. \quad \frac{9x^2 - 30xy + 25y^2 + 18xz - 30yz + 9z^2}{9x^2} \left| \frac{3x - 5y + 3z}{9x^2} \right.$$

$$\begin{array}{r|l} 6x & -30xy + 25y^2 \\ 6x - 5y & -30xy + 25y^2 \\ \hline 6x - 10y & 18xz - 30yz + 9z^2 \\ 6x - 10y + 3z & 18xz - 30yz + 9z^2 \end{array}$$

$$20. \quad \frac{x^2 + 2x - 1}{x^2} - \frac{2}{x} + \frac{1}{x^2} \left| \frac{x + 1 - \frac{1}{x}}{x^2} \right.$$

$$\begin{array}{r|l} 2x & 2x - 1 \\ 2x + 1 & 2x + 1 \\ \hline 2x + 2 & -2 - \frac{2}{x} + \frac{1}{x^2} \\ 2x + 2 - \frac{1}{x} & -2 - \frac{2}{x} + \frac{1}{x^2} \end{array}$$

$$\begin{array}{l}
 21. \quad \frac{x^4 + x^3 + \frac{13x^2}{20} + \frac{x}{5} + \frac{1}{25}}{x^4} \left| \frac{x^2 + \frac{x}{2} + \frac{1}{5}}{x^2} \right. \\
 \frac{2x^2}{2x^2 + \frac{x}{2}} \left| \frac{x^3 + \frac{13x^2}{20}}{x^3 + \frac{x^2}{4}} \right. \\
 \frac{2x^2 + x}{2x^2 + x} \left| \frac{\frac{2x^2}{5} + \frac{x}{5} + \frac{1}{25}}{\frac{2x^2}{5} + \frac{x}{5} + \frac{1}{25}} \right. \\
 \frac{2x^2 + x + \frac{1}{5}}{2x^2 + x + \frac{1}{5}} \left| \frac{\frac{2x^2}{5} + \frac{x}{5} + \frac{1}{25}}{\frac{2x^2}{5} + \frac{x}{5} + \frac{1}{25}} \right.
 \end{array}$$

$$\begin{array}{l}
 22. \quad \frac{\frac{a^4}{4} + a^3x + \frac{4a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}}{\frac{a^4}{4}} \left| \frac{\frac{a^2}{2} + ax + \frac{x^2}{3}}{\frac{a^2}{2} + ax + \frac{x^2}{3}} \right. \\
 \frac{a^2}{a^2 + ax} \left| \frac{a^3x + \frac{4a^2x^2}{3}}{a^3x + \frac{a^2x^2}{3}} \right. \\
 \frac{a^2 + 2ax}{a^2 + 2ax} \left| \frac{\frac{a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}}{\frac{a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}} \right. \\
 \frac{a^2 + 2ax + \frac{x^2}{3}}{a^2 + 2ax + \frac{x^2}{3}} \left| \frac{\frac{a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}}{\frac{a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}} \right.
 \end{array}$$

$$23. \quad \frac{x^8 + 4x^7 - 2x^6 - 20x^5 - 3x^4 + 32x^3 + 4x^2 - 16x + 4}{x^8} \left| \frac{x^4 + 2x^3 - 3x^2 - 4x + 2}{x^4 + 2x^3 - 3x^2 - 4x + 2} \right.$$

$$\begin{array}{l}
 \frac{2x^4}{2x^4 + 2x^3} \left| \frac{4x^7 - 2x^6}{4x^7 + 4x^6} \right. \\
 \frac{2x^4 + 4x^3}{2x^4 + 4x^3 - 3x^2} \left| \frac{-6x^6 - 20x^5 - 3x^4}{-6x^6 - 12x^5 + 9x^4} \right. \\
 \frac{2x^4 + 4x^3 - 6x^2}{2x^4 + 4x^3 - 6x^2 - 4x} \left| \frac{-8x^5 - 12x^4 + 32x^3 + 4x^2}{-8x^5 - 16x^4 + 24x^3 + 16x^2} \right. \\
 \frac{2x^4 + 4x^3 - 6x^2 - 8x}{2x^4 + 4x^3 - 6x^2 - 8x + 2} \left| \frac{4x^4 + 8x^3 - 12x^2 - 16x + 4}{4x^4 + 8x^3 - 12x^2 - 16x + 4} \right.
 \end{array}$$

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$$\begin{array}{l}
 25. \quad \frac{1 - a}{1} \left| \frac{1 - \frac{1}{2}a - \frac{1}{8}a^2 - \frac{1}{16}a^3}{1 - \frac{1}{2}a - \frac{1}{8}a^2 - \frac{1}{16}a^3} \right. \\
 \frac{2}{2 - \frac{1}{2}a} \left| \frac{-a}{-a + \frac{1}{4}a^2} \right. \\
 \frac{2 - a}{2 - a - \frac{1}{8}a^2} \left| \frac{-\frac{1}{4}a^2}{-\frac{1}{4}a^2 + \frac{1}{8}a^3 + \frac{1}{16}a^4} \right. \\
 \frac{2 - a - \frac{1}{4}a^2}{2 - a - \frac{1}{4}a^2} \left| \frac{-\frac{1}{8}a^3 - \frac{1}{16}a^4}{-\frac{1}{8}a^3 - \frac{1}{16}a^4} \right.
 \end{array}$$

26.

$$\begin{array}{r|l}
 a^2 + 1 & a + \frac{1}{2a} - \frac{1}{8a^3} + \frac{1}{16a^5} \\
 \hline
 2a & 1 \\
 2a + \frac{1}{2a} & 1 + \frac{1}{4a^2} \\
 \hline
 2a + \frac{1}{a} & -\frac{1}{4a^2} \\
 2a + \frac{1}{a} - \frac{1}{8a^3} & -\frac{1}{4a^2} - \frac{1}{8a^4} + \frac{1}{64a^6} \\
 \hline
 2a + \frac{1}{a} - \frac{1}{4a^3} & \frac{1}{8a^4} - \frac{1}{64a^6}
 \end{array}$$

27.

$$\begin{array}{r|l}
 x^2 - 1 & x - \frac{1}{2x} - \frac{1}{8x^3} - \frac{1}{16x^5} \\
 \hline
 2x & -1 \\
 2x - \frac{1}{2x} & -1 + \frac{1}{4x^2} \\
 \hline
 2x - \frac{1}{x} & -\frac{1}{4x^2} \\
 2x - \frac{1}{x} - \frac{1}{8x^3} & -\frac{1}{4x^2} + \frac{1}{8x^4} + \frac{1}{64x^6} \\
 \hline
 2x - \frac{1}{x} - \frac{1}{4x^3} & -\frac{1}{8x^4} - \frac{1}{64x^6}
 \end{array}$$

28.

$$\begin{array}{r|l}
 4 - a & 2 - \frac{a}{4} - \frac{a^2}{64} - \frac{a^3}{512} \\
 \hline
 4 & -a \\
 4 - \frac{a}{4} & -a + \frac{a^2}{16} \\
 \hline
 4 - \frac{a}{2} & -\frac{a^2}{16} \\
 4 - \frac{a}{2} - \frac{a^2}{64} & -\frac{a^2}{16} + \frac{a^3}{128} + \frac{a^4}{4096} \\
 \hline
 4 - \frac{a}{2} - \frac{a^2}{32} & -\frac{a^3}{128} - \frac{a^4}{4096}
 \end{array}$$

29.

$$\begin{array}{r|l}
 y^2 + 3 & y + \frac{3}{2y} - \frac{9}{8y^3} + \frac{27}{16y^5} \\
 \hline
 2y & 3 \\
 2y + \frac{3}{2y} & 3 + \frac{9}{4y^2} \\
 \hline
 2y + \frac{3}{y} & -\frac{9}{4y^2} \\
 2y + \frac{3}{y} - \frac{9}{8y^3} & -\frac{9}{4y^2} - \frac{27}{8y^4} + \frac{81}{64y^6} \\
 \hline
 2y + \frac{3}{y} - \frac{9}{4y^3} & \frac{27}{8y^4} - \frac{81}{64y^6}
 \end{array}$$

30.

$$\begin{array}{r|l}
 \frac{a^2 + 2b}{a^2} & \frac{a + \frac{b}{a} - \frac{b^2}{2a^3} + \frac{b^3}{2a^5}}{a^2} \\
 \hline
 2a & 2b \\
 2a + \frac{b}{a} & 2b + \frac{b^2}{a^2} \\
 \hline
 2a + \frac{2b}{a} & -\frac{b^2}{a^2} \\
 2a + \frac{2b}{a} - \frac{b^2}{2a^3} & -\frac{b^2}{a^2} - \frac{b^3}{a^4} + \frac{b^4}{4a^6} \\
 \hline
 2a + \frac{2b}{a} - \frac{b^2}{a^3} & \frac{b^3}{a^4} - \frac{b^4}{4a^6}
 \end{array}$$

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7. $1.74.24 \mid 132$
 $\frac{1}{1}$

$$\begin{array}{r|l}
 10 \times 2 = 20 & 74 \\
 20 + 3 = 23 & 69 \\
 \hline
 130 \times 2 = 260 & 5 \ 24 \\
 260 + 2 = 262 & 5 \ 24
 \end{array}$$

8. $1.93.21 \mid 139$
 $\frac{1}{1}$

$$\begin{array}{r|l}
 10 \times 2 = 20 & 93 \\
 20 + 3 = 23 & 69 \\
 \hline
 130 \times 2 = 260 & 24 \ 21 \\
 260 + 9 = 269 & 24 \ 21
 \end{array}$$

9. $5.71.21 \mid 239$
 $\frac{4}{4}$

$$\begin{array}{r|l}
 20 \times 2 = 40 & 1 \ 71 \\
 40 + 3 = 43 & 1 \ 29 \\
 \hline
 230 \times 2 = 460 & 42 \ 21 \\
 460 + 9 = 469 & 42 \ 21
 \end{array}$$

10. $4.20.25 \mid 205$
 $\frac{4}{4}$

$$\begin{array}{r|l}
 200 \times 2 = 400 & 20 \ 25 \\
 400 + 5 = 405 & 20 \ 25
 \end{array}$$

11. $9.54.81 \mid 309$
 $\frac{9}{9}$

$$\begin{array}{r|l}
 300 \times 2 = 600 & 54 \ 81 \\
 600 + 9 = 609 & 54 \ 81
 \end{array}$$

12. $18.66.24 \mid 432$
 $\frac{16}{16}$

$$\begin{array}{r|l}
 40 \times 2 = 80 & 2 \ 66 \\
 80 + 3 = 83 & 2 \ 49 \\
 \hline
 430 \times 2 = 860 & 17 \ 24 \\
 860 + 2 = 862 & 17 \ 24
 \end{array}$$

13. See next page.

14. $13.46.89 \mid 367$
 $\frac{9}{9}$

$$\begin{array}{r|l}
 30 \times 2 = 60 & 4 \ 46 \\
 60 + 6 = 66 & 3 \ 96 \\
 \hline
 360 \times 2 = 720 & 50 \ 89 \\
 720 + 7 = 727 & 50 \ 89
 \end{array}$$

15. $12.53.16 \mid 354$
 $\frac{9}{9}$

$$\begin{array}{r|l}
 30 \times 2 = 60 & 3 \ 53 \\
 60 + 5 = 65 & 3 \ 25 \\
 \hline
 350 \times 2 = 700 & 28 \ 16 \\
 700 + 4 = 704 & 28 \ 16
 \end{array}$$

16. $45.56.25 \mid 675$
 $\frac{36}{36}$

$$\begin{array}{r|l}
 60 \times 2 = 120 & 9 \ 56 \\
 120 + 7 = 127 & 8 \ 89 \\
 \hline
 670 \times 2 = 1340 & 67 \ 25 \\
 1340 + 5 = 1345 & 67 \ 25
 \end{array}$$

17. $99.20.16 \mid 996$
 $\frac{81}{81}$

$$\begin{array}{r|l}
 90 \times 2 = 180 & 18 \ 20 \\
 180 + 9 = 189 & 17 \ 01 \\
 \hline
 990 \times 2 = 1980 & 1 \ 19 \ 16 \\
 1980 + 6 = 1986 & 1 \ 19 \ 16
 \end{array}$$

18. $24.80.04 \mid 49.8$
 $\frac{16}{16}$

$$\begin{array}{r|l}
 40 \times 2 = 80 & 8 \ 80 \\
 80 + 9 = 89 & 8 \ 01 \\
 \hline
 490 \times 2 = 980 & 79 \ 04 \\
 980 + 8 = 988 & 79 \ 04
 \end{array}$$

$$13. \quad \begin{array}{r} 16.56.49 \quad | \quad 407 \\ 16 \end{array}$$

$$\begin{array}{r} 400 \times 2 = 800 \quad | \quad 56 \quad 49 \\ 800 + 7 = 807 \quad | \quad 56 \quad 49 \end{array}$$

$$20. \quad \begin{array}{r} .00.12.25 \quad | \quad .035 \\ 9 \end{array}$$

$$\begin{array}{r} 30 \times 2 = 60 \quad | \quad 3 \quad 25 \\ 60 + 5 = 65 \quad | \quad 3 \quad 25 \end{array}$$

$$19. \quad \begin{array}{r} 10.95.61 \quad | \quad 3.31 \\ 9 \end{array}$$

$$\begin{array}{r} 30 \times 2 = 60 \quad | \quad 1 \quad 95 \\ 60 + 3 = 63 \quad | \quad 1 \quad 89 \\ \hline 330 \times 2 = 660 \quad | \quad 6 \quad 61 \\ 660 + 1 = 661 \quad | \quad 6 \quad 61 \end{array}$$

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$$21. \quad \begin{array}{r} 1.02.01 \quad | \quad 101 \\ 1 \end{array}$$

$$\begin{array}{r} 100 \times 2 = 200 \quad | \quad 02 \quad 01 \\ 200 + 1 = 201 \quad | \quad 2 \quad 01 \end{array}$$

$$22. \quad \begin{array}{r} 9.54.81 \quad | \quad 309 \\ 9 \end{array}$$

$$\begin{array}{r} 300 \times 2 = 600 \quad | \quad 54 \quad 81 \\ 600 + 9 = 609 \quad | \quad 54 \quad 81 \end{array}$$

$$23. \quad \begin{array}{r} 36.36.09 \quad | \quad 603 \\ 36 \end{array}$$

$$\begin{array}{r} 600 \times 2 = 1200 \quad | \quad 36 \quad 09 \\ 1200 + 3 = 1203 \quad | \quad 36 \quad 09 \end{array}$$

$$24. \quad \begin{array}{r} 13.32.25 \quad | \quad 36.5 \\ 9 \end{array}$$

$$\begin{array}{r} 30 \times 2 = 60 \quad | \quad 4 \quad 32 \\ 60 + 6 = 66 \quad | \quad 3 \quad 96 \\ \hline 360 \times 2 = 720 \quad | \quad 36 \quad 25 \\ 720 + 5 = 725 \quad | \quad 36 \quad 25 \end{array}$$

$$25. \quad \begin{array}{r} 1.01.00.25 \quad | \quad 10.05 \\ 1 \end{array}$$

$$\begin{array}{r} 1000 \times 2 = 2000 \quad | \quad 01 \quad 00 \quad 25 \\ 2000 + 5 = 2005 \quad | \quad 1 \quad 00 \quad 25 \end{array}$$

$$26. \quad \begin{array}{r} 1.11.09.16 \quad | \quad 10.54 \\ 1 \end{array}$$

$$\begin{array}{r} 100 \times 2 = 200 \quad | \quad 11 \quad 09 \\ 200 + 5 = 205 \quad | \quad 10 \quad 25 \\ \hline 1050 \times 2 = 2100 \quad | \quad 84 \quad 16 \\ 2100 + 4 = 2104 \quad | \quad 84 \quad 16 \end{array}$$

$$27. \quad \begin{array}{r} 5.40.56.25 \quad | \quad 23.25 \\ 4 \end{array}$$

$$\begin{array}{r} 20 \times 2 = 40 \quad | \quad 1 \quad 40 \\ 40 + 3 = 43 \quad | \quad 1 \quad 29 \\ \hline 230 \times 2 = 460 \quad | \quad 11 \quad 56 \\ 460 + 2 = 462 \quad | \quad 9 \quad 24 \\ \hline 2320 \times 2 = 4640 \quad | \quad 2 \quad 32 \quad 25 \\ 4640 + 5 = 4645 \quad | \quad 2 \quad 32 \quad 25 \end{array}$$

$$28. \quad \begin{array}{r} 1.01.80.81 \quad | \quad 1.009 \\ 1 \end{array}$$

$$\begin{array}{r} 1000 \times 2 = 2000 \quad | \quad 01 \quad 80 \quad 81 \\ 2000 + 9 = 2009 \quad | \quad 1 \quad 80 \quad 81 \end{array}$$

$$29. \quad \begin{array}{r} 13.00.32.36 \quad | \quad 3606 \\ 9 \end{array}$$

$$\begin{array}{r} 30 \times 2 = 60 \quad | \quad 4 \quad 00 \\ 60 + 6 = 66 \quad | \quad 3 \quad 96 \\ \hline 3600 \times 2 = 7200 \quad | \quad 4 \quad 32 \quad 36 \\ 7200 + 6 = 7206 \quad | \quad 4 \quad 32 \quad 36 \end{array}$$

$$38. \quad \begin{array}{r} 3.00.00.00.00 \quad | \quad 1.7320 + \\ 1 \end{array}$$

$$\begin{array}{r} 10 \times 2 = 20 \quad | \quad 2 \quad 00 \\ 20 + 7 = 27 \quad | \quad 1 \quad 89 \\ \hline 170 \times 2 = 340 \quad | \quad 11 \quad 00 \\ 340 + 3 = 343 \quad | \quad 10 \quad 29 \\ \hline 1730 \times 2 = 3460 \quad | \quad 71 \quad 00 \\ 3460 + 2 = 3462 \quad | \quad 69 \quad 24 \\ \hline 17320 \times 2 = 34640 \quad | \quad 1 \quad 76 \quad 00 \end{array}$$

Since the square root of 3 is 1.7320+ and the square root of 4 is 2, the square root of $\frac{3}{4}$ to four decimal places is $\frac{1.7320}{2}$, or .8660.

39. $\frac{4}{5} = .8$.

$$\begin{array}{r}
 .80 \cdot 00 \cdot 00 \cdot 00 \mid \underline{.8944} \\
 64 \\
 80 \times 2 = 160 \mid 16 \ 00 \\
 160 + 9 = 169 \mid 15 \ 21 \\
 \hline
 890 \times 2 = 1780 \mid 79 \ 00 \\
 1780 + 4 = 1784 \mid 71 \ 36 \\
 \hline
 8940 \times 2 = 17880 \mid 7 \ 64 \ 00 \\
 17880 + 4 = 17884 \mid 7 \ 15 \ 36
 \end{array}$$

40. See next page.

41. $.60 \cdot 00 \cdot 00 \cdot 00 \mid \underline{.7745}$

$$\begin{array}{r}
 49 \\
 70 \times 2 = 140 \mid 11 \ 00 \\
 140 + 7 = 147 \mid 10 \ 29 \\
 \hline
 770 \times 2 = 1540 \mid 71 \ 00 \\
 1540 + 4 = 1544 \mid 61 \ 76 \\
 \hline
 7740 \times 2 = 15480 \mid 9 \ 24 \ 00 \\
 15480 + 5 = 15485 \mid 7 \ 74 \ 25
 \end{array}$$

42. $\frac{5}{6} = .83333333 +$.

$$\begin{array}{r}
 .83 \cdot 33 \cdot 33 \cdot 33 + \mid \underline{.9128} \\
 81 \\
 90 \times 2 = 180 \mid 2 \ 33 \\
 180 + 1 = 181 \mid 1 \ 81 \\
 \hline
 910 \times 2 = 1820 \mid 52 \ 33 \\
 1820 + 2 = 1822 \mid 36 \ 44 \\
 \hline
 9120 \times 2 = 18240 \mid 15 \ 89 \ 33 \\
 18240 + 8 = 18248 \mid 14 \ 59 \ 84
 \end{array}$$

43. $\frac{2}{3} = .22222222 +$.

$$\begin{array}{r}
 .22 \cdot 22 \cdot 22 \cdot 22 + \mid \underline{.4714} \\
 16 \\
 40 \times 2 = 80 \mid 6 \ 22 \\
 80 + 7 = 87 \mid 6 \ 09 \\
 \hline
 470 \times 2 = 940 \mid 13 \ 22 \\
 940 + 1 = 941 \mid 9 \ 41 \\
 \hline
 4710 \times 2 = 9420 \mid 3 \ 81 \ 22 \\
 9420 + 4 = 9424 \mid 3 \ 76 \ 96
 \end{array}$$

44. $\frac{7}{8} = .875$.

$$\begin{array}{r}
 .87 \cdot 50 \cdot 00 \cdot 00 \mid \underline{.9354} \\
 81 \\
 90 \times 2 = 180 \mid 6 \ 50 \\
 180 + 3 = 183 \mid 5 \ 49 \\
 \hline
 930 \times 2 = 1860 \mid 1 \ 01 \ 00 \\
 1860 + 5 = 1865 \mid 93 \ 25 \\
 \hline
 9350 \times 2 = 18700 \mid 7 \ 75 \ 00 \\
 18700 + 4 = 18704 \mid 7 \ 48 \ 16
 \end{array}$$

40. $\frac{5}{8} = .625$.

$$\begin{array}{r}
 .62 \cdot 50 \cdot 00 \cdot 00 \mid .7905 \\
 \underline{49} \\
 70 \times 2 = 140 \mid 13 \ 50 \\
 140 + 9 = 149 \mid 13 \ 41 \\
 \hline
 7900 \times 2 = 15800 \mid 9 \ 00 \ 00 \\
 15800 + 5 = 15805 \mid 7 \ 90 \ 25
 \end{array}$$

45.

$$\begin{array}{r}
 5.00 \cdot 00 \cdot 00 \cdot 00 \mid 2.2360 + \\
 \underline{4} \\
 20 \times 2 = 40 \mid 1 \ 00 \\
 40 + 2 = 42 \mid 84 \\
 \hline
 220 \times 2 = 440 \mid 16 \ 00 \\
 440 + 3 = 443 \mid 13 \ 29 \\
 \hline
 2230 \times 2 = 4460 \mid 2 \ 71 \ 00 \\
 4460 + 6 = 4466 \mid 2 \ 67 \ 96 \\
 \hline
 22360 \times 2 = 44720 \mid 3 \ 04 \ 00
 \end{array}$$

Since the square root of 5 is 2.2360+ and the square root of 16 is 4, the square root of $\frac{5}{16}$ to four decimal places is $\frac{2.2360}{4}$, or .5590.

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3.

$$\begin{array}{r}
 x^3 - 3x^2y + 3xy^2 - y^3 \mid x - y \\
 \hline
 x^3 \\
 3x^2 - 3xy + y^2 \mid -3x^2y + 3xy^2 - y^3 \\
 \hline
 3x^2 - 3xy + y^2 \mid -3x^2y + 3xy^2 - y^3
 \end{array}$$

4.

$$\begin{array}{r}
 m^3 - 9m^2 + 27m - 27 \mid m - 3 \\
 \hline
 m^3 \\
 3m^2 - 9m + 9 \mid -9m^2 + 27m - 27 \\
 \hline
 3m^2 - 9m + 9 \mid -9m^2 + 27m - 27
 \end{array}$$

5.

$$\begin{array}{r}
 8m^3 - 60m^2n + 150mn^2 - 125n^3 \mid 2m - 5n \\
 \hline
 8m^3 \\
 12m^2 - 30mn + 25n^2 \mid -60m^2n + 150mn^2 - 125n^3 \\
 \hline
 12m^2 - 30mn + 25n^2 \mid -60m^2n + 150mn^2 - 125n^3
 \end{array}$$

6.

$$\begin{array}{r}
 27x^3 - 189x^2y + 441xy^2 - 343y^3 \mid 3x - 7y \\
 \hline
 27x^3 \\
 27x^2 - 63xy + 49y^2 \mid -189x^2y + 441xy^2 - 343y^3 \\
 \hline
 27x^2 - 63xy + 49y^2 \mid -189x^2y + 441xy^2 - 343y^3
 \end{array}$$

7.

$$\begin{array}{r}
 125a^3 + 675a^2x + 1215ax^2 + 729x^3 \mid 5a + 9x \\
 \hline
 125a^3 \\
 75a^2 + 135ax + 81x^2 \mid 675a^2x + 1215ax^2 + 729x^3 \\
 \hline
 75a^2 + 135ax + 81x^2 \mid 675a^2x + 1215ax^2 + 729x^3
 \end{array}$$

8.

$$\begin{array}{r}
 1000p^6 - 300p^4q + 30p^2q^2 - q^3 \mid 10p^2 - q \\
 \hline
 1000p^6 \\
 300p^4 - 30p^2q + q^2 \mid -300p^4q + 30p^2q^2 - q^3 \\
 \hline
 300p^4 - 30p^2q + q^2 \mid -300p^4q + 30p^2q^2 - q^3
 \end{array}$$

$$9. \quad \frac{m^6 + 6m^5 + 15m^4 + 20m^3 + 15m^2 + 6m + 1}{m^6} \mid m^2 + 2m + 1$$

| | |
|--|---|
| $\frac{3m^4}{3m^4 + 6m^3 + 4m^2}$ | $\frac{6m^5 + 15m^4 + 20m^3}{6m^5 + 12m^4 + 8m^3}$ |
| $\frac{3m^4 + 12m^3 + 12m^2}{3m^4 + 12m^3 + 15m^2 + 6m + 1}$ | $\frac{3m^4 + 12m^3 + 15m^2 + 6m + 1}{3m^4 + 12m^3 + 15m^2 + 6m + 1}$ |

$$10. \quad \frac{x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1}{x^6} \mid x^2 - 2x + 1$$

| | |
|--|---|
| $\frac{3x^4}{3x^4 - 6x^3 + 4x^2}$ | $\frac{-6x^5 + 15x^4 - 20x^3}{-6x^5 + 12x^4 - 8x^3}$ |
| $\frac{3x^4 - 12x^3 + 12x^2}{3x^4 - 12x^3 + 15x^2 - 6x + 1}$ | $\frac{3x^4 - 12x^3 + 15x^2 - 6x + 1}{3x^4 - 12x^3 + 15x^2 - 6x + 1}$ |

$$11. \quad \frac{x^6 + 3x^5 + 9x^4 + 13x^3 + 18x^2 + 12x + 8}{x^6} \mid x^2 + x + 2$$

| | |
|--|---|
| $\frac{3x^4}{3x^4 + 3x^3 + x^2}$ | $\frac{3x^5 + 9x^4 + 13x^3}{3x^5 + 3x^4 + x^3}$ |
| $\frac{3x^4 + 6x^3 + 3x^2}{3x^4 + 6x^3 + 9x^2 + 6x + 4}$ | $\frac{6x^4 + 12x^3 + 18x^2 + 12x + 8}{6x^4 + 12x^3 + 18x^2 + 12x + 8}$ |

$$12. \quad \frac{x^6 + 12x^5 + 63x^4 + 184x^3 + 315x^2 + 300x + 125}{x^6} \mid x^2 + 4x + 5$$

| | |
|--|---|
| $\frac{3x^4}{3x^4 + 12x^3 + 16x^2}$ | $\frac{12x^5 + 63x^4 + 184x^3}{12x^5 + 48x^4 + 64x^3}$ |
| $\frac{3x^4 + 24x^3 + 48x^2}{3x^4 + 24x^3 + 63x^2 + 60x + 25}$ | $\frac{15x^4 + 120x^3 + 315x^2 + 300x + 125}{15x^4 + 120x^3 + 315x^2 + 300x + 125}$ |

$$13. \quad \frac{x^6 + 6x^5 - 18x^4 - 112x^3 + 180x^2 + 600x - 1000}{x^6} \mid x^2 + 2x - 10$$

| | |
|---|---|
| $\frac{3x^4}{3x^4 + 6x^3 + 4x^2}$ | $\frac{6x^5 - 18x^4 - 112x^3}{6x^5 + 12x^4 + 8x^3}$ |
| $\frac{3x^4 + 12x^3 + 12x^2}{3x^4 + 12x^3 - 18x^2 - 60x + 100}$ | $\frac{-30x^4 - 120x^3 + 180x^2 + 600x - 1000}{-30x^4 - 120x^3 + 180x^2 + 600x - 1000}$ |

$$14. \quad \frac{x^3 - 12x^2 + 54x - 112 + \frac{108}{x} - \frac{48}{x^2} + \frac{8}{x^3}}{x^3} \mid x - 4 + \frac{2}{x}$$

| | |
|--|---|
| $\frac{3x^2}{3x^2 - 12x + 16}$ | $\frac{-12x^2 + 54x - 112}{-12x^2 + 48x - 64}$ |
| $\frac{3x^2 - 24x + 48}{3x^2 - 24x + 54 - \frac{24}{x} + \frac{4}{x^2}}$ | $\frac{6x - 48 + \frac{108}{x} - \frac{48}{x^2} + \frac{8}{x^3}}{6x - 48 + \frac{108}{x} - \frac{48}{x^2} + \frac{8}{x^3}}$ |

$$15. \quad \frac{1 - 6a + 21a^2 - 44a^3 + 63a^4 - 54a^5 + 27a^6}{1} \mid 1 - 2a + 3a^2$$

| | |
|--|---|
| $\frac{3}{3 - 6a + 4a^2}$ | $\frac{-6a + 21a^2 - 44a^3}{-6a + 12a^2 - 8a^3}$ |
| $\frac{3 - 12a + 12a^2}{3 - 12a + 21a^2 - 18a^3 + 9a^4}$ | $\frac{9a^2 - 36a^3 + 63a^4 - 54a^5 + 27a^6}{9a^2 - 36a^3 + 63a^4 - 54a^5 + 27a^6}$ |

$$\begin{array}{r|l}
 16. & 64 - 144p + 156p^2 - 99p^3 + 39p^4 - 9p^5 + p^6 \mid 4 - 3p + p^2 \\
 & 64 \\
 \hline
 & \begin{array}{r|l}
 48 & -144p + 156p^2 - 99p^3 \\
 48 - 36p + 9p^2 & -144p + 108p^2 - 27p^3 \\
 \hline
 48 - 72p + 27p^2 & 48p^2 - 72p^3 + 39p^4 - 9p^5 + p^6 \\
 48 - 72p + 39p^2 - 9p^3 + p^4 & 48p^2 - 72p^3 + 39p^4 - 9p^5 + p^6
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 17. & \frac{a^3b^3x^9}{c^3} - \frac{3a^2bx^8}{c} + \frac{3acx^7}{b} - \frac{c^3x^6}{b^3} \mid \frac{abx^3}{c} - \frac{cx^2}{b} \\
 & \frac{a^3b^3x^9}{c^3} \\
 \hline
 & \begin{array}{r|l}
 \frac{3a^2b^2x^6}{c^2} & -\frac{3a^2bx^8}{c} + \frac{3acx^7}{b} - \frac{c^3x^6}{b^3} \\
 \frac{3a^2b^2x^6}{c^2} - 3ax^5 + \frac{c^2x^4}{b^2} & -\frac{3a^2bx^8}{c} + \frac{3acx^7}{b} - \frac{c^3x^6}{b^3}
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 18. & \frac{x^6 + 6x^4 + 15x^2 + 20}{x^6} + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6} \mid x^2 + 2 + \frac{1}{x^2} \\
 & \begin{array}{r|l}
 3x^4 & 6x^4 + 15x^2 + 20 \\
 3x^4 + 6x^2 + 4 & 6x^4 + 12x^2 + 8 \\
 \hline
 3x^4 + 12x^2 + 12 & 3x^2 + 12 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6} \\
 3x^4 + 12x^2 + 15 + \frac{6}{x^2} + \frac{1}{x^4} & 3x^2 + 12 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}
 \end{array}
 \end{array}$$

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$$\begin{array}{r|l}
 3. & 29 \cdot 791 \mid 31 \\
 & 27 \\
 \hline
 & \begin{array}{r|l}
 3(30)^2 = 2700 & 2 \ 791 \\
 3(30 \times 1) = 90 & \\
 1^2 = 1 & \\
 \hline
 & 2791 \mid 2 \ 791
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 4. & 54 \cdot 872 \mid 38 \\
 & 27 \\
 \hline
 & \begin{array}{r|l}
 3(30)^2 = 2700 & 27 \ 872 \\
 3(30 \times 8) = 720 & \\
 8^2 = 64 & \\
 \hline
 & 3484 \mid 27 \ 872
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 5. & 110 \cdot 592 \mid 48 \\
 & 64 \\
 \hline
 & \begin{array}{r|l}
 3(40)^2 = 4800 & 46 \ 592 \\
 3(40 \times 8) = 960 & \\
 8^2 = 64 & \\
 \hline
 & 5824 \mid 46 \ 592
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 6. & 300 \cdot 763 \mid 67 \\
 & 216 \\
 \hline
 & \begin{array}{r|l}
 3(60)^2 = 10800 & 84 \ 763 \\
 3(60 \times 7) = 1260 & \\
 7^2 = 49 & \\
 \hline
 & 12109 \mid 84 \ 763
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 7. & 681 \cdot 472 \mid 88 \\
 & 512 \\
 \hline
 & \begin{array}{r|l}
 3(80)^2 = 19200 & 169 \ 472 \\
 3(80 \times 8) = 1920 & \\
 8^2 = 64 & \\
 \hline
 & 21184 \mid 169 \ 472
 \end{array}
 \end{array}$$

$$\begin{array}{r|l}
 8. & 941 \cdot 192 \mid 98 \\
 & 729 \\
 \hline
 & \begin{array}{r|l}
 3(90)^2 = 24300 & 212 \ 192 \\
 3(90 \times 8) = 2160 & \\
 8^2 = 64 & \\
 \hline
 & 26524 \mid 212 \ 192
 \end{array}
 \end{array}$$

| 9. | 2.406.104 134 | 10. | 69.426.531 411 |
|--------------------------|-----------------|--------------------------|------------------|
| | 1 | | 64 |
| $3(10)^2 = 300$ | 1 406 | $3(40)^2 = 4800$ | 5 426 |
| $3(10 \times 3) = 90$ | | $3(40 \times 1) = 120$ | |
| $3^2 = 9$ | | $1^2 = 1$ | |
| | 399 1 197 | | 4921 4 921 |
| $3(130)^2 = 50700$ | 209 104 | $3(410)^2 = 504300$ | 505 531 |
| $3(130 \times 4) = 1560$ | | $3(410 \times 1) = 1230$ | |
| $4^2 = 16$ | | $1^2 = 1$ | |
| | 52276 209 104 | | 505531 505 531 |

11.

| | |
|--------------------------|------------------|
| | 28.372.625 305 |
| | 27 |
| $3(30)^2 = 2700$ | 1 372 |
| $3(300)^2 = 270000$ | 1 372 625 |
| $3(300 \times 5) = 4500$ | |
| $5^2 = 25$ | |
| | 274525 1 372 625 |

12.

| | |
|--------------------------|-------------------|
| | 48.228.544 3.64 |
| | 27 |
| $3(30)^2 = 2700$ | 21 228 |
| $3(30 \times 6) = 540$ | |
| $6^2 = 36$ | |
| | 3276 19 656 |
| $3(360)^2 = 388800$ | 1 572 544 |
| $3(360 \times 4) = 4320$ | |
| $4^2 = 16$ | |
| | 393136 1 572 544 |

13.

| | |
|--------------------------|-------------------|
| | 17.173.512 25.8 |
| | 8 |
| $3(20)^2 = 1200$ | 9 173 |
| $3(20 \times 5) = 300$ | |
| $5^2 = 25$ | |
| | 1525 7 625 |
| $3(250)^2 = 187500$ | 1 548 512 |
| $3(250 \times 8) = 6000$ | |
| $8^2 = 64$ | |
| | 193564 1 548 512 |

14.

| | |
|--------------------------|-------------------|
| | 95.443.993 4.57 |
| | 64 |
| $3(40)^2 = 4800$ | 31 443 |
| $3(40 \times 5) = 600$ | |
| $5^2 = 25$ | |
| | 5425 27 125 |
| $3(450)^2 = 607500$ | 4 318 993 |
| $3(450 \times 7) = 9450$ | |
| $7^2 = 49$ | |
| | 616999 4 318 993 |

15. See next column.

16. .001·906·624 | .124

| | | |
|-------------------|---------|---------|
| | 1 | |
| $3(10)^2$ | = 300 | 906 |
| $3(10 \times 2)$ | = 60 | |
| 2^2 | = 4 | |
| | 364 | 728 |
| $3(120)^2$ | = 43200 | 178 624 |
| $3(120 \times 4)$ | = 1440 | |
| 4^2 | = 16 | |
| | 44656 | 178 624 |

18.

15. .000·024·389 | .029

| | | |
|------------------|--------|--------|
| | 8 | |
| $3(20)^2$ | = 1200 | 16 389 |
| $3(20 \times 9)$ | = 540 | |
| 9^2 | = 81 | |
| | 1821 | 16 389 |

17. .000·912·673 | .097

| | | |
|------------------|---------|---------|
| | 729 | |
| $3(90)^2$ | = 24300 | 183 673 |
| $3(90 \times 7)$ | = 1890 | |
| 7^2 | = 49 | |
| | 26239 | 183 673 |

.259·694·072 | .638

| | | |
|-------------------|-----------|-----------|
| | 216 | |
| $3(60)^2$ | = 10800 | 43 694 |
| $3(60 \times 3)$ | = 540 | |
| 3^2 | = 9 | |
| | 11349 | 34 047 |
| $3(630)^2$ | = 1190700 | 9 647 072 |
| $3(630 \times 8)$ | = 15120 | |
| 8^2 | = 64 | |
| | 1205884 | 9 647 072 |

19.

926·859·375 | 9.75

| | | |
|-------------------|-----------|------------|
| | 729 | |
| $3(90)^2$ | = 24300 | 197 859 |
| $3(90 \times 7)$ | = 1890 | |
| 7^2 | = 49 | |
| | 26239 | 183 673 |
| $3(970)^2$ | = 2822700 | 14 186 375 |
| $3(970 \times 5)$ | = 14550 | |
| 5^2 | = 25 | |
| | 2837275 | 14 186 375 |

20.

514·500·058·197 | 80.18

| | | |
|--------------------|-------------|-------------|
| | 512 | |
| $3(80)^2$ | = 19200 | 2 500 |
| $3(800)^2$ | = 1920000 | 2 500 058 |
| $3(800 \times 1)$ | = 2400 | |
| 1^2 | = 1 | |
| | 1922401 | 1 922 401 |
| $3(8010)^2$ | = 192480300 | 577 657 197 |
| $3(8010 \times 3)$ | = 72090 | |
| 3^2 | = 9 | |
| | 192552399 | 577 657 197 |

21.

| | | | | |
|--------------------|---|---------|---------------|--------------|
| | | | 2.000.000.000 | <u>1.259</u> |
| | | | 1 | |
| $3(10)^2$ | = | 300 | 1 000 | |
| $3(10 \times 2)$ | = | 60 | | |
| 2^2 | = | 4 | | |
| | | | 364 | 728 |
| $3(120)^2$ | = | 43200 | 272 000 | |
| $3(120 \times 5)$ | = | 1800 | | |
| 5^2 | = | 25 | | |
| | | | 45025 | 225 125 |
| $3(1250)^2$ | = | 4687500 | 46 875 000 | |
| $3(1250 \times 9)$ | = | 33750 | | |
| 9^2 | = | 81 | | |
| | | | 4721331 | 42 491 979 |

22.

| | | | | |
|--------------------|---|---------|---------------|--------------|
| | | | 5.000.000.000 | <u>1.709</u> |
| | | | 1 | |
| $3(10)^2$ | = | 300 | 4 000 | |
| $3(10 \times 7)$ | = | 210 | | |
| 7^2 | = | 49 | | |
| | | | 559 | 3 913 |
| $3(170)^2$ | = | 86700 | 87 000 | |
| $3(1700)^2$ | = | 8670000 | 87 000 000 | |
| $3(1700 \times 9)$ | = | 45900 | | |
| 9^2 | = | 81 | | |
| | | | 8715981 | 78 443 829 |

23.

| | | | | |
|-------------------|---|---------|--------------|-------------|
| | | | .800.000.000 | <u>.928</u> |
| | | | 729 | |
| $3(90)^2$ | = | 24300 | 71 000 | |
| $3(90 \times 2)$ | = | 540 | | |
| 2^2 | = | 4 | | |
| | | | 24844 | 49 688 |
| $3(920)^2$ | = | 2539200 | 21 312 000 | |
| $3(920 \times 8)$ | = | 22080 | | |
| 8^2 | = | 64 | | |
| | | | 2561344 | 20 490 752 |

24.

| | | | | |
|-------------------|---|--------|--------------|-------------|
| | | | .160.000.000 | <u>.542</u> |
| | | | 125 | |
| $3(50)^2$ | = | 7500 | 35 000 | |
| $3(50 \times 4)$ | = | 600 | | |
| 4^2 | = | 16 | | |
| | | | 8116 | 32 464 |
| $3(540)^2$ | = | 874800 | 2 536 000 | |
| $3(540 \times 2)$ | = | 3240 | | |
| 2^2 | = | 4 | | |
| | | | 878044 | 1 756 088 |

25. Since, Ex. 22, the cube root of 5 is 1.709+, and since the cube root of 64 is 4, the cube root of $\frac{5}{64}$ to three decimal places is $\frac{1.709}{4}$, or .427.

26. $\frac{2}{3} = .66666666 +$.

$$\begin{array}{r}
 .666-666-666 + \underline{.873} \\
 512 \\
 \hline
 3(80)^2 = 19200 \quad 154 \ 666 \\
 3(80 \times 7) = 1680 \\
 7^2 = 49 \\
 \hline
 20929 \quad 146 \ 503 \\
 3(870)^2 = 2270700 \quad 8 \ 163 \ 666 \\
 3(870 \times 3) = 7830 \\
 3^2 = 9 \\
 \hline
 2278539 \quad 6 \ 835 \ 617
 \end{array}$$

27. $\frac{7}{8} = .875$.

$$\begin{array}{r}
 .875-000-000 \underline{.956} \\
 729 \\
 \hline
 3(90)^2 = 24300 \quad 146 \ 000 \\
 3(90 \times 5) = 1350 \\
 5^2 = 25 \\
 \hline
 25675 \quad 128 \ 375 \\
 3(950)^2 = 2707500 \quad 17 \ 625 \ 000 \\
 3(950 \times 6) = 17100 \\
 6^2 = 36 \\
 \hline
 2724636 \quad 16 \ 347 \ 816
 \end{array}$$

28. $\frac{1}{16} = .1875$.

$$\begin{array}{r}
 .187-500-000 \underline{.572} \\
 125 \\
 \hline
 3(50)^2 = 7500 \quad 62 \ 500 \\
 3(50 \times 7) = 1050 \\
 7^2 = 49 \\
 \hline
 8599 \quad 60 \ 193 \\
 3(570)^2 = 974700 \quad 2 \ 307 \ 000 \\
 3(570 \times 2) = 3420 \\
 2^2 = 4 \\
 \hline
 978124 \quad 1 \ 956 \ 248
 \end{array}$$

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1. $16 - 32x + 24x^2 - 8x^3 + x^4 \mid 4 - 4x + x^2$

$$\begin{array}{r}
 16 \\
 \hline
 8 \quad \mid -32x + 24x^2 \\
 8 - 4x \quad \mid -32x + 16x^2 \\
 \hline
 8 - 8x \quad \mid 8x^2 - 8x^3 + x^4 \\
 8 - 8x + x^2 \quad \mid 8x^2 - 8x^3 + x^4
 \end{array}$$

$$\begin{array}{r}
 4 - 4x + x^2 \mid 2 - x \\
 4 \\
 \hline
 4 \quad \mid -4x + x^2 \\
 4 - x \quad \mid -4x + x^2
 \end{array}$$

Hence, the fourth root of $16 - 32x + 24x^2 - 8x^3 + x^4$ is $2 - x$.

2. $x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4 \mid x^2 + 6xy + 9y^2$

$$\begin{array}{r}
 2x^2 \\
 \hline
 2x^2 + 6xy \mid 12x^3y + 54x^2y^2 \\
 2x^2 + 6xy \mid 12x^3y + 36x^2y^2 \\
 \hline
 2x^2 + 12xy \mid 18x^2y^2 + 108xy^3 + 81y^4 \\
 2x^2 + 12xy + 9y^2 \mid 18x^2y^2 + 108xy^3 + 81y^4
 \end{array}$$

$$\begin{array}{r}
 x^2 + 6xy + 9y^2 \mid x + 3y \\
 x^2 \\
 \hline
 2x \mid 6xy + 9y^2 \\
 2x + 3y \mid 6xy + 9y^2
 \end{array}$$

Hence, the fourth root of $x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4$ is $x + 3y$.

3. $16m^4 - 32m^3 + 24m^2 - 8m + 1 \mid 4m^2 - 4m + 1$

$$\begin{array}{r}
 16m^4 \\
 \hline
 8m^2 \quad \mid -32m^3 + 24m^2 \\
 8m^2 - 4m \quad \mid -32m^3 + 16m^2 \\
 \hline
 8m^2 - 8m \quad \mid 8m^2 - 8m + 1 \\
 8m^2 - 8m + 1 \quad \mid 8m^2 - 8m + 1
 \end{array}$$

$$\begin{array}{r}
 4m^2 - 4m + 1 \mid 2m - 1 \\
 4m^2 \\
 \hline
 4m \quad \mid -4m + 1 \\
 4m - 1 \quad \mid -4m + 1
 \end{array}$$

Hence, the fourth root of $16m^4 - 32m^3 + 24m^2 - 8m + 1$ is $2m - 1$.

$$4. \quad \frac{32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1}{32x^5} \mid \frac{2x + 1}{2x + 1}$$

$$\begin{array}{l} 5(2x)^4 = 80x^4 \\ 5(2x)^4 + 10(2x)^3(1) + 10(2x)^2(1)^2 + 5(2x)(1)^3 + (1)^4 \\ = 80x^4 + 80x^3 + 40x^2 + 10x + 1 \end{array}$$

$$5. \quad \frac{a^{10} + 15a^8 + 90a^6 + 270a^4 + 405a^2 + 243}{a^{10}} \mid \frac{a^2 + 3}{a^2 + 3}$$

$$\begin{array}{l} 5(a^2)^4 = 5a^8 \\ 5(a^2)^4 + 10(a^2)^3(3) + 10(a^2)^2(3)^2 + 5(a^2)(3)^3 + (3)^4 \\ = 5a^8 + 30a^6 + 90a^4 + 135a^2 + 81 \end{array}$$

6. See next page.

$$7. \quad \frac{64x^8 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729}{64x^8} \mid \frac{4x^2 - 12x + 9}{4x^2 - 12x + 9}$$

$$\begin{array}{l} 48x^4 \\ 48x^4 - 144x^3 + 144x^2 \\ 48x^4 - 288x^3 + 432x^2 \\ 48x^4 - 288x^3 + 540x^2 - 324x + 81 \end{array} \mid \begin{array}{l} -576x^5 + 2160x^4 - 4320x^3 \\ -576x^5 + 1728x^4 - 1728x^3 \\ 432x^4 - 2592x^3 + 4860x^2 - 2916x + 729 \\ 432x^4 - 2592x^3 + 4860x^2 - 2916x + 729 \end{array}$$

The square root of $4x^2 - 12x + 9$ is found to be $2x - 3$.

Hence, the sixth root of $64x^8 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$ is $2x - 3$.

$$8. \quad \frac{x^6 + 6acx^5 + 15a^2c^2x^4 + 20a^3c^3x^3 + 15a^4c^4x^2 + 6a^5c^5x + a^6c^6}{x^6} \mid \frac{x^2 + 2acx + a^2c^2}{x^2 + 2acx + a^2c^2}$$

$$\begin{array}{l} 3x^4 \\ 3x^4 + 6acx^3 + 4a^2c^2x^2 \\ 3x^4 + 12acx^3 + 12a^2c^2x^2 \\ 3x^4 + 12acx^3 + 15a^2c^2x^2 + 6a^3c^3x + a^4c^4 \end{array} \mid \begin{array}{l} 6acx^5 + 15a^2c^2x^4 + 20a^3c^3x^3 \\ 6acx^5 + 12a^2c^2x^3 + 8a^3c^3x^2 \\ 3a^2c^2x^4 + 12a^3c^3x^3 + 15a^4c^4x^2 + 6a^5c^5x + a^6c^6 \\ 3a^2c^2x^4 + 12a^3c^3x^3 + 15a^4c^4x^2 + 6a^5c^5x + a^6c^6 \end{array}$$

The square root of $x^2 + 2acx + a^2c^2$ is found to be $x + ac$.

Hence, the sixth root of $x^6 + 6acx^5 + 15a^2c^2x^4 + 20a^3c^3x^3 + 15a^4c^4x^2 + 6a^5c^5x + a^6c^6$ is $x + ac$.

$$\begin{array}{r|l}
 6. & x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64 \quad | \quad x^2 - 4x + 4 \\
 & x^6 \\
 \hline
 3x^4 & -12x^5 + 60x^4 - 160x^3 \\
 3x^4 - 12x^3 + 16x^2 & -12x^5 + 48x^4 - 64x^3 \\
 \hline
 3x^4 - 24x^3 + 48x^2 & 12x^4 - 96x^3 + 240x^2 - 192x + 64 \\
 3x^4 - 24x^3 + 60x^2 - 48x + 16 & 12x^4 - 96x^3 + 240x^2 - 192x + 64 \\
 \hline
 \end{array}$$

The square root of $x^2 - 4x + 4$ is found to be $x - 2$.

Hence, the sixth root of $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$ is $x - 2$.

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$$\begin{array}{l}
 2. \quad 3375 = 3^3 \times 5^3; \\
 \therefore \sqrt[3]{3375} = 3 \times 5 = 15.
 \end{array}$$

$$\begin{array}{l}
 3. \quad 1296 = 2^4 \times 3^4; \\
 \therefore \sqrt[4]{1296} = 2 \times 3 = 6.
 \end{array}$$

$$\begin{array}{l}
 4. \quad 531441 = 3^6 \times 3^6; \\
 \therefore \sqrt[6]{531441} = 3 \times 3 = 9.
 \end{array}$$

$$\begin{array}{l}
 5. \quad 759375 = 3^5 \times 5^5; \\
 \therefore \sqrt[5]{759375} = 3 \times 5 = 15.
 \end{array}$$

$$\begin{array}{l}
 6. \quad 4084101 = 3^5 \times 7^5; \\
 \therefore \sqrt[5]{4084101} = 3 \times 7 = 21.
 \end{array}$$

$$\begin{array}{l}
 7. \quad 262144 = 2^6 \times 2^6 \times 2^6; \\
 \therefore \sqrt[6]{262144} = 2 \times 2 \times 2 = 8.
 \end{array}$$

$$\begin{array}{r|l}
 9. & 5.06.25 \quad | \quad 225 \\
 & 4 \\
 \hline
 20 \times 2 = 40 & 1 \quad 06 \\
 40 + 2 = 42 & 84 \\
 \hline
 220 \times 2 = 440 & 22 \quad 25 \\
 440 + 5 = 445 & 22 \quad 25 \\
 \hline
 \end{array}$$

$$\begin{array}{r|l}
 & 2.25 \quad | \quad 15 \\
 & 1 \\
 \hline
 10 \times 2 = 20 & 1 \quad 25 \\
 20 + 5 = 25 & 1 \quad 25 \\
 \hline
 \end{array}$$

Hence, $\sqrt[4]{50625} = 15$.

$$\begin{array}{r|l}
 10. & 4.66.56 \quad | \quad 216 \\
 & 4 \\
 \hline
 20 \times 2 = 40 & 66 \\
 40 + 1 = 41 & 41 \\
 \hline
 210 \times 2 = 420 & 25 \quad 56 \\
 420 + 6 = 426 & 25 \quad 56 \\
 \hline
 \end{array}$$

$$\begin{array}{l}
 \sqrt[3]{216} = 6. \\
 \text{Hence, } \sqrt[6]{46656} = 6.
 \end{array}$$

$$\begin{array}{r|l}
 11. & 53.14.41 \quad | \quad 729 \\
 & 49 \\
 \hline
 70 \times 2 = 140 & 4 \quad 14 \\
 140 + 2 = 142 & 2 \quad 84 \\
 \hline
 720 \times 2 = 1440 & 1 \quad 30 \quad 41 \\
 1440 + 9 = 1449 & 1 \quad 30 \quad 41 \\
 \hline
 \end{array}$$

$$\begin{array}{l}
 \sqrt[3]{729} = 9. \\
 \text{Hence, } \sqrt[6]{531441} = 9.
 \end{array}$$

$$\begin{array}{r|l}
 12. & 5.76.48.01 \quad | \quad 2401 \\
 & 4 \\
 \hline
 20 \times 2 = 40 & 1 \quad 76 \\
 40 + 4 = 44 & 1 \quad 76 \\
 \hline
 240 \times 2 = 480 & 48 \\
 2400 \times 2 = 4800 & 48 \quad 01 \\
 4800 + 1 = 4801 & 48 \quad 01 \\
 \hline
 \end{array}$$

$$\begin{array}{r|l}
 & 24.01 \quad | \quad 49 \\
 & 16 \\
 \hline
 40 \times 2 = 80 & 8 \quad 01 \\
 80 + 9 = 89 & 8 \quad 01 \\
 \hline
 \sqrt{49} = 7. \\
 \text{Hence, } \sqrt[8]{5764801} = 7.
 \end{array}$$

13. See next page.

14.

| | | | |
|--------------------|---|----------|-----------------------|
| | | | 10·604·499·373 2197 |
| | | 8 | |
| $3(20)^2$ | = | 1200 | 2 604 |
| $3(20 \times 1)$ | = | 60 | |
| 1^2 | = | 1 | |
| | | 1261 | 1 261 |
| $3(210)^2$ | = | 132300 | 1 343 499 |
| $3(210 \times 9)$ | = | 5670 | |
| 9^2 | = | 81 | |
| | | 138051 | 1 242 459 |
| $3(2190)^2$ | = | 14388300 | 101 040 373 |
| $3(2190 \times 7)$ | = | 45990 | |
| 7^2 | = | 49 | |
| | | 14434339 | 101 040 373 |

| | | |
|------------------|---|-------------|
| | | 2·197 13 |
| | | 1 |
| $3(10)^2$ | = | 300 1 197 |
| $3(10 \times 3)$ | = | 90 |
| 3^2 | = | 9 |
| | | 399 1 197 |

Hence, $\sqrt[9]{10604499373} = 13$.

2.

$$\begin{array}{r|l}
 x^4 + 6x^3 + 11x^2 + 6x - 8 & x^2 + 3x + 1 \\
 \hline
 x^4 & \\
 \hline
 2x^2 & 6x^3 + 11x^2 \\
 2x^2 + 3x & 6x^3 + 9x^2 \\
 \hline
 2x^2 + 6x & 2x^2 + 6x - 8 \\
 2x^2 + 6x + 1 & 2x^2 + 6x + 1 \\
 \hline
 & -9
 \end{array}$$

Therefore, $x^4 + 6x^3 + 11x^2 + 6x - 8$
 $= (x^2 + 3x + 1)^2 - 9$
 $= (x^2 + 3x + 1 + 3)(x^2 + 3x + 1 - 3)$
 $= (x^2 + 3x + 4)(x^2 + 3x - 2).$

3.

$$\begin{array}{r|l}
 x^6 + 2x^5 + 5x^4 + 8x^3 + 8x^2 + 8x + 3 & x^3 + x^2 + 2x + 2 \\
 \hline
 x^6 & \\
 \hline
 2x^5 & 2x^5 + 5x^4 \\
 2x^5 + x^2 & 2x^5 + x^4 \\
 \hline
 2x^3 + 2x^2 & 4x^4 + 8x^3 + 8x^2 \\
 2x^3 + 2x^2 + 2x & 4x^4 + 4x^3 + 4x^2 \\
 \hline
 2x^3 + 2x^2 + 4x & 4x^3 + 4x^2 + 8x + 3 \\
 2x^3 + 2x^2 + 4x + 2 & 4x^3 + 4x^2 + 8x + 4 \\
 \hline
 & -1
 \end{array}$$

Therefore, $x^6 + 2x^5 + 5x^4 + 8x^3 + 8x^2 + 8x + 3$
 $= (x^3 + x^2 + 2x + 2)^2 - 1$
 $= (x^3 + x^2 + 2x + 2 + 1)(x^3 + x^2 + 2x + 2 - 1)$
 $= (x^3 + x^2 + 2x + 3)(x^3 + x^2 + 2x + 1).$

13. $24 \cdot 13 \cdot 75 \cdot 69 \mid 4913$

$4 \cdot 913 \mid 17$

| | |
|------------------------|---------------|
| 16 | |
| $40 \times 2 = 80$ | $8 \ 13$ |
| $80 + 9 = 89$ | $8 \ 01$ |
| $490 \times 2 = 980$ | $12 \ 75$ |
| $980 + 1 = 981$ | $9 \ 81$ |
| $4910 \times 2 = 9820$ | $2 \ 94 \ 69$ |
| $9820 + 3 = 9823$ | $2 \ 94 \ 69$ |

| | |
|-------------------------|-----------------|
| $3 (10)^2 = 300$ | $3 \ 913$ |
| $3 (10 \times 7) = 210$ | |
| $7^2 = 49$ | |
| | $559 \ 3 \ 913$ |

Hence, $\sqrt[6]{24137569} = 17$.

4.
$$\frac{x^6 - 4x^5 + 6x^4 + 6x^3 - 19x^2 + 10x + 9}{x^6} \mid x^3 - 2x^2 + x + 5$$

| | |
|------------------------|----------------------------|
| $2x^3$ | $-4x^5 + 6x^4$ |
| $2x^3 - 2x^2$ | $-4x^5 + 4x^4$ |
| $2x^3 - 4x^2$ | $2x^4 + 6x^3 - 19x^2$ |
| $2x^3 - 4x^2 + x$ | $2x^4 - 4x^3 + x^2$ |
| $2x^3 - 4x^2 + 2x$ | $10x^3 - 20x^2 + 10x + 9$ |
| $2x^3 - 4x^2 + 2x + 5$ | $10x^3 - 20x^2 + 10x + 25$ |
| -16 | |

Therefore, $x^6 - 4x^5 + 6x^4 + 6x^3 - 19x^2 + 10x + 9$
 $= (x^3 - 2x^2 + x + 5)^2 - 16$
 $= (x^3 - 2x^2 + x + 5 + 4)(x^3 - 2x^2 + x + 5 - 4)$
 $= (x^3 - 2x^2 + x + 9)(x^3 - 2x^2 + x + 1).$

5.
$$\frac{4x^6 + 12x^5 + 25x^4 + 40x^3 + 40x^2 + 32x + 15}{4x^6} \mid 2x^3 + 3x^2 + 4x + 4$$

| | |
|------------------------|----------------------------|
| $4x^3$ | $12x^5 + 25x^4$ |
| $4x^3 + 3x^2$ | $12x^5 + 9x^4$ |
| $4x^3 + 6x^2$ | $16x^4 + 40x^3 + 40x^2$ |
| $4x^3 + 6x^2 + 4x$ | $16x^4 + 24x^3 + 16x^2$ |
| $4x^3 + 6x^2 + 8x$ | $16x^3 + 24x^2 + 32x + 15$ |
| $4x^3 + 6x^2 + 8x + 4$ | $16x^3 + 24x^2 + 32x + 16$ |
| -1 | |

Therefore, $4x^6 + 12x^5 + 25x^4 + 40x^3 + 40x^2 + 32x + 15$
 $= (2x^3 + 3x^2 + 4x + 4)^2 - 1$
 $= (2x^3 + 3x^2 + 4x + 4 + 1)(2x^3 + 3x^2 + 4x + 4 - 1)$
 $= (2x^3 + 3x^2 + 4x + 5)(2x^3 + 3x^2 + 4x + 3).$

THEORY OF EXPONENTS

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11. $8^{\frac{1}{2}} = 2.$

12. $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = 2^2 = 4.$

13. $8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}.$

14. $(-8)^{\frac{1}{3}} = -2.$

15. $(64)^{\frac{3}{2}} = (64^{\frac{1}{2}})^3 = 4^3 = 64.$

16. $32^{\frac{2}{3}} = (32^{\frac{1}{3}})^2 = 2^2 = 4.$

17. $25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = (\pm 5)^3 = \pm 125.$

18. $81^{\frac{2}{3}} = (81^{\frac{1}{3}})^2 = (\pm 3)^2 = \pm 27.$

19. $64^{-\frac{2}{3}} = \frac{1}{(64^{\frac{1}{3}})^2} = \frac{1}{4^2} = \frac{1}{16}.$

20. $(-8)^{-\frac{4}{3}} = \frac{1}{(-8)^{\frac{4}{3}}} = \frac{1}{(-2)^4} = \frac{1}{16}.$

21. $(-32)^{-\frac{2}{3}} = \frac{1}{(-32)^{\frac{2}{3}}} = \frac{1}{(-2)^3} = -\frac{1}{8}.$

22. $16^{-\frac{3}{2}} = \frac{1}{16^{\frac{3}{2}}} = \frac{1}{(\pm 2)^3} = \pm \frac{1}{8}.$

$$\begin{aligned}
 23. \quad & \sqrt[3]{x^2} + x^{\frac{1}{3}} + 8^{\frac{1}{3}} + 3x^{\frac{2}{3}} - 5\sqrt[3]{x} - \sqrt[3]{27^2} \\
 &= x^{\frac{2}{3}} + x^{\frac{1}{3}} + 4 + 3x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 3^2 \\
 &= 4x^{\frac{2}{3}} - 4x^{\frac{1}{3}} - 5.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & 4\sqrt[5]{x} + 5x^0 - 3x^{-\frac{1}{5}} + 2\sqrt[5]{x^{-1}} - 8^{\frac{2}{5}} - 2x^{\frac{1}{5}} \\
 &= 4x^{\frac{1}{5}} + 5 - 3x^{-\frac{1}{5}} + 2x^{-\frac{1}{5}} - 4 - 2x^{\frac{1}{5}} \\
 &= 2x^{\frac{1}{5}} + 1 - x^{-\frac{1}{5}}.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \sqrt[3]{a^2} - \sqrt[3]{a^2b} + \sqrt[3]{ab^2} - b + a + 4\sqrt[3]{a^2b} - 4a^{\frac{1}{3}}b^{\frac{2}{3}} + \sqrt[3]{b^3} \\
 &= a^{\frac{2}{3}} - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} - b + a + 4a^{\frac{2}{3}}b^{\frac{1}{3}} - 4a^{\frac{1}{3}}b^{\frac{2}{3}} + b \\
 &= a + a^{\frac{2}{3}} + 3a^{\frac{2}{3}}b^{\frac{1}{3}} - 3a^{\frac{1}{3}}b^{\frac{2}{3}}.
 \end{aligned}$$

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$$16. (a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) = (a^{\frac{1}{2}})^2 - (b^{\frac{1}{2}})^2 = a - b.$$

$$17. (x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} - y^{\frac{2}{3}}) = (x^{\frac{2}{3}})^2 - (y^{\frac{2}{3}})^2 = x^{\frac{4}{3}} - y^{\frac{4}{3}}.$$

$$18. (x^{-\frac{1}{2}} + 10)(x^{-\frac{1}{2}} - 1) = (x^{-\frac{1}{2}})^2 + 9x^{-\frac{1}{2}} - 10 = x^{-1} + 9x^{-\frac{1}{2}} - 10.$$

$$19. (x^{\frac{3}{2}} - 4)(x^{\frac{3}{2}} + 5) = (x^{\frac{3}{2}})^2 + x^{\frac{3}{2}} - 20 = x^{\frac{3}{2}} + x^{\frac{3}{2}} - 20.$$

$$\begin{aligned}
 20. \quad & \frac{x^{\frac{2}{3}} - y^{\frac{2}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} \\
 & \frac{x - x^{\frac{1}{3}}y^{\frac{2}{3}}}{x - x^{\frac{1}{3}}y^{\frac{2}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} - y} \\
 & \frac{x - x^{\frac{1}{3}}y^{\frac{2}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} - y}{x - x^{\frac{1}{3}}y^{\frac{2}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} - y}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & \frac{x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}}{x^{\frac{1}{3}} + y^{-\frac{1}{3}}} \\
 & \frac{x^{\frac{2}{3}} - x^{\frac{2}{3}}y^{-\frac{1}{3}} + x^{\frac{1}{3}}y^{-\frac{2}{3}}}{x^{\frac{1}{3}} + y^{-\frac{1}{3}}} \\
 & \frac{x^{\frac{2}{3}} - x^{\frac{2}{3}}y^{-\frac{1}{3}} + x^{\frac{1}{3}}y^{-\frac{2}{3}} + y^{-\frac{2}{3}}}{x^{\frac{1}{3}} + y^{-\frac{1}{3}}}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & \frac{a^{\frac{2}{3}} - b^{-\frac{2}{3}} + a^{\frac{1}{3}}b^{-\frac{1}{3}} + 1}{a^{\frac{1}{3}} - b^{\frac{1}{3}}} \\
 & \frac{a - a^{\frac{1}{3}}b^{-\frac{2}{3}} + a^{\frac{2}{3}}b^{-\frac{1}{3}} + a^{\frac{1}{3}}}{a - a^{\frac{1}{3}}b^{-\frac{2}{3}} + a^{\frac{2}{3}}b^{-\frac{1}{3}} + a^{\frac{1}{3}}} \\
 & \frac{a - a^{\frac{1}{3}}b^{-\frac{2}{3}} + a^{\frac{2}{3}}b^{-\frac{1}{3}} - a^{\frac{1}{3}}b^0 - a^{\frac{2}{3}}b^{\frac{1}{3}} + b^{-\frac{1}{3}} - b^{\frac{1}{3}}}{a - a^{\frac{1}{3}}b^{-\frac{2}{3}} + a^{\frac{2}{3}}b^{-\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{-\frac{1}{3}} - b^{\frac{1}{3}}}
 \end{aligned}$$

23.

$$\begin{aligned}
 & \frac{1 - x + x^2}{x^{-3} + x^{-2} + x^{-1}} \\
 & \frac{x^{-3} - x^{-2} + x^{-1} + x^{-2} - x^{-1} + x^0}{x^{-3} + x^{-2} + x^{-1} + x^{-2} - x^{-1} + x^0} \\
 & \frac{1 - x + x^2}{x^{-3} + x^{-2} + x^{-1} + x^{-2} - x^{-1} + x^0}
 \end{aligned}$$

24.

$$\begin{aligned}
 & \frac{a^{-1} + b^{-\frac{1}{2}} + c^{\frac{1}{2}}}{a^{-1} + b^{-\frac{1}{2}} + 2c^{\frac{1}{2}}} \\
 & \frac{a^{-2} + a^{-1}b^{-\frac{1}{2}} + a^{-1}c^{\frac{1}{2}}}{a^{-2} + a^{-1}b^{-\frac{1}{2}} + a^{-1}c^{\frac{1}{2}}} \\
 & \frac{a^{-2} + a^{-1}b^{-\frac{1}{2}} + a^{-1}c^{\frac{1}{2}} + b^{-1} + b^{-\frac{1}{2}}c^{\frac{1}{2}}}{a^{-2} + a^{-1}b^{-\frac{1}{2}} + a^{-1}c^{\frac{1}{2}} + b^{-1} + b^{-\frac{1}{2}}c^{\frac{1}{2}}} \\
 & \frac{a^{-2} + 2a^{-1}b^{-\frac{1}{2}} + 3a^{-1}c^{\frac{1}{2}} + b^{-1} + 3b^{-\frac{1}{2}}c^{\frac{1}{2}} + 2c}{a^{-2} + 2a^{-1}b^{-\frac{1}{2}} + 3a^{-1}c^{\frac{1}{2}} + b^{-1} + 3b^{-\frac{1}{2}}c^{\frac{1}{2}} + 2c}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \frac{a^2b^2 - ab^3 + b^4}{a^{-2}b^{-2} + a^{-3}b^{-1} + a^{-4}} \\
 & \frac{a^0b^0 - a^{-1}b + a^{-2}b^2}{a^{-2}b^{-2} + a^{-3}b^{-1} + a^{-4}} \\
 & \frac{1}{1 + a^{-2}b^2 + a^{-4}b^4}
 \end{aligned}$$

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$$36. (a - b) \div (a^{\frac{1}{2}} + b^{\frac{1}{2}}) = [(a^{\frac{1}{2}})^2 - (b^{\frac{1}{2}})^2] \div [(a^{\frac{1}{2}}) + (b^{\frac{1}{2}})] = a^{\frac{1}{2}} - b^{\frac{1}{2}}.$$

$$37. (a - b) \div (a^{\frac{1}{3}} - b^{\frac{1}{3}}) = [(a^{\frac{1}{3}})^3 - (b^{\frac{1}{3}})^3] \div [(a^{\frac{1}{3}}) - (b^{\frac{1}{3}})] \\ = (a^{\frac{1}{3}})^2 + a^{\frac{1}{3}}b^{\frac{1}{3}} + (b^{\frac{1}{3}})^2 = a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}.$$

$$38. (a + b) \div (a^{\frac{1}{5}} + b^{\frac{1}{5}}) = [(a^{\frac{1}{5}})^5 + (b^{\frac{1}{5}})^5] \div [(a^{\frac{1}{5}}) + (b^{\frac{1}{5}})] \\ = (a^{\frac{1}{5}})^4 - (a^{\frac{1}{5}})^3(b^{\frac{1}{5}}) + (a^{\frac{1}{5}})^2(b^{\frac{1}{5}})^2 - (a^{\frac{1}{5}})(b^{\frac{1}{5}})^3 + (b^{\frac{1}{5}})^4 \\ = a^{\frac{4}{5}} - a^{\frac{3}{5}}b^{\frac{1}{5}} + a^{\frac{2}{5}}b^{\frac{2}{5}} - a^{\frac{1}{5}}b^{\frac{3}{5}} + b^{\frac{4}{5}}.$$

$$39. \begin{array}{r} a^2 + b^2 \\ a^2 + a^{\frac{2}{3}}b^{\frac{2}{3}} \\ \hline -a^{\frac{4}{3}}b^{\frac{2}{3}} + b^2 \\ -a^{\frac{4}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}b^{\frac{4}{3}} \\ \hline a^{\frac{2}{3}}b^{\frac{4}{3}} + b^2 \\ a^{\frac{2}{3}}b^{\frac{4}{3}} + b^2 \end{array} \quad \begin{array}{r} a^{\frac{2}{3}} + b^{\frac{2}{3}} \\ a^{\frac{4}{3}} - a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}} \end{array}$$

$$40. \begin{array}{r} x - 1 \\ x + x^{\frac{2}{3}} + x^{\frac{1}{3}} \\ \hline -x^{\frac{2}{3}} - x^{\frac{1}{3}} - 1 \\ -x^{\frac{2}{3}} - x^{\frac{1}{3}} - 1 \end{array} \quad \begin{array}{r} x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1 \\ x^{\frac{1}{3}} - 1 \end{array}$$

$$42. \begin{array}{r} x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}} \\ x^{\frac{1}{2}} - x^0 \\ \hline -1 + x^{-\frac{1}{2}} \\ -x^0 + x^{-\frac{1}{2}} \end{array} \quad \begin{array}{r} x^{\frac{1}{2}} - x^{-\frac{1}{2}} \\ x^{\frac{1}{2}} - x^{-\frac{1}{2}} \end{array}$$

$$41. \begin{array}{r} a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{-\frac{1}{3}} + b^{-1} \\ a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{-\frac{1}{3}} \\ \hline a^{\frac{1}{3}}b^{-\frac{1}{3}} + b^{-1} \\ a^{\frac{1}{3}}b^{-\frac{1}{3}} + b^{-1} \end{array} \quad \begin{array}{r} a^{\frac{1}{3}} + b^{-\frac{1}{3}} \\ a^{\frac{1}{3}} + b^{-\frac{1}{3}} \end{array}$$

$$43. \begin{array}{r} x^{-2} - 4x^{-1} + 3 \\ x^{-2} - 3x^{-1} \\ \hline -x^{-1} + 3 \\ -x^{-1} + 3 \end{array} \quad \begin{array}{r} x^{-1} - 3 \\ x^{-1} - 1 \end{array}$$

$$44. \begin{array}{r} 4x^2y^{-1} - 5y + x^{-2}y^3 \\ 4x^2y^{-1} - 4y \\ \hline -y + x^{-2}y^3 \\ -x^0y + x^{-2}y^3 \end{array} \quad \begin{array}{r} x^2 - y^2 \\ 4y^{-1} - x^{-2}y \end{array}$$

$$45. \begin{array}{r} a^2 - b^3 \\ a^2 + a^{\frac{2}{3}}b^{\frac{2}{3}} \\ \hline -a^{\frac{2}{3}}b^{\frac{2}{3}} - b^3 \\ -a^{\frac{2}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}b \\ \hline a^{\frac{2}{3}}b - b^3 \\ a^{\frac{2}{3}}b + ab^{\frac{2}{3}} \\ \hline -ab^{\frac{2}{3}} - b^3 \\ -ab^{\frac{2}{3}} - a^{\frac{2}{3}}b^2 \\ \hline a^{\frac{2}{3}}b^2 - b^3 \\ a^{\frac{2}{3}}b^2 + a^{\frac{1}{3}}b^{\frac{2}{3}} \\ \hline -a^{\frac{1}{3}}b^{\frac{2}{3}} - b^3 \\ -a^{\frac{1}{3}}b^{\frac{2}{3}} - b^3 \end{array} \quad \begin{array}{r} a^{\frac{1}{3}} + b^{\frac{1}{3}} \\ a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + ab - a^{\frac{2}{3}}b^{\frac{2}{3}} + a^{\frac{1}{3}}b^2 - b^{\frac{5}{3}} \end{array}$$

$$55. (a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 = (a^{\frac{1}{2}})^2 - 2(a^{\frac{1}{2}})(b^{\frac{1}{2}}) + (b^{\frac{1}{2}})^2 = a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b.$$

$$56. (a^{\frac{1}{2}} + b^{\frac{1}{2}})^3 = (a^{\frac{1}{2}})^3 + 3(a^{\frac{1}{2}})^2(b^{\frac{1}{2}}) + 3(a^{\frac{1}{2}})(b^{\frac{1}{2}})^2 + (b^{\frac{1}{2}})^3 \\ = a^{\frac{3}{2}} + 3ab^{\frac{1}{2}} + 3a^{\frac{1}{2}}b^{\frac{3}{2}} + b.$$

$$57. (a^{-1} - b^{\frac{2}{3}})^3 = (a^{-1})^3 - 3(a^{-1})^2(b^{\frac{2}{3}}) + 3(a^{-1})(b^{\frac{2}{3}})^2 - (b^{\frac{2}{3}})^3 \\ = a^{-3} - 3a^{-2}b^{\frac{2}{3}} + 3a^{-1}b^{\frac{4}{3}} - b^2.$$

$$58. (x^{-\frac{1}{2}} - y^{\frac{1}{3}})^4 = (x^{-\frac{1}{2}})^4 - 4(x^{-\frac{1}{2}})^3(y^{\frac{1}{3}}) + 6(x^{-\frac{1}{2}})^2(y^{\frac{1}{3}})^2 - 4(x^{-\frac{1}{2}})(y^{\frac{1}{3}})^3 + (y^{\frac{1}{3}})^4 \\ = x^{-2} - 4x^{-\frac{3}{2}}y^{\frac{1}{3}} + 6x^{-1}y - 4x^{-\frac{1}{2}}y^{\frac{2}{3}} + y^{\frac{4}{3}}.$$

$$59. (a^{-\frac{1}{2}} + \frac{1}{2})^3 = (a^{-\frac{1}{2}})^3 + 3(a^{-\frac{1}{2}})^2(\frac{1}{2}) + 3(a^{-\frac{1}{2}})(\frac{1}{2})^2 + (\frac{1}{2})^3 \\ = a^{-\frac{3}{2}} + \frac{3}{2}a^{-1} + \frac{3}{4}a^{-\frac{1}{2}} + \frac{1}{8}.$$

$$60. (1 - x^{\frac{2}{3}})^4 = (1)^4 - 4(1)^3(x^{\frac{2}{3}}) + 6(1)^2(x^{\frac{2}{3}})^2 - 4(1)(x^{\frac{2}{3}})^3 + (x^{\frac{2}{3}})^4 \\ = 1 - 4x^{\frac{2}{3}} + 6x^{\frac{4}{3}} - 4x^{\frac{6}{3}} + x^{\frac{8}{3}}.$$

$$61. \sqrt[4]{x^{\frac{2}{3}}} = (x^{\frac{2}{3}})^{\frac{1}{4}} = x^{\frac{1}{6}}.$$

$$64. \sqrt[3]{a^{-\frac{1}{2}}b^{-3}} = (a^{-\frac{1}{2}}b^{-3})^{\frac{1}{3}} = a^{-\frac{1}{6}}b^{-1}.$$

$$62. \sqrt[4]{a^{-\frac{1}{2}}} = (a^{-\frac{1}{2}})^{\frac{1}{4}} = a^{-\frac{1}{8}}.$$

$$65. \sqrt{x^{\frac{4}{3}}y^{-8}} = (x^{\frac{4}{3}}y^{-8})^{\frac{1}{2}} = x^{\frac{2}{3}}y^{-4}.$$

$$63. \sqrt[4]{a^{-\frac{3}{2}}} = (a^{-\frac{3}{2}})^{\frac{1}{4}} = a^{-\frac{3}{8}}.$$

$$66. \sqrt[n]{a^x b^y} = (a^x b^y)^{\frac{1}{n}} = a^{\frac{x}{n}} b^{\frac{y}{n}}.$$

$$67. (\frac{1}{8}\frac{1}{3}a^{-\frac{2}{3}}b^{\frac{1}{3}})^{-\frac{2}{3}} = (\frac{1}{8}\frac{1}{3})^{-\frac{2}{3}}(a^{-\frac{2}{3}})^{-\frac{2}{3}}(b^{\frac{1}{3}})^{-\frac{2}{3}} = 216ab^{-\frac{2}{3}}.$$

$$68. (\frac{1}{9}m^{-1}n^{-\frac{1}{2}})^{\frac{1}{2}} = (\frac{1}{9})^{\frac{1}{2}}(m^{-1})^{\frac{1}{2}}(n^{-\frac{1}{2}})^{\frac{1}{2}} = \frac{1}{3}m^{-\frac{1}{2}}n^{-\frac{1}{4}}.$$

$$69. (4x^{2n}y^{-3}z^4)^{\frac{2}{3}} = 4^{\frac{2}{3}}(x^{2n})^{\frac{2}{3}}(y^{-3})^{\frac{2}{3}}(z^4)^{\frac{2}{3}} = 32x^{\frac{4n}{3}}y^{-2}z^{\frac{8}{3}}.$$

$$70. \frac{x^2 + 2x^{\frac{3}{2}} + 3x + 4x^{\frac{1}{2}} + 3 + 2x^{-\frac{1}{2}} + x^{-1}}{x^2} \left| \frac{x + x^{\frac{1}{2}} + 1 + x^{-\frac{1}{2}}}{x^2} \right.$$

$$\begin{array}{l|l} 2x & 2x^{\frac{3}{2}} + 3x \\ 2x + x^{\frac{1}{2}} & 2x^{\frac{3}{2}} + x \end{array}$$

$$\begin{array}{l|l} 2x + 2x^{\frac{1}{2}} & 2x + 4x^{\frac{1}{2}} + 3 \\ 2x + 2x^{\frac{1}{2}} + 1 & 2x + 2x^{\frac{1}{2}} + 1 \end{array}$$

$$\begin{array}{l|l} 2x + 2x^{\frac{1}{2}} + 2 & 2x^{\frac{1}{2}} + 2 + 2x^{-\frac{1}{2}} + x^{-1} \\ 2x + 2x^{\frac{1}{2}} + 2 + x^{-\frac{1}{2}} & 2x^{\frac{1}{2}} + 2 + 2x^{-\frac{1}{2}} + x^{-1} \end{array}$$

$$71. \frac{x^2 - 2xy^{\frac{1}{2}} + y + 4xz^{-1} - 4y^{\frac{1}{2}}z^{-1} + 4z^{-2}}{x^2} \left| \frac{x - y^{\frac{1}{2}} + 2z^{-1}}{x^2} \right.$$

$$\begin{array}{l|l} 2x & -2xy^{\frac{1}{2}} + y \\ 2x - y^{\frac{1}{2}} & -2xy^{\frac{1}{2}} + y \end{array}$$

$$\begin{array}{l|l} 2x - 2y^{\frac{1}{2}} & 4xz^{-1} - 4y^{\frac{1}{2}}z^{-1} + 4z^{-2} \\ 2x - 2y^{\frac{1}{2}} + 2z^{-1} & 4xz^{-1} - 4y^{\frac{1}{2}}z^{-1} + 4z^{-2} \end{array}$$

$$72. \frac{a - 4a^{\frac{1}{2}}b^{\frac{1}{2}} + 4b^{\frac{2}{3}} + 6a^{\frac{1}{3}}c^{\frac{1}{2}} - 12b^{\frac{1}{3}}c^{\frac{1}{2}} + 9c^{\frac{1}{2}}}{a} \left| \frac{a^{\frac{1}{2}} - 2b^{\frac{1}{2}} + 3c^{\frac{1}{2}}}{a} \right.$$

$$\begin{array}{l|l} 2a^{\frac{1}{2}} & -4a^{\frac{1}{2}}b^{\frac{1}{2}} + 4b^{\frac{2}{3}} \\ 2a^{\frac{1}{2}} - 2b^{\frac{1}{2}} & -4a^{\frac{1}{2}}b^{\frac{1}{2}} + 4b^{\frac{2}{3}} \end{array}$$

$$\begin{array}{l|l} 2a^{\frac{1}{2}} - 4b^{\frac{1}{2}} & 6a^{\frac{1}{3}}c^{\frac{1}{2}} - 12b^{\frac{1}{3}}c^{\frac{1}{2}} + 9c^{\frac{1}{2}} \\ 2a^{\frac{1}{2}} - 4b^{\frac{1}{2}} + 3c^{\frac{1}{2}} & 6a^{\frac{1}{3}}c^{\frac{1}{2}} - 12b^{\frac{1}{3}}c^{\frac{1}{2}} + 9c^{\frac{1}{2}} \end{array}$$

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$$73. \quad \begin{array}{r|l} a^2 + 6a^{\frac{1}{2}} + 12a^{\frac{3}{2}} + 8 & \underline{a^{\frac{3}{2}} + 2} \\ a^2 & \\ \hline 3a^{\frac{1}{2}} & \\ 3a^{\frac{1}{2}} + 6a^{\frac{3}{2}} + 4 & \left| \begin{array}{l} 6a^{\frac{3}{2}} + 12a^{\frac{3}{2}} + 8 \\ 6a^{\frac{3}{2}} + 12a^{\frac{3}{2}} + 8 \end{array} \right. \end{array}$$

$$74. \quad \begin{array}{r|l} a - 3a^{\frac{2}{3}}b^{\frac{2}{3}} + 3a^{\frac{1}{3}}b^{\frac{4}{3}} - b^2 & \underline{a^{\frac{1}{3}} - b^{\frac{2}{3}}} \\ a & \\ \hline 3a^{\frac{2}{3}} & \\ 3a^{\frac{2}{3}} - 3a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}} & \left| \begin{array}{l} -3a^{\frac{2}{3}}b^{\frac{2}{3}} + 3a^{\frac{1}{3}}b^{\frac{4}{3}} - b^2 \\ -3a^{\frac{2}{3}}b^{\frac{2}{3}} + 3a^{\frac{1}{3}}b^{\frac{4}{3}} - b^2 \end{array} \right. \end{array}$$

$$75. \quad \begin{array}{r|l} 8x^{-1} - 12x^{-\frac{2}{3}}y + 6x^{-\frac{1}{3}}y^2 - y^3 & \underline{2x^{-\frac{1}{3}} - y} \\ 8x^{-1} & \\ \hline 12x^{-\frac{2}{3}} & \\ 12x^{-\frac{2}{3}} - 6x^{-\frac{1}{3}}y + y^2 & \left| \begin{array}{l} -12x^{-\frac{2}{3}}y + 6x^{-\frac{1}{3}}y^2 - y^3 \\ -12x^{-\frac{2}{3}}y + 6x^{-\frac{1}{3}}y^2 - y^3 \end{array} \right. \end{array}$$

$$76. \quad \begin{array}{r|l} x^{\frac{3}{2}} - 6x + 15x^{\frac{1}{2}} - 20 + 15x^{-\frac{1}{2}} - 6x^{-1} + x^{-\frac{3}{2}} & \underline{x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}}} \\ x^{\frac{3}{2}} & \\ \hline 3x & \\ 3x - 6x^{\frac{1}{2}} + 4 & \left| \begin{array}{l} -6x + 15x^{\frac{1}{2}} - 20 \\ -6x + 12x^{\frac{1}{2}} - 8 \end{array} \right. \\ \hline 3x - 12x^{\frac{1}{2}} + 12 & \\ 3x - 12x^{\frac{1}{2}} + 15 - 6x^{-\frac{1}{2}} + x^{-1} & \left| \begin{array}{l} 3x^{\frac{1}{2}} - 12 + 15x^{-\frac{1}{2}} - 6x^{-1} + x^{-\frac{3}{2}} \\ 3x^{\frac{1}{2}} - 12 + 15x^{-\frac{1}{2}} - 6x^{-1} + x^{-\frac{3}{2}} \end{array} \right. \end{array}$$

$$78. \quad \S 128, \quad a^{-2} - b^{-2} = (a^{-1} + b^{-1})(a^{-1} - b^{-1}) = \left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{a} - \frac{1}{b}\right).$$

$$79. \quad \S 128, \quad 9 - x^{-2} = (3 + x^{-1})(3 - x^{-1}) = \left(3 + \frac{1}{x}\right)\left(3 - \frac{1}{x}\right).$$

$$80. \quad \S 128, \quad 16 - a^{-4} = (4 + a^{-2})(2 + a^{-1})(2 - a^{-1}) \\ = \left(4 + \frac{1}{a^2}\right)\left(2 + \frac{1}{a}\right)\left(2 - \frac{1}{a}\right).$$

$$81. \quad \S 133, \quad 27 - b^{-3} = (3 - b^{-1})(9 + 3b^{-1} + b^{-2}) \\ = \left(3 - \frac{1}{b}\right)\left(9 + \frac{3}{b} + \frac{1}{b^2}\right).$$

$$82. \quad \S 132, \quad b^{-3} + y^{-3} = (b^{-1} + y^{-1})(b^{-2} - b^{-1}y^{-1} + y^{-2}) \\ = \left(\frac{1}{b} + \frac{1}{y}\right)\left(\frac{1}{b^2} - \frac{1}{by} + \frac{1}{y^2}\right).$$

$$83. \quad \S 133, \quad x^3 - x^{-3} = (x - x^{-1})(x^2 + x^0 + x^{-2}) \\ = \left(x - \frac{1}{x}\right)\left(x^2 + 1 + \frac{1}{x^2}\right)$$

$$\S 138, \quad = \left(x - \frac{1}{x}\right)\left(x + 1 + \frac{1}{x}\right)\left(x - 1 + \frac{1}{x}\right).$$

84. § 125, $a^2 + 2 + a^{-2} = (a + a^{-1})(a + a^{-1}) = \left(a + \frac{1}{a}\right)\left(a + \frac{1}{a}\right).$
85. § 125, $b^4 - 8 + 16b^{-4} = (b^2 - 4b^{-2})(b^2 - 4b^{-2})$
 § 128, $= (b + 2b^{-1})(b - 2b^{-1})(b + 2b^{-1})(b - 2b^{-1})$
 $= \left(b + \frac{2}{b}\right)\left(b - \frac{2}{b}\right)\left(b + \frac{2}{b}\right)\left(b - \frac{2}{b}\right).$
86. § 131, $12 - x^{-1} - x^{-2} = (4 + x^{-1})(3 - x^{-1}) = \left(4 + \frac{1}{x}\right)\left(3 - \frac{1}{x}\right).$
87. § 131, $2 - 3x^{-1} - 2x^{-2} = (2 + x^{-1})(1 - 2x^{-1}) = \left(2 + \frac{1}{x}\right)\left(1 - \frac{2}{x}\right).$
88. $x^{\frac{1}{2}} = 2.$
 $(x^{\frac{1}{2}})^3 = 2^3.$
 $x = 8.$
89. $x^{\frac{2}{3}} = 8.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = 8^{\frac{3}{2}}.$
 $x = 16.$
90. $x^{\frac{2}{3}} = 4.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = 4^{\frac{3}{2}}.$
 $x = 32.$
91. $x^{\frac{2}{3}} = 16.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = 16^{\frac{3}{2}}.$
 $x = 8.$
92. $\frac{1}{3}x^{\frac{2}{3}} = 9.$
 $x^{\frac{2}{3}} = 27.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = 27^{\frac{3}{2}}.$
 $x = 9.$
93. $x^{-\frac{1}{2}} = 5.$
 $(x^{-\frac{1}{2}})^{-2} = 5^{-2}.$
 $x = \frac{1}{25}.$
94. $\frac{1}{4}x^{\frac{2}{3}} = 25.$
 $x^{\frac{2}{3}} = 100.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = 100^{\frac{3}{2}}.$
 $x = 1000.$
95. $2x^{-\frac{2}{3}} = \frac{1}{3^{\frac{1}{2}}}.$
 $x^{-\frac{2}{3}} = \frac{1}{6^{\frac{1}{2}}}.$
 $(x^{-\frac{2}{3}})^{-\frac{3}{2}} = \left(\frac{1}{6^{\frac{1}{2}}}\right)^{-\frac{3}{2}}.$
 $x = 64^{\frac{3}{2}} = 16.$
96. $x^{-\frac{1}{2}} = 6.$
 $(x^{-\frac{1}{2}})^{-2} = 6^{-2}.$
 $x = \frac{1}{6^2} = \frac{1}{36}.$
97. $x^{-\frac{2}{3}} = 16.$
 $(x^{-\frac{2}{3}})^{-\frac{3}{2}} = 16^{-\frac{3}{2}}.$
 $x = \frac{1}{16^{\frac{3}{2}}} = \frac{1}{64}.$
98. $25x^{-\frac{2}{3}} = 1.$
 $x^{-\frac{2}{3}} = \frac{1}{25}.$
 $(x^{-\frac{2}{3}})^{-\frac{3}{2}} = \left(\frac{1}{25}\right)^{-\frac{3}{2}}.$
 $x = 25^{\frac{3}{2}} = 125.$
99. $x^{\frac{2}{3}} = 243.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = 243^{\frac{3}{2}}.$
 $x = 9.$
100. $x^{\frac{5}{6}} + 32 = 0.$
 $x^{\frac{5}{6}} = -32.$
 $(x^{\frac{5}{6}})^{\frac{6}{5}} = (-32)^{\frac{6}{5}}.$
 $x = -8.$
101. $x^{\frac{2}{3}} + a^6 = 0.$
 $x^{\frac{2}{3}} = -a^6.$
 $(x^{\frac{2}{3}})^{\frac{3}{2}} = (-a^6)^{\frac{3}{2}}.$
 $x = a^9.$
102. $x^{\frac{5}{6}} - 64 = 0.$
 $x^{\frac{5}{6}} = 64.$
 $(x^{\frac{5}{6}})^{\frac{6}{5}} = 64^{\frac{6}{5}}.$
 $x = 32.$
103. $x^{-\frac{2}{3}} + 27 = 0.$
 $x^{-\frac{2}{3}} = -27.$
 $(x^{-\frac{2}{3}})^{-\frac{3}{2}} = (-27)^{-\frac{3}{2}}.$
 $x = \left(\frac{1}{-27}\right)^{\frac{3}{2}} = \frac{1}{9}.$

RADICALS

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3. $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}.$
4. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}.$
5. $\sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{8} \times \sqrt[3]{2} = 2\sqrt[3]{2}.$
6. $\sqrt{128} = \sqrt{64 \times 2} = \sqrt{64} \times \sqrt{2} = 8\sqrt{2}.$
7. $\sqrt[3]{250} = \sqrt[3]{125 \times 2} = \sqrt[3]{125} \times \sqrt[3]{2} = 5\sqrt[3]{2}.$

8. $\sqrt[4]{32} = \sqrt[4]{16 \times 2} = \sqrt[4]{16} \times \sqrt[4]{2} = 2\sqrt[4]{2}.$
9. $\sqrt{600} = \sqrt{100 \times 6} = \sqrt{100} \times \sqrt{6} = 10\sqrt{6}.$
10. $\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}.$
11. $\sqrt[5]{160} = \sqrt[5]{32 \times 5} = \sqrt[5]{32} \times \sqrt[5]{5} = 2\sqrt[5]{5}.$
12. $\sqrt[3]{3000} = \sqrt[3]{1000 \times 3} = \sqrt[3]{1000} \times \sqrt[3]{3} = 10\sqrt[3]{3}.$
13. $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = \sqrt[3]{27} \times \sqrt[3]{3} = 3\sqrt[3]{3}.$
14. $\sqrt[3]{189} = \sqrt[3]{27 \times 7} = \sqrt[3]{27} \times \sqrt[3]{7} = 3\sqrt[3]{7}.$
15. $\sqrt{162} = \sqrt{81 \times 2} = \sqrt{81} \times \sqrt{2} = 9\sqrt{2}.$
16. $\sqrt{18a^2} = \sqrt{9a^2 \times 2} = \sqrt{9a^2} \times \sqrt{2} = 3a\sqrt{2}.$
17. $\sqrt{25b} = \sqrt{25 \times b} = \sqrt{25} \times \sqrt{b} = 5\sqrt{b}.$
18. $\sqrt{98c^3} = \sqrt{49c^2 \times 2c} = \sqrt{49c^2} \times \sqrt{2c} = 7c\sqrt{2c}.$
19. $\sqrt{50a} = \sqrt{25 \times 2a} = \sqrt{25} \times \sqrt{2a} = 5\sqrt{2a}.$
20. $\sqrt[5]{640} = \sqrt[5]{32 \times 20} = \sqrt[5]{32} \times \sqrt[5]{20} = 2\sqrt[5]{20}.$
21. $\sqrt{84} = \sqrt{4 \times 21} = \sqrt{4} \times \sqrt{21} = 2\sqrt{21}.$
22. $\sqrt[3]{72} = \sqrt[3]{8 \times 9} = \sqrt[3]{8} \times \sqrt[3]{9} = 2\sqrt[3]{9}.$
23. $\sqrt[3]{192} = \sqrt[3]{64 \times 3} = \sqrt[3]{64} \times \sqrt[3]{3} = 4\sqrt[3]{3}.$
24. $\sqrt{800} = \sqrt{400 \times 2} = \sqrt{400} \times \sqrt{2} = 20\sqrt{2}.$
25. $\sqrt[3]{3645} = \sqrt[3]{729 \times 5} = \sqrt[3]{729} \times \sqrt[3]{5} = 9\sqrt[3]{5}.$
26. $\sqrt{735} = \sqrt{49 \times 15} = \sqrt{49} \times \sqrt{15} = 7\sqrt{15}.$
27. $\sqrt{243a^5x^{10}} = \sqrt{81a^4x^{10}} \times \sqrt{3a} = 9a^2x^5\sqrt{3a}.$
28. $\sqrt[3]{128a^6b^4} = \sqrt[3]{64a^6b^3} \times \sqrt[3]{2b} = 4a^2b\sqrt[3]{2b}.$
29. $\sqrt{405a^5y^2} = \sqrt{81a^4y^2} \times \sqrt{5a} = 9a^2y\sqrt{5a}.$
30. $\sqrt{375x^6y^3} = \sqrt{25x^6y^2} \times \sqrt{15y} = 5x^3y\sqrt{15y}.$
31. $(245a^6y^4)^{\frac{1}{2}} = (49a^6y^4)^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 7a^3y^2\sqrt{5}.$
32. $(135x^4y^5)^{\frac{1}{3}} = (27x^3y^3)^{\frac{1}{3}} \times (5xy^2)^{\frac{1}{3}} = 3xy\sqrt[3]{5xy^2}.$
33. $(a^3 + 5a^2)^{\frac{1}{2}} = \sqrt{a^2(a+5)} = \sqrt{a^2} \times \sqrt{a+5} = a\sqrt{a+5}.$
34. $(16x - 16)^{\frac{1}{2}} = \sqrt{16(x-1)} = \sqrt{16} \times \sqrt{x-1} = 4\sqrt{x-1}.$
35. $\sqrt{18x-9} = \sqrt{9(2x-1)} = \sqrt{9} \times \sqrt{2x-1} = 3\sqrt{2x-1}.$
36. $\sqrt[3]{x^6-2x^3} = \sqrt[3]{x^3(x^3-2)} = \sqrt[3]{x^3} \times \sqrt[3]{x^3-2} = x\sqrt[3]{x^3-2}.$
37. $\sqrt{8-20b^2} = \sqrt{4(2-5b^2)} = \sqrt{4} \times \sqrt{2-5b^2} = 2\sqrt{2-5b^2}.$
38. $5\sqrt{4a^2+4} = 5\sqrt{4(a^2+1)} = 5\sqrt{4} \times \sqrt{a^2+1} = 10\sqrt{a^2+1}.$
39. $\sqrt{5x^2-10xy+5y^2} = \sqrt{5(x-y)^2} = \sqrt{(x-y)^2} \times \sqrt{5} = (x-y)\sqrt{5}.$
40. $\sqrt{4a^3-24a^2x+36ax^2} = \sqrt{4a(a-3x)^2} = \sqrt{4(a-3x)^2} \times \sqrt{a}$
 $= 2(a-3x)\sqrt{a}.$

$$41. (3am^2 + 6am + 3a)^{\frac{1}{2}} = \sqrt{3a(m+1)^2} = \sqrt{(m+1)^2} \times \sqrt{3a} \\ = (m+1)\sqrt{3a}.$$

$$42. (x^4y - 3x^3y^2 + 3x^2y^3 - xy^4)^{\frac{1}{3}} = \sqrt[3]{xy(x-y)^3} = \sqrt[3]{(x-y)^3} \times \sqrt[3]{xy} \\ = (x-y)\sqrt[3]{xy}.$$

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$$44. \sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4}} \times \sqrt{2} = \frac{1}{2}\sqrt{2}.$$

$$45. \sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{9}} \times \sqrt{3} = \frac{1}{3}\sqrt{3}.$$

$$46. \sqrt{\frac{1}{5}} = \sqrt{\frac{5}{25}} = \sqrt{\frac{1}{25}} \times \sqrt{5} = \frac{1}{5}\sqrt{5}.$$

$$47. \sqrt{\frac{1}{8}} = \sqrt{\frac{2}{16}} = \sqrt{\frac{1}{16}} \times \sqrt{2} = \frac{1}{4}\sqrt{2}.$$

$$48. \sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9}} \times \sqrt{6} = \frac{1}{3}\sqrt{6}.$$

$$49. \sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \sqrt{\frac{1}{16}} \times \sqrt{6} = \frac{1}{4}\sqrt{6}.$$

$$50. \sqrt[3]{\frac{5}{12}} = \sqrt[3]{\frac{90}{216}} = \sqrt[3]{\frac{1}{216}} \times \sqrt[3]{90} = \frac{1}{6}\sqrt[3]{90}.$$

$$51. \sqrt[3]{\frac{4}{5}} = \sqrt[3]{\frac{100}{125}} = \sqrt[3]{\frac{1}{125}} \times \sqrt[3]{100} = \frac{1}{5}\sqrt[3]{100}.$$

$$52. \sqrt{\frac{2a^3}{b}} = \sqrt{\frac{2a^3b}{b^2}} = \sqrt{\frac{a^2}{b^2}} \times \sqrt{2ab} = \frac{a}{b}\sqrt{2ab}.$$

$$53. \sqrt{\frac{5x^4y^2}{2a^2}} = \sqrt{\frac{10x^4y^2}{4a^2}} = \sqrt{\frac{x^4y^2}{4a^2}} \times \sqrt{10} = \frac{x^2y}{2a}\sqrt{10}.$$

$$54. \sqrt[4]{\frac{x}{y}} = \sqrt[4]{\frac{xy^3}{y^4}} = \sqrt[4]{\frac{1}{y^4}} \times \sqrt[4]{xy^3} = \frac{1}{y}\sqrt[4]{xy^3}.$$

$$55. \sqrt{\frac{1}{x^3}} = \sqrt{\frac{x}{x^4}} = \sqrt{\frac{1}{x^4}} \times \sqrt{x} = \frac{1}{x^2}\sqrt{x}.$$

$$56. \sqrt{\frac{2}{3y^5}} = \sqrt{\frac{6y}{9y^6}} = \sqrt{\frac{1}{9y^6}} \times \sqrt{6y} = \frac{1}{3y^3}\sqrt{6y}.$$

$$57. \sqrt{\frac{4a}{3x^2}} = \sqrt{\frac{12a}{9x^2}} = \sqrt{\frac{4}{9x^2}} \times \sqrt{3a} = \frac{2}{3x}\sqrt{3a}.$$

$$58. \sqrt{\frac{3x}{50a^3y}} = \sqrt{\frac{6axy}{100a^4y^2}} = \sqrt{\frac{1}{100a^4y^2}} \times \sqrt{6axy} = \frac{1}{10a^2y}\sqrt{6axy}.$$

$$59. \sqrt[3]{\frac{a}{3b^2}} = \sqrt[3]{\frac{9ab}{27b^3}} = \sqrt[3]{\frac{1}{27b^3}} \times \sqrt[3]{9ab} = \frac{1}{3b}\sqrt[3]{9ab}.$$

$$60. (a+b)\sqrt{\frac{a+b}{a-b}} = (a+b)\sqrt{\frac{a^2-b^2}{(a-b)^2}} \\ = (a+b)\sqrt{\frac{1}{(a-b)^2}} \times \sqrt{a^2-b^2} = \frac{a+b}{a-b}\sqrt{a^2-b^2}.$$

$$\begin{aligned}
 61. \quad \frac{2y}{x-2y} \sqrt{\frac{x-2y}{2y}} &= \frac{2y}{x-2y} \sqrt{\frac{2y(x-2y)}{4y^2}} \\
 &= \frac{2y}{x-2y} \sqrt{\frac{1}{4y^2}} \times \sqrt{2xy-4y^2} \\
 &= \frac{1}{x-2y} \sqrt{2xy-4y^2}.
 \end{aligned}$$

$$\begin{aligned}
 62. \quad (1-x^3) \sqrt{\frac{1-x+x^2}{1+x+x^2}} &= (1-x^3) \sqrt{\frac{(1+x+x^2)(1-x+x^2)}{(1+x+x^2)^2}} \\
 &= (1-x)(1+x+x^2) \sqrt{\frac{1}{(1+x+x^2)^2}} \times \sqrt{1+x^2+x^4} \\
 &= (1-x) \sqrt{1+x^2+x^4}.
 \end{aligned}$$

$$\begin{aligned}
 63. \quad \frac{(a+b)^2}{a-b} \sqrt[3]{\frac{a+b}{(a-b)^2}} &= \frac{(a+b)^2}{a-b} \sqrt[3]{\frac{(a-b)(a+b)}{(a-b)^3}} \\
 &= \frac{(a+b)^2}{a-b} \sqrt[3]{\frac{1}{(a-b)^3}} \times \sqrt[3]{a^2-b^2} \\
 &= \frac{(a+b)^2}{(a-b)^2} \sqrt[3]{a^2-b^2}.
 \end{aligned}$$

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3. $\sqrt[4]{36} = \sqrt[4]{6^2} = 6^{\frac{1}{2}} = 6^{\frac{1}{2}} = \sqrt{6}.$
4. $\sqrt[4]{25} = \sqrt[4]{5^2} = 5^{\frac{1}{2}} = 5^{\frac{1}{2}} = \sqrt{5}.$
5. $\sqrt[4]{144} = \sqrt[4]{12^2} = 12^{\frac{1}{2}} = 12^{\frac{1}{2}} = \sqrt{4 \times 3} = 2\sqrt{3}.$
6. $\sqrt[6]{81} = \sqrt[6]{9^2} = 9^{\frac{1}{3}} = 9^{\frac{1}{3}} = \sqrt[3]{9}.$
7. $\sqrt[4]{1600} = \sqrt[4]{40^2} = 40^{\frac{1}{2}} = 40^{\frac{1}{2}} = \sqrt{4 \times 10} = 2\sqrt{10}.$
8. $\sqrt[6]{27a^3} = \sqrt[6]{(3a)^3} = (3a)^{\frac{1}{2}} = (3a)^{\frac{1}{2}} = \sqrt{3a}.$
9. $\sqrt[6]{343} = \sqrt[6]{7^3} = 7^{\frac{1}{2}} = 7^{\frac{1}{2}} = \sqrt{7}.$
10. $\sqrt[4]{289} = \sqrt[4]{17^2} = 17^{\frac{1}{2}} = 17^{\frac{1}{2}} = \sqrt{17}.$
11. $\sqrt[4]{9a^2b^2c^6} = \sqrt[4]{9a^2b^2c^2 \cdot c^4} = c \sqrt[4]{(3abc)^2} = c(3abc)^{\frac{1}{2}}$
 $= c(3abc)^{\frac{1}{2}} = c\sqrt{3abc}.$
12. $\sqrt[4]{121a^6x^4} = \sqrt[4]{121a^2 \cdot a^4x^4} = ax \sqrt[4]{(11a)^2} = ax(11a)^{\frac{1}{2}}$
 $= ax(11a)^{\frac{1}{2}} = ax\sqrt{11a}.$
13. $\sqrt[6]{a^4b^2c^4d^8} = \sqrt[6]{a^4b^2c^4d^2 \cdot d^6} = d \sqrt[6]{(a^2bc^2d)^2}$
 $= d(a^2bc^2d)^{\frac{1}{3}} = d(a^2bc^2d)^{\frac{1}{3}} = d\sqrt[3]{a^2bc^2d}.$
14. $\sqrt[4]{(x^2-2xy+y^2)} = \sqrt[4]{(x-y)^2} = (x-y)^{\frac{1}{2}} = (x-y)^{\frac{1}{2}} = \sqrt{x-y}.$

2. $2\sqrt{2} = \sqrt{4}\sqrt{2} = \sqrt{8}.$
3. $3\sqrt{5} = \sqrt{9}\sqrt{5} = \sqrt{45}.$
4. $5\sqrt{2} = \sqrt{25}\sqrt{2} = \sqrt{50}.$
5. $3^4\sqrt{2} = \sqrt[4]{81}\sqrt[4]{2} = \sqrt[4]{162}.$
6. $3^3\sqrt[3]{3} = \sqrt[3]{27}\sqrt[3]{3} = \sqrt[3]{81}.$
7. $4\sqrt{5} = \sqrt{16}\sqrt{5} = \sqrt{80}.$
8. $\frac{1}{2}\sqrt{8} = \sqrt{\frac{1}{4}}\sqrt{8} = \sqrt{2}.$
9. $a^2\sqrt[3]{b} = \sqrt[3]{a^6}\sqrt[3]{b} = \sqrt[3]{a^6b}.$
10. $\frac{1}{2}\sqrt{2} = \sqrt{\frac{1}{4}}\sqrt{2} = \sqrt{\frac{1}{2}}.$
11. $\frac{3}{4}\sqrt{x^5} = \sqrt{\frac{9}{16}}\sqrt{x^5} = \sqrt{\frac{9}{16}x^5}.$
12. $\frac{1}{2}\sqrt{bc} = \sqrt{\frac{1}{4}}\sqrt{bc} = \sqrt{\frac{1}{4}bc}.$
13. $\frac{3}{4}\sqrt{\frac{7}{9}} = \sqrt{\frac{9}{16}}\sqrt{\frac{7}{9}} = \sqrt{\frac{7}{16}}.$
14. $\frac{4}{5}\sqrt[3]{4\frac{3}{8}} = \sqrt[3]{\frac{16}{25}}\sqrt[3]{\frac{35}{8}} = \sqrt[3]{\frac{14}{5}}.$
15. $\frac{3}{2}\sqrt{\frac{3}{2}\frac{3}{2}a^2} = \sqrt{\frac{9}{4}}\sqrt{\frac{3}{2}\frac{3}{2}a^2} = \sqrt{\frac{11}{4}a^2}.$
16. $\frac{2}{3}\sqrt[3]{1\frac{1}{8}} = \sqrt[3]{\frac{8}{27}}\sqrt[3]{\frac{9}{8}} = \sqrt[3]{\frac{1}{3}}.$
17. $\frac{2}{3}\sqrt[3]{4\frac{3}{8}} = \sqrt[3]{\frac{16}{27}}\sqrt[3]{\frac{27}{8}} = \sqrt[3]{\frac{2}{3}}.$
18. $\frac{x+y}{x-y}\sqrt{\frac{x-y}{x+y}} = \sqrt{\frac{(x+y)^2}{(x-y)^2}}\sqrt{\frac{x-y}{x+y}} = \sqrt{\frac{x+y}{x-y}}.$
19. $\frac{a+4}{a-4}\sqrt{1-\frac{8}{a+4}} = \sqrt{\frac{(a+4)^2}{(a-4)^2}}\sqrt{\frac{a-4}{a+4}} = \sqrt{\frac{a+4}{a-4}}.$
20. $\frac{1}{ab}(a-b)^{\frac{1}{2}} = \left(\frac{1}{a^2b^2}\right)^{\frac{1}{2}}(a-b)^{\frac{1}{2}} = \sqrt{\frac{a-b}{a^2b^2}}.$

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2. $\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{2}{4}} = \sqrt[4]{4}.$
 $\sqrt[4]{3} = \sqrt[4]{3}.$
3. $\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{125}.$
 $\sqrt[3]{6} = 6^{\frac{1}{3}} = 6^{\frac{2}{6}} = \sqrt[6]{36}.$
4. See next column.
5. $\sqrt[6]{10} = \sqrt[6]{10}.$
 $\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{8}.$
 $\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{25}.$
6. $\sqrt[6]{4} = 4^{\frac{1}{6}} = 4^{\frac{2}{12}} = \sqrt[12]{16}.$
 $\sqrt[4]{2} = 2^{\frac{1}{4}} = 2^{\frac{3}{12}} = \sqrt[12]{8}.$
 $\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{5}{10}} = \sqrt[10]{729}.$
7. $\sqrt[10]{13} = \sqrt[10]{13}.$
 $\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{5}{10}} = \sqrt[10]{3125}.$
 $\sqrt[5]{4} = 4^{\frac{1}{5}} = 4^{\frac{2}{10}} = \sqrt[10]{16}.$
4. $\sqrt[4]{7} = \sqrt[4]{7}.$
 $\sqrt{10} = 10^{\frac{1}{2}} = 10^{\frac{2}{4}} = \sqrt[4]{100}.$
8. $\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{2}{4}} = \sqrt[4]{729}.$
 $\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{625}.$
 $\sqrt[4]{\frac{1}{27}} = \frac{1}{3}(3)^{\frac{1}{4}} = \frac{1}{3}(3)^{\frac{3}{12}} = \frac{1}{3}\sqrt[12]{27}.$
9. $\sqrt{ab} = (ab)^{\frac{1}{2}} = (ab)^{\frac{2}{4}} = \sqrt[4]{a^2b^2}.$
 $\sqrt[3]{ab^2} = (ab^2)^{\frac{1}{3}} = (ab^2)^{\frac{2}{6}} = \sqrt[6]{a^2b^4}.$
 $\sqrt[4]{2} = 2^{\frac{1}{4}} = 2^{\frac{3}{12}} = \sqrt[12]{8}.$
10. $\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{2}{4}} = \sqrt[4]{a^2}.$
 $\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}.$
 $\sqrt[4]{x} = x^{\frac{1}{4}} = x^{\frac{3}{12}} = \sqrt[12]{x^3}.$
 $\sqrt[6]{y} = y^{\frac{1}{6}} = y^{\frac{2}{12}} = \sqrt[12]{y^2}.$

11. $\sqrt[3]{a+b} = (a+b)^{\frac{1}{3}} = (a+b)^{\frac{2}{3}} = \sqrt[6]{(a+b)^2}$.
 $\sqrt{x+y} = (x+y)^{\frac{1}{2}} = (x+y)^{\frac{3}{2}} = \sqrt[6]{(x+y)^3}$.
12. $\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6} = \frac{1}{3}(6)^{\frac{1}{2}} = \frac{1}{3}(6)^{\frac{3}{2}} = \frac{1}{3}\sqrt[6]{216}$.
 $\sqrt[3]{\frac{1}{16}x} = \frac{1}{4}\sqrt[3]{4x} = \frac{1}{4}(4x)^{\frac{1}{3}} = \frac{1}{4}(4x)^{\frac{2}{3}} = \frac{1}{4}\sqrt[6]{16x^2}$.
 $2\sqrt{5} = 2(5)^{\frac{1}{2}} = 2(5)^{\frac{3}{2}} = 2\sqrt[6]{125}$.
13. $\sqrt[n]{x} = x^{\frac{1}{n}} = x^{\frac{2}{2n}} = \sqrt[2n]{x^2}$.
 $\sqrt{xy} = (xy)^{\frac{1}{2}} = (xy)^{\frac{n}{2n}} = \sqrt[2n]{x^ny^n}$.
 $\sqrt[n]{x^2y^2} = (xy)^{\frac{2}{n}} = (xy)^{\frac{4}{2n}} = \sqrt[2n]{x^4y^4}$.
14. $(a+b)\sqrt{a-b} = (a+b)(a-b)^{\frac{1}{2}} = (a+b)(a-b)^{\frac{3}{2}} = (a+b)\sqrt[6]{(a-b)^3}$.
 $\sqrt[3]{a-b} = (a-b)^{\frac{1}{3}} = (a-b)^{\frac{2}{3}} = \sqrt[6]{(a-b)^2}$.
15. $\sqrt{a+b} = (a+b)^{\frac{1}{2}} = (a+b)^{\frac{2}{4}} = \sqrt[4]{(a+b)^2}$.
 $\sqrt[4]{a^2+b^2} = \sqrt[4]{a^2+b^2}$.
 $\sqrt{a-b} = (a-b)^{\frac{1}{2}} = (a-b)^{\frac{2}{4}} = \sqrt[4]{(a-b)^2}$.

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2. $\sqrt{50} + \sqrt{18} + \sqrt{98} = 5\sqrt{2} + 3\sqrt{2} + 7\sqrt{2} = 15\sqrt{2}$.
3. $\sqrt{27} + \sqrt{12} + \sqrt{75} = 3\sqrt{3} + 2\sqrt{3} + 5\sqrt{3} = 10\sqrt{3}$.
4. $\sqrt{20} + \sqrt{80} + \sqrt{45} = 2\sqrt{5} + 4\sqrt{5} + 3\sqrt{5} = 9\sqrt{5}$.
5. $\sqrt{28} + \sqrt{63} + \sqrt{700} = 2\sqrt{7} + 3\sqrt{7} + 10\sqrt{7} = 15\sqrt{7}$.
6. $\sqrt[3]{250} + \sqrt[3]{16} + \sqrt[3]{54} = 5\sqrt[3]{2} + 2\sqrt[3]{2} + 3\sqrt[3]{2} = 10\sqrt[3]{2}$.
7. $\sqrt[3]{128} + \sqrt[3]{686} + \sqrt[3]{\frac{1}{4}} = 4\sqrt[3]{2} + 7\sqrt[3]{2} + \frac{1}{2}\sqrt[3]{2} = \frac{23}{2}\sqrt[3]{2}$.
8. $\sqrt[3]{135} + \sqrt[3]{320} + \sqrt[3]{625} = 3\sqrt[3]{5} + 4\sqrt[3]{5} + 5\sqrt[3]{5} = 12\sqrt[3]{5}$.
9. $\sqrt[3]{500} + \sqrt[3]{108} + \sqrt[3]{-32} = 5\sqrt[3]{4} + 3\sqrt[3]{4} - 2\sqrt[3]{4} = 6\sqrt[3]{4}$.
10. $\sqrt{\frac{1}{2}} + \sqrt{12\frac{1}{2}} + \sqrt{\frac{1}{8}} + \sqrt{1\frac{1}{8}} = \frac{1}{2}\sqrt{2} + \frac{5}{2}\sqrt{2} + \frac{1}{4}\sqrt{2} + \frac{3}{4}\sqrt{2} = 4\sqrt{2}$.
11. $\sqrt{\frac{1}{3}} + \sqrt{75} + \frac{2}{3}\sqrt{3} + \sqrt{12} = \frac{1}{3}\sqrt{3} + 5\sqrt{3} + \frac{2}{3}\sqrt{3} + 2\sqrt{3} = 8\sqrt{3}$.
12. $\sqrt{\frac{3}{4}} + \frac{1}{3}\sqrt{3} + \frac{7}{6}\sqrt{9} + \sqrt{147} = \frac{1}{2}\sqrt{3} + \frac{1}{3}\sqrt{3} + \frac{7}{6}\sqrt{3} + 7\sqrt{3} = 9\sqrt{3}$.
13. $\sqrt[3]{40} + \sqrt{28} + \sqrt[6]{25} + \sqrt{175} = 2\sqrt[3]{5} + 2\sqrt{7} + \sqrt[3]{5} + 5\sqrt{7}$
 $= 3\sqrt[3]{5} + 7\sqrt{7}$.
14. $\sqrt[3]{375} + \sqrt{44} + \sqrt[3]{192} + \sqrt{99} = 5\sqrt[3]{3} + 2\sqrt{11} + 4\sqrt[3]{3} + 3\sqrt{11}$
 $= 9\sqrt[3]{3} + 5\sqrt{11}$.

$$15. \sqrt{245} - \sqrt{405} + \sqrt{45} = 7\sqrt{5} - 9\sqrt{5} + 3\sqrt{5} = \sqrt{5}.$$

$$16. \sqrt{12} + 3\sqrt{75} - 2\sqrt{27} = 2\sqrt{3} + 15\sqrt{3} - 6\sqrt{3} = 11\sqrt{3}.$$

$$17. 5\sqrt{72} + 3\sqrt{18} - \sqrt{50} = 30\sqrt{2} + 9\sqrt{2} - 5\sqrt{2} = 34\sqrt{2}.$$

$$18. \sqrt[3]{128} + \sqrt[3]{686} - \sqrt[3]{54} = 4\sqrt[3]{2} + 7\sqrt[3]{2} - 3\sqrt[3]{2} = 8\sqrt[3]{2}.$$

$$19. \sqrt{112} - \sqrt{343} + \sqrt{448} = 4\sqrt{7} - 7\sqrt{7} + 8\sqrt{7} = 5\sqrt{7}.$$

$$20. \sqrt[3]{135} - \sqrt[3]{625} + \sqrt[3]{320} = 3\sqrt[3]{5} - 5\sqrt[3]{5} + 4\sqrt[3]{5} = 2\sqrt[3]{5}.$$

$$21. \sqrt[3]{\frac{8}{5}} + \sqrt[3]{\frac{1}{5}} + \sqrt[3]{\frac{5}{5}} = 2\sqrt[3]{\frac{1}{5}} + \sqrt[3]{\frac{1}{5}} + 3\sqrt[3]{\frac{1}{5}} = 6\sqrt[3]{\frac{1}{5}} = \frac{6}{5}\sqrt[3]{25}.$$

$$22. \sqrt[3]{864} - \sqrt[3]{4000} + \sqrt[3]{32} = 6\sqrt[3]{4} - 10\sqrt[3]{4} + 2\sqrt[3]{4} = -2\sqrt[3]{4}.$$

$$23. \sqrt[3]{128x} + \sqrt[3]{375x} - \sqrt[3]{54x} = 4\sqrt[3]{2x} + 5\sqrt[3]{3x} - 3\sqrt[3]{2x} \\ = \sqrt[3]{2x} + 5\sqrt[3]{3x}.$$

$$24. \sqrt{\frac{a}{x^2}} + \sqrt{\frac{a}{y^2}} - \sqrt{\frac{a}{z^2}} = \frac{1}{x}\sqrt{a} + \frac{1}{y}\sqrt{a} - \frac{1}{z}\sqrt{a} \\ = \left(\frac{1}{x} + \frac{1}{y} - \frac{1}{z}\right)\sqrt{a}.$$

$$25. \sqrt{\frac{ax^4}{by^2}} - \sqrt{\frac{16ax^2}{by^2}} + \sqrt{\frac{4ax^2}{by^2}} \\ = \sqrt{\frac{abx^4}{b^2y^2}} - \sqrt{\frac{16abx^2}{b^2y^2}} + \sqrt{\frac{4abx^2}{b^2y^2}} \\ = \frac{x^2}{by}\sqrt{ab} - \frac{4x}{by}\sqrt{ab} + \frac{2x}{by}\sqrt{ab} = \frac{x^2 - 2x}{by}\sqrt{ab}.$$

$$26. \sqrt{\frac{a}{bc}} + \sqrt{\frac{b}{ac}} + \sqrt{\frac{c}{ab}} = \frac{1}{bc}\sqrt{abc} + \frac{1}{ac}\sqrt{abc} + \frac{1}{ab}\sqrt{abc} \\ = \left(\frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}\right)\sqrt{abc} = \frac{a+b+c}{abc}\sqrt{abc}.$$

$$27. \sqrt{(a+b)^2c} - \sqrt{(a-b)^2c} = (a+b)\sqrt{c} - (a-b)\sqrt{c} = 2b\sqrt{c}.$$

$$28. 6\sqrt[3]{\frac{9}{27}} + 4\sqrt[3]{\frac{1}{16}} - 8\sqrt[3]{\frac{675}{320}} = 6\sqrt[3]{\frac{4}{9}} + 4\sqrt[3]{\frac{1}{8}} - 8\sqrt[3]{\frac{135}{64}} \\ = 4\sqrt[3]{5} + 2\sqrt[3]{5} - 6\sqrt[3]{5} = 0.$$

$$29. \sqrt[5]{-96x^4} + 2\sqrt[5]{3x^4} - \sqrt[5]{5x} + \sqrt[5]{40x^4} \\ = -2\sqrt[5]{3x^4} + 2\sqrt[5]{3x^4} - \sqrt[5]{5x} + 2x\sqrt[5]{5x} \\ = (2x - 1)\sqrt[5]{5x}.$$

$$30. \sqrt[3]{abx} - \sqrt[3]{a^2b^2x^2} + \sqrt[3]{8a^3b^3x^3} \\ = \sqrt[3]{abx} - \sqrt[3]{abx} + \sqrt[3]{2abx} = \sqrt[3]{2abx}.$$

$$\begin{aligned}
 31. \quad & \sqrt{3x^3 + 30x^2 + 75x} - \sqrt{3x^3 - 6x^2 + 3x} \\
 &= \sqrt{3x(x+5)^2} - \sqrt{3x(x-1)^2} \\
 &= [x+5 - (x-1)] \sqrt{3x} = 6\sqrt{3x}.
 \end{aligned}$$

$$\begin{aligned}
 32. \quad & \sqrt{5a^5 + 30a^4 + 45a^3} - \sqrt{5a^5 - 40a^4 + 80a^3} \\
 &= \sqrt{5a^3(a+3)^2} - \sqrt{5a^3(a-4)^2} \\
 &= [a+3 - (a-4)] \sqrt{5a^3} \\
 &= 7\sqrt{5a^3} = 7a\sqrt{5a}.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad & \sqrt{50} + \sqrt[6]{9} - 4\sqrt{\frac{1}{2}} + \sqrt[3]{-24} + \sqrt[9]{27} - \sqrt[4]{64} \\
 &= 5\sqrt{2} + \sqrt[3]{3} - 2\sqrt{2} - 2\sqrt[3]{3} + \sqrt[3]{3} - 2\sqrt{2} = \sqrt{2}.
 \end{aligned}$$

$$\begin{aligned}
 34. \quad & \sqrt{\frac{2}{3}} + 6\sqrt{\frac{5}{4}} - \frac{1}{5}\sqrt{18} + \sqrt[4]{36} - \sqrt[8]{\frac{16}{81}} + \sqrt[6]{125} - 2\sqrt{\frac{2}{25}} \\
 &= \frac{1}{3}\sqrt{6} + 3\sqrt{5} - \frac{3}{5}\sqrt{2} + \sqrt{6} - \frac{1}{3}\sqrt{6} + \sqrt{5} - \frac{2}{5}\sqrt{2} \\
 &= \sqrt{6} + 4\sqrt{5} - \sqrt{2}.
 \end{aligned}$$

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4. $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4.$
5. $\sqrt{2} \times \sqrt{6} = \sqrt{12} = 2\sqrt{3}.$
6. $\sqrt{3} \times \sqrt{15} = \sqrt{45} = 3\sqrt{5}.$
7. $\sqrt{3} \times \sqrt{48} = \sqrt{144} = 12.$
8. $2\sqrt{5} \times 3\sqrt{10} = 6\sqrt{50} = 30\sqrt{2}.$
9. $3\sqrt{20} \times 2\sqrt{2} = 6\sqrt{40} = 12\sqrt{10}.$
10. $\sqrt{2} \times 3\sqrt{3} = \sqrt[6]{8} \times 3\sqrt[6]{9} = 3\sqrt[6]{72}.$
11. $\sqrt[3]{2} \times 2\sqrt{5} = \sqrt[6]{4} \times 2\sqrt[6]{125} = 2\sqrt[6]{500}.$
12. $\sqrt[3]{3} \times 3\sqrt{3} = \sqrt[6]{9} \times 3\sqrt[6]{27} = 3\sqrt[6]{243}.$
13. $2\sqrt{6} \times \sqrt{18} = 2\sqrt{108} = 12\sqrt{3}.$
14. $2\sqrt[3]{3} \times 3\sqrt[3]{45} = 6\sqrt[3]{135} = 18\sqrt[3]{5}.$
15. $2\sqrt[4]{6} \times 3\sqrt{6} = 2\sqrt[4]{6} \times 3\sqrt[4]{36} = 6\sqrt[4]{216}.$
16. $3\sqrt{3} \times 2\sqrt[3]{5} = 3\sqrt[6]{27} \times 2\sqrt[6]{25} = 6\sqrt[6]{675}.$
17. $\sqrt[4]{5} \times \sqrt[6]{10} = \sqrt[12]{125} \times \sqrt[12]{100} = \sqrt[12]{12500}.$
18. $2\sqrt[3]{250} \times \sqrt{2} = 10\sqrt[6]{4} \times \sqrt[6]{8} = 10\sqrt[6]{32}.$
19. $2\sqrt[3]{24} \times \sqrt[3]{18} = 4\sqrt[3]{3} \times \sqrt[3]{18} = 4\sqrt[3]{54} = 12\sqrt[3]{2}.$
20. $\sqrt{28} \times 3\sqrt{7} = 2\sqrt{7} \times 3\sqrt{7} = 2 \times 3 \times 7 = 42.$
21. $2\sqrt[5]{2} \times \sqrt[10]{512} = 2(2)^{\frac{1}{5}} \times (2)^{\frac{5}{10}} = 2 \times 2 \times 2^{\frac{1}{10}} = 4\sqrt[10]{2}.$
22. $\sqrt{ab} \times \sqrt{bc} \times \sqrt{cd} \times \sqrt{da} = \sqrt{a^2b^2c^2d^2} = abcd.$
23. $\sqrt{x^3y^2} \times \sqrt{12x} \times \sqrt{75xy^2} = xy\sqrt{x} \times 2\sqrt{3x} \times 5y\sqrt{3x}$
 $= 10xy^2\sqrt{9x^3} = 30x^2y^2\sqrt{x}.$

$$\begin{aligned}
 24. \quad \sqrt[3]{2ab} \times \sqrt[3]{abc} \times \sqrt[4]{4a^2b^2} &= \sqrt[6]{8a^3b^3} \times \sqrt[6]{a^2b^2c^2} \times \sqrt[6]{8a^3b^3} \\
 &= \sqrt[6]{64a^8b^8c^2} = 2ab\sqrt[6]{a^2b^2c^2} \\
 &= 2ab\sqrt[3]{abc}.
 \end{aligned}$$

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$$\begin{aligned}
 25. \quad \sqrt{mn} \times \sqrt[4]{m^2n} \times \sqrt[8]{mn^4} &= \sqrt[8]{m^4n^4} \times \sqrt[8]{m^4n^2} \times \sqrt[8]{mn^4} \\
 &= \sqrt[8]{m^9n^{10}} = mn\sqrt[8]{mn^2}.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \sqrt[3]{2axy} \times \sqrt[3]{xy} \times \sqrt[4]{a^2xy} &= \sqrt[12]{64a^6x^6y^6} \times \sqrt[12]{x^4y^4} \times \sqrt[12]{a^6x^3y^3} \\
 &= \sqrt[12]{64a^{12}x^{13}y^{13}} = axy\sqrt[12]{64xy}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \sqrt{x^{-1}y} \times \sqrt[3]{x^{-2}y^2} \times \sqrt{x^{-3}y^3} &= \sqrt{x^{-1}y} \times \sqrt{x^{-8}y^3} \times \sqrt[3]{x^{-2}y^2} \\
 &= \sqrt{x^{-4}y^4} \times \sqrt[3]{x^{-2}y^2} \\
 &= x^{-2}y^2\sqrt[3]{xy^2 \cdot x^{-3}} \\
 &= x^{-2}y^2\sqrt[3]{xy^2}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \sqrt{a-b} \times \sqrt[4]{a^2b^2} \times \sqrt[4]{(a-b)^{-2}} &= \sqrt{a-b} \times \sqrt{(a-b)^{-1}} \times \sqrt{ab} \\
 &= \sqrt{(a-b)^0} \times \sqrt{ab} = \sqrt{ab}.
 \end{aligned}$$

$$29. \quad \sqrt{\frac{2}{3}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{3}{4}} = \sqrt{\frac{2}{5}} = \frac{1}{5}\sqrt{10}.$$

$$30. \quad \sqrt{\frac{1}{2}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{7}{2}} = \sqrt{\frac{7}{5}} = \frac{1}{5}\sqrt{35}.$$

$$31. \quad \sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{3}{4}} \times \sqrt{\frac{1}{2}} = \sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{1}{2}} = \sqrt[6]{\frac{1}{4}} \times \sqrt[6]{\frac{1}{8}} = \sqrt[6]{\frac{1}{32}} = \frac{1}{2}\sqrt[6]{2}.$$

$$32. \quad \sqrt[3]{\frac{1}{3}} \times \sqrt[3]{\frac{3}{2}} \times \sqrt{\frac{2}{3}} = \sqrt[6]{\frac{1}{9}} \times \sqrt[6]{\frac{3}{2}} \times \sqrt[6]{\frac{8}{27}} = \sqrt[6]{\frac{1}{81}} = \sqrt[3]{\frac{2}{9}} = \frac{1}{3}\sqrt[3]{6}.$$

$$33. \quad \sqrt[6]{\frac{3}{2}} \times \sqrt[3]{\frac{4}{3}} \times \sqrt{\frac{3}{2}} = \sqrt[6]{\frac{3}{2}} \times \sqrt[6]{\frac{16}{9}} \times \sqrt[6]{\frac{27}{8}} = \sqrt[6]{9} = \sqrt[3]{3}.$$

$$34. \quad \sqrt[4]{\frac{2}{3}} \times \sqrt[3]{\frac{3}{4}} \times \sqrt[4]{\frac{6}{8}} = \sqrt[4]{\frac{2}{3} \times \frac{6}{8}} \times \sqrt[4]{\frac{9}{16}} = \sqrt[4]{\frac{9}{32}} = \sqrt[4]{\frac{72}{256}} = \frac{1}{4}\sqrt[4]{72}.$$

$$\begin{aligned}
 36. \quad &\sqrt{5} + \sqrt{3} \\
 &\underline{\sqrt{5} - \sqrt{3}} \\
 &5 + \sqrt{15} \\
 &\underline{-\sqrt{15} - 3} \\
 &5 \qquad -3 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 37. \quad &\sqrt{7} + \sqrt{2} \\
 &\underline{\sqrt{7} - \sqrt{2}} \\
 &7 + \sqrt{14} \\
 &\underline{-\sqrt{14} - 2} \\
 &7 \qquad -2 \\
 &= 5.
 \end{aligned}$$

$$\begin{aligned}
 38. \quad &\sqrt{6} - \sqrt{5} \\
 &\underline{\sqrt{6} - \sqrt{5}} \\
 &6 - \sqrt{30} \\
 &\underline{-\sqrt{30} + 5} \\
 &6 - 2\sqrt{30} + 5 \\
 &= 11 - 2\sqrt{30}.
 \end{aligned}$$

$$\begin{aligned}
 39. \quad &5 - \sqrt{5} \\
 &\underline{1 + \sqrt{5}} \\
 &5 - \sqrt{5} \\
 &\underline{5\sqrt{5} - 5} \\
 &5 + 4\sqrt{5} - 5 \\
 &= 4\sqrt{5}.
 \end{aligned}$$

$$\begin{array}{r}
 40. \quad 4\sqrt{7} + 1 \\
 \underline{4\sqrt{7} - 1} \\
 112 + 4\sqrt{7} \\
 \quad - 4\sqrt{7} - 1 \\
 \hline
 112 \qquad - 1 \\
 = 111.
 \end{array}$$

$$\begin{array}{r}
 41. \quad 2\sqrt{2} + \sqrt{3} \\
 \underline{4\sqrt{2} + \sqrt{3}} \\
 16 + 4\sqrt{6} \\
 \qquad 2\sqrt{6} + 3 \\
 \hline
 16 + 6\sqrt{6} + 3 \\
 = 19 + 6\sqrt{6}.
 \end{array}$$

$$\begin{array}{r}
 42. \quad 2\sqrt{3} + 3\sqrt{5} \\
 \underline{3\sqrt{3} + 2\sqrt{5}} \\
 18 + 9\sqrt{15} \\
 \qquad 4\sqrt{15} + 30 \\
 \hline
 18 + 13\sqrt{15} + 30 \\
 = 48 + 13\sqrt{15}.
 \end{array}$$

$$\begin{array}{r}
 43. \quad 3a + \sqrt{5} \\
 \underline{2a - \sqrt{5}} \\
 6a^2 + 2a\sqrt{5} \\
 \qquad - 3a\sqrt{5} - 5 \\
 \hline
 6a^2 - a\sqrt{5} - 5
 \end{array}$$

44.

$$\begin{array}{r}
 2\sqrt{6} - 3\sqrt{5} \\
 \underline{4\sqrt{3} - \sqrt{10}} \\
 24\sqrt{2} - 12\sqrt{15} \\
 \qquad - 4\sqrt{15} + 15\sqrt{2} \\
 \hline
 24\sqrt{2} - 16\sqrt{15} + 15\sqrt{2} \\
 = 39\sqrt{2} - 16\sqrt{15}.
 \end{array}$$

45.

$$\begin{array}{r}
 a^2 - ab\sqrt{2} + b^2 \\
 \underline{a^2 + ab\sqrt{2} + b^2} \\
 a^4 - a^3b\sqrt{2} + a^2b^2 \\
 \qquad a^3b\sqrt{2} - 2a^2b^2 + ab^3\sqrt{2} \\
 \qquad \qquad a^2b^2 - ab^3\sqrt{2} + b^4 \\
 \hline
 a^4 \qquad \qquad \qquad + b^4
 \end{array}$$

46.

$$\begin{array}{r}
 x - \sqrt{xyz} + yz \\
 \underline{\sqrt{x} + \sqrt{yz}} \\
 x\sqrt{x} - x\sqrt{yz} + yz\sqrt{x} \\
 \qquad x\sqrt{yz} - yz\sqrt{x} + yz\sqrt{yz} \\
 \hline
 x\sqrt{x} \qquad \qquad \qquad + yz\sqrt{yz}
 \end{array}$$

47.

$$\begin{array}{r}
 x\sqrt{x} - x\sqrt{y} + y\sqrt{x} - y\sqrt{y} \\
 \underline{\sqrt{x} + \sqrt{y}} \\
 x^2 - x\sqrt{xy} + xy - y\sqrt{xy} \\
 \qquad x\sqrt{xy} - xy + y\sqrt{xy} - y^2 \\
 \hline
 x^2 \qquad \qquad \qquad - y^2
 \end{array}$$

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$$4. \quad \sqrt{50} \div \sqrt{8} = \sqrt{\frac{50}{8}} = \sqrt{\frac{25}{4}} = \frac{5}{2}.$$

$$5. \quad \sqrt{72} \div 2\sqrt{6} = \frac{1}{2}\sqrt{12} = \sqrt{3}.$$

$$6. \quad 4\sqrt{5} \div \sqrt{40} = 4\sqrt{\frac{1}{8}} = \sqrt{2}.$$

$$7. \quad 6\sqrt{7} \div \sqrt{126} = 6\sqrt{\frac{1}{18}} = \sqrt{2}.$$

$$8. \quad \sqrt[3]{4} \div \sqrt{2} = \sqrt[6]{16} \div \sqrt[6]{8} = \sqrt[6]{2}.$$

$$10. \quad 7\sqrt{75} \div 5\sqrt{28} = 35\sqrt{3} \div 10\sqrt{7} = \frac{7}{2}\sqrt{\frac{3}{7}} = \frac{1}{2}\sqrt{21}.$$

$$11. \quad \sqrt[3]{16} \div \sqrt[6]{32} = \sqrt[6]{256} \div \sqrt[6]{32} = \sqrt[6]{8} = \sqrt{2}.$$

12. $2\sqrt[3]{12} \div \sqrt{8} = 2\sqrt[6]{144} \div 2\sqrt{2}$
 $= \sqrt[6]{144} \div \sqrt[6]{8} = \sqrt[6]{18}.$
13. $\sqrt[3]{ax} \div \sqrt{xy} = \sqrt[6]{a^2x^2} \div \sqrt[6]{x^3y^3}$
 $= \sqrt[6]{\frac{a^2}{xy^3}} = \frac{1}{xy} \sqrt[6]{a^2x^5y^3}.$
14. $\sqrt{2ab^3} \div \sqrt[4]{a^4b^4} = b\sqrt{2ab} \div ab = \frac{1}{a} \sqrt{2ab}.$
15. $\sqrt[3]{a^2x^2} \div \sqrt{2ax} = \sqrt[6]{a^4x^4} \div \sqrt[6]{8a^3x^3} = \sqrt[6]{\frac{1}{8}ax} = \frac{1}{2}\sqrt[6]{8ax}.$
16. $\sqrt[3]{9a^2b^2} \div \sqrt{3ab} = \sqrt[6]{81a^4b^4} \div \sqrt[6]{27a^3b^3} = \sqrt[6]{3ab}.$
17. $\sqrt[4]{4x^2y^2} \div \sqrt[3]{2xy} = \sqrt[6]{8x^3y^3} \div \sqrt[6]{4x^2y^2} = \sqrt[6]{2xy}.$
18. $\sqrt{a-b} \div \sqrt{a+b} = \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{a^2-b^2}{(a+b)^2}} = \frac{1}{a+b} \sqrt{a^2-b^2}.$
19. $3\sqrt[3]{\frac{2}{3}} \div \sqrt[3]{\frac{3}{4}} = 3\sqrt[6]{\frac{4}{3}} \div \frac{1}{2}\sqrt[6]{27} = 6\sqrt[6]{\frac{4}{27}} = 2\sqrt[6]{12}.$
20. $\frac{\sqrt{3} \mid \sqrt{15} - \sqrt{3}}{\sqrt{5} - 1}$
21. $\frac{\sqrt{2} \mid \sqrt{6} - 2\sqrt{3} + 4}{\sqrt{3} - 2\sqrt{\frac{3}{2}} + \sqrt{8}}$
 $= \sqrt{3} - \sqrt{6} + 2\sqrt{2}.$
22. $\frac{\frac{1}{3}\sqrt{6} \mid \sqrt{2} + 2 + \frac{1}{3}\sqrt{42}}{3\sqrt{\frac{1}{3}} + 3\sqrt{\frac{2}{3}} + \sqrt{7}}$
 $= \sqrt{3} + \sqrt{6} + \sqrt{7}.$
23. $(5\sqrt{2} + 5\sqrt{3}) \div (\sqrt{10} + \sqrt{15}) = (\sqrt{50} + \sqrt{75}) \div (\sqrt{10} + \sqrt{15}) = \sqrt{5}.$
24. $\frac{5 + 5\sqrt{30} + 36}{5 + 2\sqrt{30}} \mid \frac{\sqrt{5} + 2\sqrt{6}}{\sqrt{5} + 3\sqrt{6}}$
 $\frac{3\sqrt{30} + 36}{3\sqrt{30} + 36}$

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4. $(3\sqrt{ab})^2 = 3^2(ab)^{\frac{2}{2}} = 9ab.$
5. $(2\sqrt[3]{3x})^2 = 2^2 \cdot 3^{\frac{2}{3}}x^{\frac{2}{3}} = 4\sqrt[3]{9x^2}.$
6. $(x\sqrt[3]{2x^3})^2 = (x^2\sqrt[3]{2})^2 = x^4 \cdot 2^{\frac{2}{3}} = x^4\sqrt[3]{4}.$
7. $(n^2\sqrt{4b})^2 = n^4(4b)^{\frac{2}{2}} = 4bn^4.$
8. $(a\sqrt[4]{a^2b})^2 = a^2(a^2)^{\frac{2}{4}}b^{\frac{2}{4}} = a^2 \cdot a \cdot b^{\frac{1}{2}} = a^3\sqrt{b}.$
9. $(2\sqrt{5})^3 = 2^3 \cdot 5^{\frac{3}{2}} = 8 \cdot 5 \cdot 5^{\frac{1}{2}} = 40\sqrt{5}.$
10. $(3\sqrt{2})^3 = 3^3 \cdot 2^{\frac{3}{2}} = 27 \cdot 2 \cdot 2^{\frac{1}{2}} = 54\sqrt{2}.$

11. $(2\sqrt[3]{a^2})^3 = 2^3 (a^2)^3 = 8 a^2.$
12. $(\sqrt[4]{a^2 b^3})^3 = (a^2 b^3)^{\frac{3}{4}} = a^{\frac{3}{2}} b^{\frac{9}{4}} = ab^2 \cdot a^{\frac{1}{2}} b^{\frac{1}{4}} = ab^2 \sqrt[4]{a^2 b}.$
13. $(\sqrt[6]{4 n^3})^3 = (4 n^3)^{\frac{3}{6}} = (4 n^3)^{\frac{1}{2}} = \sqrt{4 n^3} = 2 n \sqrt{n}.$
14. $(-2\sqrt{2 ab})^4 = 16 (2 ab)^2 = 16 (2 ab)^2 = 64 a^2 b^2.$
15. $(-\sqrt[2]{2} \sqrt[3]{x})^3 = -(2)^{\frac{3}{2}} x^{\frac{3}{2}} = -2 \cdot 2^{\frac{1}{2}} x^{\frac{3}{2}} = -2\sqrt{2} x.$
16. $(-\sqrt[2]{2} \sqrt[3]{ax^2})^4 = +2^{\frac{4}{2}} (ax^2)^{\frac{4}{3}} = 4 ax^2 (ax^2)^{\frac{2}{3}} = 4 ax^2 \sqrt[3]{ax^2}.$
17. $(-2\sqrt{x} \sqrt[3]{y})^5 = (-2)^5 x^{\frac{5}{2}} y^{\frac{5}{3}} = -32 x^2 y \cdot x^{\frac{1}{2}} y^{\frac{2}{3}}$
 $= -32 x^2 y \cdot x^{\frac{2}{3}} y^{\frac{4}{3}} = -32 x^{\frac{8}{3}} y^{\frac{10}{3}}.$
18. $(-3 a^{\frac{2}{3}} x^{\frac{3}{2}})^6 = (-3)^6 a^{3n} x^{2n} = 729 a^{3n} x^{2n}.$
19. $(2 + \sqrt{6})^2 = 2^2 + 2 \cdot 2\sqrt{6} + (\sqrt{6})^2$
 $= 4 + 4\sqrt{6} + 6$
 $= 10 + 4\sqrt{6}.$
20. $(2 + \sqrt{2})^2 = 2^2 + 2 \cdot 2\sqrt{2} + (\sqrt{2})^2$
 $= 4 + 4\sqrt{2} + 2$
 $= 6 + 4\sqrt{2}.$
21. $(2 + \sqrt{5})^3 = 2^3 + 3(2)^2\sqrt{5} + 3 \cdot 2(\sqrt{5})^2 + (\sqrt{5})^3$
 $= 8 + 12\sqrt{5} + 30 + 5\sqrt{5}$
 $= 38 + 17\sqrt{5}.$
22. $(2 - \sqrt{3})^3 = 2^3 - 3(2)^2\sqrt{3} + 3 \cdot 2(\sqrt{3})^2 - (\sqrt{3})^3$
 $= 8 - 12\sqrt{3} + 18 - 3\sqrt{3}$
 $= 26 - 15\sqrt{3}.$
23. $(\sqrt{7} - \sqrt{6})^2 = (\sqrt{7})^2 - 2\sqrt{7}\sqrt{6} + (\sqrt{6})^2$
 $= 7 - 2\sqrt{42} + 6$
 $= 13 - 2\sqrt{42}.$
24. $(2\sqrt{2} - \sqrt{3})^2 = (2\sqrt{2})^2 - 2 \cdot 2\sqrt{2}\sqrt{3} + (\sqrt{3})^2$
 $= 8 - 4\sqrt{6} + 3$
 $= 11 - 4\sqrt{6}.$
25. $(\sqrt{x} \pm 1)^2 = (\sqrt{x})^2 \pm 2\sqrt{x} \cdot 1 + 1^2$
 $= x \pm 2\sqrt{x} + 1.$
26. $(\sqrt{a} - \sqrt{b})^3 = (\sqrt{a})^3 - 3(\sqrt{a})^2\sqrt{b} + 3\sqrt{a}(\sqrt{b})^2 - (\sqrt{b})^3$
 $= a\sqrt{a} - 3a\sqrt{b} + 3b\sqrt{a} - b\sqrt{b}.$
27. $(\sqrt{x} \pm 1)^3 = (\sqrt{x})^3 \pm 3(\sqrt{x})^2 \cdot 1 + 3\sqrt{x}(1)^2 \pm (1)^3$
 $= x\sqrt{x} \pm 3x + 3\sqrt{x} \pm 1.$

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30. $\sqrt{\sqrt{2}} = (2^{\frac{1}{2}})^{\frac{1}{2}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$. 32. $\sqrt{\sqrt[6]{x^2}} = (x^{\frac{2}{6}})^{\frac{1}{2}} = x^{\frac{1}{6}} = \sqrt[6]{x}$.
31. $\sqrt{\sqrt[3]{5}} = (5^{\frac{1}{3}})^{\frac{1}{2}} = 5^{\frac{1}{6}} = \sqrt[6]{5}$. 33. $\sqrt{\sqrt[3]{x^n}} = (x^{\frac{n}{3}})^{\frac{1}{2}} = x^{\frac{n}{6}} = \sqrt[6]{x^n}$.
34. $\sqrt{\sqrt[5]{x^{12}}} = (x^{\frac{12}{5}})^{\frac{1}{2}} = x^{\frac{6}{5}} = x^1 x^{\frac{1}{5}} = x\sqrt[5]{x}$.
35. $\sqrt{\sqrt[n]{a^n x^2}} = [(a^n x^2)^{\frac{1}{n}}]^{\frac{1}{2}} = (a^n x^2)^{\frac{1}{2n}} = \sqrt[2n]{a^n x^2}$.
36. $\sqrt[3]{\sqrt{2}x} = [(2x)^{\frac{1}{2}}]^{\frac{1}{3}} = (2x)^{\frac{1}{6}} = \sqrt[6]{2x}$.
37. $\sqrt[3]{\sqrt{7}a^3} = [(7a^3)^{\frac{1}{2}}]^{\frac{1}{3}} = (7a^3)^{\frac{1}{6}} = \sqrt[6]{7a^3}$.
38. $\sqrt[3]{\sqrt[4]{8m^3x^3}} = (8^{\frac{1}{4}}m^{\frac{3}{4}}x^{\frac{3}{4}})^{\frac{1}{3}} = (2^{\frac{3}{4}}m^{\frac{3}{4}}x^{\frac{3}{4}})^{\frac{1}{3}} = 2^{\frac{1}{4}}m^{\frac{1}{4}}x^{\frac{1}{4}} = \sqrt[4]{2mx}$.
39. $\sqrt[3]{-27\sqrt{x^6}} = \sqrt[3]{-27x^3} = -3x$.
40. $\sqrt[3]{-64\sqrt[5]{a^3y^3}} = (-64)^{\frac{1}{3}}[(ay)^{\frac{3}{5}}]^{\frac{1}{3}} = -4(ay)^{\frac{1}{5}} = -4\sqrt[5]{ay}$.
41. $\sqrt[3]{-\sqrt{a^n b^n}} = -[(ab)^{\frac{n}{2}}]^{\frac{1}{3}} = -(ab)^{\frac{n}{6}} = -\sqrt[6]{a^n b^n}$.
42. $\sqrt{\sqrt[3]{4a^2x^4}} = \sqrt{\sqrt[3]{(2ax^2)^2}} = [(2ax^2)^{\frac{2}{3}}]^{\frac{1}{2}} = (2ax^2)^{\frac{1}{3}} = \sqrt[3]{2ax^2}$.
43. $\sqrt[3]{\sqrt{a^{12}x^4}} = \sqrt[3]{a^6x^2} = a^2\sqrt[3]{x^2}$.
44. $(\sqrt[3]{8a^3x^3})^{\frac{1}{2}} = [\sqrt{(2ax)^3}]^{\frac{1}{2}} = [(2ax)^{\frac{3}{2}}]^{\frac{1}{2}} = (2ax)^{\frac{3}{4}} = \sqrt[4]{2ax}$.
45. $(\sqrt[m]{x^m y^m})^{\frac{1}{mn}} = [(xy)^{\frac{m}{2}}]^{\frac{1}{mn}} = (xy)^{\frac{1}{2n}} = \sqrt[2n]{xy}$.
46. $\sqrt{\left(\frac{x^n b^2}{a^{-2}y^n}\right)^{\frac{2}{n}}} = \left(\frac{x^n b^2}{a^{-2}y^n}\right)^{\frac{1}{n}} = \left(\frac{x^n a^2 b^2}{y^n}\right)^{\frac{1}{n}}$
 $= \sqrt[n]{\frac{x^n}{y^n} \cdot a^2 b^2} = \frac{x}{y} \sqrt[n]{a^2 b^2}$.

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19. By § 111, the rational binomial of lowest degree in which $\sqrt[3]{a^2} + \sqrt[3]{b}$, or $a^{\frac{2}{3}} + b^{\frac{1}{3}}$, is exactly contained is $(a^{\frac{2}{3}})^3 + (b^{\frac{1}{3}})^3$, or $a^2 + b$. Therefore, the simplest rationalizing factor is the quotient, $a^{\frac{2}{3}} - a^{\frac{2}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

20. By § 111, the rational binomial of lowest degree in which $\sqrt[3]{a} - \sqrt[5]{x}$, or $a^{\frac{1}{3}} - x^{\frac{1}{5}}$, is exactly contained is $(a^{\frac{1}{3}})^{15} - (x^{\frac{1}{5}})^{15}$, or $a^5 - x^3$. Therefore, the simplest rationalizing factor is the quotient, $a^{\frac{1}{3}} + a^{\frac{1}{3}}x^{\frac{1}{5}} + a^{\frac{1}{3}}x^{\frac{2}{5}} + a^{\frac{1}{3}}x^{\frac{3}{5}} + a^{\frac{1}{3}}x^{\frac{4}{5}} + a^{\frac{2}{3}}x + a^{\frac{2}{3}}x^{\frac{1}{5}} + a^{\frac{2}{3}}x^{\frac{2}{5}} + a^{\frac{2}{3}}x^{\frac{3}{5}} + a^{\frac{2}{3}}x^{\frac{4}{5}} + a^{\frac{4}{3}}x^2 + a^{\frac{4}{3}}x^{\frac{1}{5}} + a^{\frac{4}{3}}x^{\frac{2}{5}} + a^{\frac{4}{3}}x^{\frac{3}{5}} + a^{\frac{4}{3}}x^{\frac{4}{5}}$.

21. By § 111, the rational binomial of lowest degree in which $\sqrt{x} + \sqrt[5]{y^3}$, or $x^{\frac{1}{2}} + y^{\frac{3}{5}}$, is exactly contained is $(x^{\frac{1}{2}})^{10} - (y^{\frac{3}{5}})^{10}$, or $x^5 - y^6$. Therefore, the simplest rationalizing factor is the quotient, $x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{3}{5}} + x^{\frac{1}{2}}y^{\frac{6}{5}} - x^{\frac{3}{2}}y^{\frac{3}{5}} + x^{\frac{3}{2}}y^{\frac{6}{5}} - x^{\frac{5}{2}}y^{\frac{3}{5}} + x^{\frac{5}{2}}y^{\frac{6}{5}} - y^{\frac{3}{5}}$.

$$31. \frac{\sqrt[5]{6}}{\sqrt[5]{12}} = \frac{\sqrt[5]{216}}{\sqrt[5]{144}} = \frac{\sqrt[5]{3}}{\sqrt[5]{2}} = \frac{\sqrt[5]{3}}{\sqrt[5]{2}} \times \frac{\sqrt[5]{32}}{\sqrt[5]{32}} = \frac{\sqrt[5]{96}}{\sqrt[5]{64}} = \frac{\sqrt[5]{96}}{2}.$$

$$32. \frac{\sqrt[3]{a}}{\sqrt[3]{ax^2}} = \frac{\sqrt[3]{a} \sqrt[3]{a^2x}}{\sqrt[3]{ax^2} \sqrt[3]{a^2x}} = \frac{\sqrt[3]{a^3} \sqrt[3]{a^2x}}{\sqrt[3]{a^3x^3}} = \frac{a \sqrt[3]{a^2x}}{ax} = \frac{\sqrt[3]{a^2x}}{x}.$$

$$33. \frac{\sqrt{a+b}}{\sqrt{a-b}} = \frac{\sqrt{a+b} \sqrt{a-b}}{\sqrt{(a-b)^2}} = \frac{\sqrt{a^2-b^2}}{a-b}.$$

$$34. \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{\sqrt{x-2} \sqrt{x+2}}{\sqrt{(x+2)^2}} = \frac{\sqrt{x^2-4}}{x+2}.$$

$$36. \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{(\sqrt{3} + \sqrt{2})(\sqrt{3} + \sqrt{2})}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})} = \frac{3 + 2\sqrt{6} + 2}{3 - 2} = 5 + 2\sqrt{6}.$$

$$37. \frac{5}{\sqrt{5} - \sqrt{3}} = \frac{5(\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} = \frac{5(\sqrt{5} + \sqrt{3})}{5 - 3} = \frac{5(\sqrt{5} + \sqrt{3})}{2}.$$

$$38. \frac{\sqrt{2}-1}{\sqrt{2}+\sqrt{3}} = \frac{(\sqrt{2}-1)(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{2-\sqrt{6}-\sqrt{2}+\sqrt{3}}{2-3} \\ = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2.$$

$$39. \frac{5-3\sqrt{2}}{2-\sqrt{2}} = \frac{(5-3\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{4-\sqrt{2}}{4-2} = \frac{4-\sqrt{2}}{2}.$$

$$40. \frac{1-\sqrt{7}}{\sqrt{8}-\sqrt{7}} = \frac{(1-\sqrt{7})(\sqrt{8}+\sqrt{7})}{(\sqrt{8}-\sqrt{7})(\sqrt{8}+\sqrt{7})} = \frac{2\sqrt{2}-2\sqrt{14}+\sqrt{7}-7}{8-7} \\ = 2\sqrt{2}-2\sqrt{14}+\sqrt{7}-7.$$

$$41. \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{x+2\sqrt{xy}+y}{x-y}.$$

$$42. \frac{4\sqrt{3}+6\sqrt{3}}{3\sqrt{3}-2\sqrt{2}} = \frac{2(3\sqrt{3}+2\sqrt{2})(3\sqrt{3}+2\sqrt{2})}{(3\sqrt{3}-2\sqrt{2})(3\sqrt{3}+2\sqrt{2})} = \frac{2(27+12\sqrt{6}+8)}{27-8} \\ = \frac{70+24\sqrt{6}}{19}.$$

$$43. \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} = \frac{(\sqrt{x+1}-2)(\sqrt{x+1}-2)}{(\sqrt{x+1}+2)(\sqrt{x+1}-2)} = \frac{x+1-4\sqrt{x+1}+4}{x+1-4} \\ = \frac{x+5-4\sqrt{x+1}}{x-3}.$$

44.

$$\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{(x-\sqrt{x^2-1})(x-\sqrt{x^2-1})}{(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})} = \frac{x^2-2x\sqrt{x^2-1}+x^2-1}{x^2-(x^2-1)} \\ = 2x^2-2x\sqrt{x^2-1}-1.$$

$$\begin{aligned}
 45. \quad \frac{a\sqrt{a} - \sqrt{x+1}}{\sqrt{a^3} + \sqrt{x+1}} &= \frac{(a\sqrt{a} - \sqrt{x+1})(a\sqrt{a} - \sqrt{x+1})}{(\sqrt{a^3} + \sqrt{x+1})(\sqrt{a^3} - \sqrt{x+1})} \\
 &= \frac{a^3 - 2a\sqrt{ax+a} + x+1}{a^3 - x - 1}.
 \end{aligned}$$

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$$\begin{aligned}
 46. \quad \frac{\sqrt{x+y} - \sqrt{x-y}}{\sqrt{x+y} + \sqrt{x-y}} &= \frac{(\sqrt{x+y} - \sqrt{x-y})(\sqrt{x+y} - \sqrt{x-y})}{(\sqrt{x+y} + \sqrt{x-y})(\sqrt{x+y} - \sqrt{x-y})} \\
 &= \frac{x+y - 2\sqrt{x^2-y^2} + x-y}{x+y - (x-y)} = \frac{2x - 2\sqrt{x^2-y^2}}{2y} = \frac{x - \sqrt{x^2-y^2}}{y}.
 \end{aligned}$$

$$\begin{aligned}
 47. \quad \frac{\sqrt{a^2+a+1} - 1}{\sqrt{a^2+a+1} + 1} &= \frac{(\sqrt{a^2+a+1} - 1)(\sqrt{a^2+a+1} - 1)}{(\sqrt{a^2+a+1} + 1)(\sqrt{a^2+a+1} - 1)} \\
 &= \frac{a^2+a+1 - 2\sqrt{a^2+a+1} + 1}{a^2+a+1 - 1} = \frac{a^2+a+2 - 2\sqrt{a^2+a+1}}{a^2+a}.
 \end{aligned}$$

$$49. \quad \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2} = \frac{5 \times 1.41421}{2} = 3.5355.$$

$$50. \quad \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} = \frac{2 \times 2.23607}{5} = .8944.$$

$$51. \quad \frac{6}{\sqrt{8}} = \frac{6\sqrt{2}}{4} = \frac{6 \times 1.41421}{4} = \frac{3 \times 1.41421}{2} = 2.1213.$$

$$52. \quad \frac{10}{\sqrt{45}} = \frac{10\sqrt{5}}{15} = \frac{10 \times 2.23607}{15} = \frac{2 \times 2.23607}{3} = 1.4907.$$

$$53. \quad \frac{15}{\sqrt{50}} = \frac{15\sqrt{2}}{10} = \frac{15 \times 1.41421}{10} = \frac{3 \times 1.41421}{2} = 2.1213.$$

$$54. \quad \frac{1}{\sqrt{125}} = \frac{\sqrt{5}}{25} = \frac{2.23607}{25} = .08944.$$

$$\begin{aligned}
 55. \quad \frac{8 - \sqrt{3}}{2 - \sqrt{3}} &= \frac{(8 - \sqrt{3})(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{16 + 6\sqrt{3} - 3}{4 - 3} = 13 + 6\sqrt{3} \\
 &= 13 + 6 \times 1.73205 = 23.3923.
 \end{aligned}$$

$$\begin{aligned}
 56. \quad \frac{1 + \sqrt{2}}{2 - \sqrt{2}} &= \frac{(1 + \sqrt{2})(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = \frac{2 + 3\sqrt{2} + 2}{4 - 2} = 2 + \frac{3}{2}\sqrt{2} \\
 &= 2 + \frac{3}{2} \times 1.41421 = 4.1213.
 \end{aligned}$$

$$\begin{aligned}
 57. \quad \frac{5 + 2\sqrt{5}}{5 - 2\sqrt{5}} &= \frac{(5 + 2\sqrt{5})(5 + 2\sqrt{5})}{(5 - 2\sqrt{5})(5 + 2\sqrt{5})} = \frac{25 + 20\sqrt{5} + 20}{25 - 20} \\
 &= 9 + 4\sqrt{5} = 9 + 4 \times 2.23607 = 17.9442.
 \end{aligned}$$

$$\begin{aligned}
 59. \quad \frac{\sqrt{2} - \sqrt{5} - \sqrt{7}}{\sqrt{2} + \sqrt{5} + \sqrt{7}} &= \frac{(\sqrt{2} - \sqrt{7}) - \sqrt{5}}{(\sqrt{2} + \sqrt{5}) + \sqrt{7}} \times \frac{(\sqrt{2} - \sqrt{7}) + \sqrt{5}}{(\sqrt{2} + \sqrt{5}) - \sqrt{7}} \\
 &= \frac{2 - 2\sqrt{14} + 7 - 5}{2 + 2\sqrt{10} + 5 - 7} = \frac{4 - 2\sqrt{14}}{2\sqrt{10}} \\
 &= \frac{2 - \sqrt{14}}{\sqrt{10}} = \frac{2\sqrt{10} - 2\sqrt{35}}{10} = \frac{\sqrt{10} - \sqrt{35}}{5}.
 \end{aligned}$$

60.

$$\begin{aligned}
 \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2} - \sqrt{6}} &= \frac{\sqrt{3} + \sqrt{2}}{(\sqrt{3} + \sqrt{2}) - \sqrt{6}} \times \frac{\sqrt{3} + \sqrt{2} + \sqrt{6}}{(\sqrt{3} + \sqrt{2}) + \sqrt{6}} \\
 &= \frac{3 + 2\sqrt{6} + 2 + 3\sqrt{2} + 2\sqrt{3}}{3 + 2\sqrt{6} + 2 - 6} = \frac{2\sqrt{6} + 5 + 3\sqrt{2} + 2\sqrt{3}}{2\sqrt{6} - 1} \\
 &= \frac{(2\sqrt{6} + 5 + 3\sqrt{2} + 2\sqrt{3})(2\sqrt{6} + 1)}{(2\sqrt{6} - 1)(2\sqrt{6} + 1)} \\
 &= \frac{24 + 10\sqrt{6} + 12\sqrt{3} + 12\sqrt{2} + 2\sqrt{6} + 5 + 3\sqrt{2} + 2\sqrt{3}}{24 - 1} \\
 &= \frac{29 + 12\sqrt{6} + 14\sqrt{3} + 15\sqrt{2}}{23}.
 \end{aligned}$$

$$\begin{aligned}
 61. \quad \frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}} &= \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})} \\
 &= \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{2 + 2\sqrt{6} + 3 - 5} = \frac{\sqrt{2} + \sqrt{3} - \sqrt{5}}{2\sqrt{6}} \\
 &= \frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{2\sqrt{6} \cdot \sqrt{6}} = \frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}.
 \end{aligned}$$

$$\begin{aligned}
 62. \quad \frac{2\sqrt{2} - 3\sqrt{3} + 4\sqrt{5}}{\sqrt{2} + \sqrt{3} - \sqrt{5}} &= \frac{2\sqrt{2} - 3\sqrt{3} + 4\sqrt{5}}{(\sqrt{2} + \sqrt{3}) - \sqrt{5}} \times \frac{\sqrt{2} + \sqrt{3} + \sqrt{5}}{(\sqrt{2} + \sqrt{3}) + \sqrt{5}} \\
 &= \frac{15 - \sqrt{6} + 6\sqrt{10} + \sqrt{15}}{2\sqrt{6}} \\
 &= \frac{15\sqrt{6} - 6 + 12\sqrt{15} + 3\sqrt{10}}{12} \\
 &= \frac{5\sqrt{6} - 2 + 4\sqrt{15} + \sqrt{10}}{4}.
 \end{aligned}$$

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$$11. \quad \sqrt{12 + 4\sqrt{5}} = \sqrt{12 + 2\sqrt{20}} = \sqrt{10} + \sqrt{2}.$$

$$12. \quad \sqrt{11 + 4\sqrt{7}} = \sqrt{11 + 2\sqrt{28}} = \sqrt{7} + \sqrt{4} = \sqrt{7} + 2.$$

$$13. \quad \sqrt{12 - 6\sqrt{3}} = \sqrt{12 - 2\sqrt{27}} = \sqrt{9} - \sqrt{3} = 3 - \sqrt{3}.$$

$$14. \quad \sqrt{17 + 12\sqrt{2}} = \sqrt{17 + 2\sqrt{72}} = \sqrt{9} + \sqrt{8} = 3 + 2\sqrt{2}.$$

$$15. \sqrt{15 - 6\sqrt{6}} = \sqrt{15 - 2\sqrt{54}} = \sqrt{9} - \sqrt{6} = 3 - \sqrt{6}.$$

$$16. \sqrt{18 + 6\sqrt{5}} = \sqrt{18 + 2\sqrt{45}} = \sqrt{15} + \sqrt{3}.$$

$$17. \sqrt{a^2 + b + 2a\sqrt{b}} = \sqrt{a^2 + b + 2\sqrt{a^2b}} = \sqrt{a^2} + \sqrt{b} = a + \sqrt{b}.$$

$$18. \sqrt{2a - 2\sqrt{a^2 - b^2}} = \sqrt{(a+b) + (a-b) - 2\sqrt{(a+b)(a-b)}} \\ = \sqrt{a+b} - \sqrt{a-b}.$$

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$$2. \text{ Let } \sqrt{x} + \sqrt{y} = \sqrt{25 + 10\sqrt{6}}. \quad (1)$$

$$\text{Then, } \sqrt{x} - \sqrt{y} = \sqrt{25 - 10\sqrt{6}}. \quad (2)$$

$$\text{Multiplying (1) by (2), } x - y = \sqrt{625 - 600} = \sqrt{25} = 5. \quad (3)$$

$$\text{Squaring (1), } x + y + 2\sqrt{xy} = 25 + 10\sqrt{6}. \\ \therefore x + y = 25. \quad (4)$$

$$\text{Solving (4) and (3), } x = 15, y = 10. \\ \therefore \sqrt{x} = \sqrt{15}, \sqrt{y} = \sqrt{10}.$$

$$\text{Hence, } \sqrt{25 + 10\sqrt{6}} = \sqrt{15} + \sqrt{10}.$$

$$3. \text{ Let } \sqrt{x} + \sqrt{y} = \sqrt{19 + 6\sqrt{2}}. \quad (1)$$

$$\text{Then, } \sqrt{x} - \sqrt{y} = \sqrt{19 - 6\sqrt{2}}. \quad (2)$$

$$\text{Multiplying (1) by (2), } x - y = \sqrt{361 - 72} = \sqrt{289} = 17. \quad (3)$$

$$\text{Squaring (1), } x + y + 2\sqrt{xy} = 19 + 6\sqrt{2}. \\ \therefore x + y = 19. \quad (4)$$

$$\text{Solving (4) and (3), } x = 18, y = 1. \\ \therefore \sqrt{x} = 3\sqrt{2}, \sqrt{y} = 1.$$

$$\text{Hence, } \sqrt{19 + 6\sqrt{2}} = 3\sqrt{2} + 1.$$

$$4. \text{ Let } \sqrt{x} + \sqrt{y} = \sqrt{45 + 30\sqrt{2}}. \quad (1)$$

$$\text{Then, } \sqrt{x} - \sqrt{y} = \sqrt{45 - 30\sqrt{2}}. \quad (2)$$

$$\text{Multiplying (1) by (2), } x - y = \sqrt{2025 - 1800} = \sqrt{225} = 15. \quad (3)$$

$$\text{Squaring (1), } x + y + 2\sqrt{xy} = 45 + 30\sqrt{2}. \\ \therefore x + y = 45. \quad (4)$$

$$\text{Solving (4) and (3), } x = 30, y = 15. \\ \therefore \sqrt{x} = \sqrt{30}, \sqrt{y} = \sqrt{15}.$$

$$\text{Hence, } \sqrt{45 + 30\sqrt{2}} = \sqrt{30} + \sqrt{15}.$$

$$5. \text{ Let } \sqrt{x} - \sqrt{y} = \sqrt{35 - 14\sqrt{6}}. \quad (1)$$

$$\text{Then, } \sqrt{x} + \sqrt{y} = \sqrt{35 + 14\sqrt{6}}. \quad (2)$$

$$\text{Multiplying (1) by (2), } x - y = \sqrt{1225 - 1176} = \sqrt{49} = 7. \quad (3)$$

$$\text{Squaring (2), } x + y + 2\sqrt{xy} = 35 + 14\sqrt{6}. \\ \therefore x + y = 35. \quad (4)$$

$$\text{Solving (4) and (3), } x = 21, y = 14. \\ \therefore \sqrt{x} = \sqrt{21}, \sqrt{y} = \sqrt{14}.$$

$$\text{Hence, } \sqrt{35 - 14\sqrt{6}} = \sqrt{21} - \sqrt{14}.$$

6. Let $\sqrt{x} + \sqrt{y} = \sqrt{11 + 6\sqrt{2}}$ (1)

Then, $\sqrt{x} - \sqrt{y} = \sqrt{11 - 6\sqrt{2}}$ (2)

Multiplying (1) by (2), $x - y = \sqrt{121 - 72} = \sqrt{49} = 7$. (3)

Squaring (1), $x + y + 2\sqrt{xy} = 11 + 6\sqrt{2}$.
 $\therefore x + y = 11$. (4)

Solving (4) and (3), $x = 9, y = 2$.
 $\therefore \sqrt{x} = 3, \sqrt{y} = \sqrt{2}$.

Hence, $\sqrt{11 + 6\sqrt{2}} = 3 + \sqrt{2}$.

7. Let $\sqrt{x} - \sqrt{y} = \sqrt{24 - 8\sqrt{5}}$ (1)

Then, $\sqrt{x} + \sqrt{y} = \sqrt{24 + 8\sqrt{5}}$ (2)

Multiplying (1) by (2), $x - y = \sqrt{576 - 320} = \sqrt{256} = 16$. (3)

Squaring (2), $x + y + 2\sqrt{xy} = 24 + 8\sqrt{5}$.
 $\therefore x + y = 24$. (4)

Solving (4) and (3), $x = 20, y = 4$.
 $\therefore \sqrt{x} = 2\sqrt{5}, \sqrt{y} = 2$.

Hence, $\sqrt{24 - 8\sqrt{5}} = 2\sqrt{5} - 2$.

8. Let $\sqrt{x} + \sqrt{y} = \sqrt{16 + 6\sqrt{7}}$ (1)

Then, $\sqrt{x} - \sqrt{y} = \sqrt{16 - 6\sqrt{7}}$ (2)

Multiplying (1) by (2), $x - y = \sqrt{256 - 252} = \sqrt{4} = 2$. (3)

Squaring (1), $x + y + 2\sqrt{xy} = 16 + 6\sqrt{7}$.
 $\therefore x + y = 16$. (4)

Solving (4) and (3), $x = 9, y = 7$.
 $\therefore \sqrt{x} = 3, \sqrt{y} = \sqrt{7}$.

Hence, $\sqrt{16 + 6\sqrt{7}} = 3 + \sqrt{7}$.

9. Let $\sqrt{x} - \sqrt{y} = \sqrt{21 - 8\sqrt{5}}$ (1)

Then, $\sqrt{x} + \sqrt{y} = \sqrt{21 + 8\sqrt{5}}$ (2)

Multiplying (1) by (2), $x - y = \sqrt{441 - 320} = \sqrt{121} = 11$. (3)

Squaring (2), $x + y + 2\sqrt{xy} = 21 + 8\sqrt{5}$.
 $\therefore x + y = 21$. (4)

Solving (4) and (3), $x = 16, y = 5$.
 $\therefore \sqrt{x} = 4, \sqrt{y} = \sqrt{5}$.

Hence, $\sqrt{21 - 8\sqrt{5}} = 4 - \sqrt{5}$.

10. Let $\sqrt{x} - \sqrt{y} = \sqrt{47 - 12\sqrt{11}}$ (1)

Then, $\sqrt{x} + \sqrt{y} = \sqrt{47 + 12\sqrt{11}}$ (2)

Multiplying (1) by (2), $x - y = \sqrt{2209 - 1584} = \sqrt{625} = 25$. (3)

Squaring (2), $x + y + 2\sqrt{xy} = 47 + 12\sqrt{11}$.
 $\therefore x + y = 47$. (4)

Solving (4) and (3), $x = 36, y = 11$.
 $\therefore \sqrt{x} = 6, \sqrt{y} = \sqrt{11}$.

Hence, $\sqrt{47 - 12\sqrt{11}} = 6 - \sqrt{11}$.

11. Let

$$\sqrt{x} + \sqrt{y} = \sqrt{56 + 32\sqrt{3}}. \quad (1)$$

Then,

$$\sqrt{x} - \sqrt{y} = \sqrt{56 - 32\sqrt{3}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{3136 - 3072} = \sqrt{64} = 8. \quad (3)$$

Squaring (1),

$$x + y + 2\sqrt{xy} = 56 + 32\sqrt{3}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} x + y &= 56. \\ x &= 32, y = 24. \\ \therefore \sqrt{x} &= 4\sqrt{2}, \sqrt{y} = 2\sqrt{6}. \end{aligned}$$

Hence,

$$\sqrt{56 + 32\sqrt{3}} = 4\sqrt{2} + 2\sqrt{6}.$$

12. Let

$$\sqrt{x} - \sqrt{y} = \sqrt{35 - 12\sqrt{6}}. \quad (1)$$

Then,

$$\sqrt{x} + \sqrt{y} = \sqrt{35 + 12\sqrt{6}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{1225 - 864} = \sqrt{361} = 19. \quad (3)$$

Squaring (2),

$$x + y + 2\sqrt{xy} = 35 + 12\sqrt{6}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} x + y &= 35. \\ x &= 27, y = 8. \\ \therefore \sqrt{x} &= 3\sqrt{3}, \sqrt{y} = 2\sqrt{2}. \end{aligned}$$

Hence,

$$\sqrt{35 - 12\sqrt{6}} = 3\sqrt{3} - 2\sqrt{2}.$$

13. Let

$$\sqrt{x} - \sqrt{y} = \sqrt{56 - 12\sqrt{3}}. \quad (1)$$

Then,

$$\sqrt{x} + \sqrt{y} = \sqrt{56 + 12\sqrt{3}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{3136 - 432} = \sqrt{2704} = 52. \quad (3)$$

Squaring (2),

$$x + y + 2\sqrt{xy} = 56 + 12\sqrt{3}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} x + y &= 56. \\ x &= 54, y = 2. \\ \therefore \sqrt{x} &= 3\sqrt{6}, \sqrt{y} = \sqrt{2}. \end{aligned}$$

Hence,

$$\sqrt{56 - 12\sqrt{3}} = 3\sqrt{6} - \sqrt{2}.$$

14. Let

$$\sqrt{x} + \sqrt{y} = \sqrt{2 + \sqrt{3}}. \quad (1)$$

Then,

$$\sqrt{x} - \sqrt{y} = \sqrt{2 - \sqrt{3}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{4 - 3} = 1. \quad (3)$$

Squaring (1),

$$x + y + 2\sqrt{xy} = 2 + \sqrt{3}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} x + y &= 2. \\ x &= \frac{3}{2}, y = \frac{1}{2}. \\ \therefore \sqrt{x} &= \frac{1}{2}\sqrt{6}, \sqrt{y} = \frac{1}{2}\sqrt{2}. \end{aligned}$$

Hence,

$$\sqrt{2 + \sqrt{3}} = \frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2}.$$

15. Let

$$\sqrt{x} + \sqrt{y} = \sqrt{6 + \sqrt{35}}. \quad (1)$$

Then,

$$\sqrt{x} - \sqrt{y} = \sqrt{6 - \sqrt{35}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{36 - 35} = 1. \quad (3)$$

Squaring (1),

$$x + y + 2\sqrt{xy} = 6 + \sqrt{35}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} x + y &= 6. \\ x &= \frac{7}{2}, y = \frac{5}{2}. \\ \therefore \sqrt{x} &= \frac{1}{2}\sqrt{14}, \sqrt{y} = \frac{1}{2}\sqrt{10}. \end{aligned}$$

Hence,

$$\sqrt{6 + \sqrt{35}} = \frac{1}{2}\sqrt{14} + \frac{1}{2}\sqrt{10}.$$

16. Let

$$\sqrt{x} + \sqrt{y} = \sqrt{1 + \frac{2}{3}\sqrt{2}}. \quad (1)$$

Then,

$$\sqrt{x} - \sqrt{y} = \sqrt{1 - \frac{2}{3}\sqrt{2}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{1 - \frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}. \quad (3)$$

Squaring (1),

$$x + y + 2\sqrt{xy} = 1 + \frac{2}{3}\sqrt{2}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} \therefore x + y &= 1. \\ x &= \frac{2}{3}, y = \frac{1}{3}. \\ \therefore \sqrt{x} &= \frac{1}{3}\sqrt{6}, \sqrt{y} = \frac{1}{3}\sqrt{3}. \end{aligned}$$

Hence,

$$\sqrt{1 + \frac{2}{3}\sqrt{2}} = \frac{1}{3}\sqrt{6} + \frac{1}{3}\sqrt{3}.$$

17. Let

$$\sqrt{x} + \sqrt{y} = \sqrt{2 + \frac{4}{5}\sqrt{6}}. \quad (1)$$

Then,

$$\sqrt{x} - \sqrt{y} = \sqrt{2 - \frac{4}{5}\sqrt{6}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{4 - \frac{16}{25}} = \sqrt{\frac{4}{25}} = \frac{2}{5}. \quad (3)$$

Squaring (1),

$$x + y + 2\sqrt{xy} = 2 + \frac{4}{5}\sqrt{6}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} \therefore x + y &= 2. \\ x &= \frac{6}{5}, y = \frac{4}{5}. \\ \therefore \sqrt{x} &= \frac{1}{5}\sqrt{30}, \sqrt{y} = \frac{2}{5}\sqrt{5}. \end{aligned}$$

Hence,

$$\sqrt{2 + \frac{4}{5}\sqrt{6}} = \frac{1}{5}\sqrt{30} + \frac{2}{5}\sqrt{5}.$$

18. Let

$$\sqrt{x} + \sqrt{y} = \sqrt{30 + 20\sqrt{2}}. \quad (1)$$

Then,

$$\sqrt{x} - \sqrt{y} = \sqrt{30 - 20\sqrt{2}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{900 - 800} = \sqrt{100} = 10. \quad (3)$$

Squaring (1),

$$x + y + 2\sqrt{xy} = 30 + 20\sqrt{2}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} \therefore x + y &= 30. \\ x &= 20, y = 10. \\ \therefore \sqrt{x} &= 2\sqrt{5}, \sqrt{y} = \sqrt{10}. \end{aligned}$$

Hence,

$$\sqrt{30 + 20\sqrt{2}} = 2\sqrt{5} + \sqrt{10}.$$

19. Let

$$\sqrt{x} - \sqrt{y} = \sqrt{18 - 6\sqrt{5}}. \quad (1)$$

Then,

$$\sqrt{x} + \sqrt{y} = \sqrt{18 + 6\sqrt{5}}. \quad (2)$$

Multiplying (1) by (2),

$$x - y = \sqrt{324 - 180} = \sqrt{144} = 12. \quad (3)$$

Squaring (2),

$$x + y + 2\sqrt{xy} = 18 + 6\sqrt{5}. \quad (4)$$

Solving (4) and (3),

$$\begin{aligned} \therefore x + y &= 18. \\ x &= 15, y = 3. \\ \therefore \sqrt{x} &= \sqrt{15}, \sqrt{y} = \sqrt{3}. \end{aligned}$$

Hence,

$$\sqrt{18 - 6\sqrt{5}} = \sqrt{15} - \sqrt{3}.$$

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13.

$$\sqrt{x + 16} - \sqrt{x} = 2.$$

Transposing,

$$\sqrt{x + 16} = \sqrt{x} + 2.$$

Squaring,

$$x + 16 = x + 4\sqrt{x} + 4.$$

Canceling, etc.,

$$\sqrt{x} = 3.$$

Squaring,

$$x = 9.$$

14.

Transposing,

Squaring,

Canceling, etc.,

Squaring, etc.,

$$\sqrt{2x} - \sqrt{2x - 15} = 1.$$

$$\sqrt{2x} - 1 = \sqrt{2x - 15}.$$

$$2x - 2\sqrt{2x} + 1 = 2x - 15.$$

$$\sqrt{2x} = 8.$$

$$x = 32.$$

15.

Squaring,

Canceling $x^2 = x^2$,

Solving,

$$\sqrt{x^2 + x + 1} = 2 - x.$$

$$x^2 + x + 1 = 4 - 4x + x^2.$$

$$x + 1 = 4 - 4x.$$

$$x = \frac{3}{5}.$$

16.

Dividing by 3,

Squaring,

Canceling $x^2 = x^2$,

Solving,

$$3\sqrt{x^2 - 9} = 3x - 3.$$

$$\sqrt{x^2 - 9} = x - 1.$$

$$x^2 - 9 = x^2 - 2x + 1.$$

$$-9 = -2x + 1.$$

$$x = 5.$$

17.

Squaring,

Simplifying,

Dividing by 4 and squaring,

$$\sqrt{x + 2} = \sqrt{x + 32}.$$

$$x + 4\sqrt{x + 4} = x + 32.$$

$$4\sqrt{x} = 28.$$

$$x = 49.$$

18.

Transposing,

Squaring, § 218, Prin. 1,

Canceling, etc.,

Dividing by 10 and squaring,

$$5 - \sqrt{x + 5} = \sqrt{x}.$$

$$-\sqrt{x + 5} = \sqrt{x} - 5.$$

$$+ (x + 5) = x - 10\sqrt{x} + 25.$$

$$10\sqrt{x} = 20.$$

$$x = 4.$$

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19.

Dividing by \sqrt{x} ,

Transposing, etc.,

Dividing by 2 and squaring,

$$2\sqrt{x} - x = x - 8\sqrt{x}.$$

$$2 - \sqrt{x} = \sqrt{x} - 8.$$

$$2\sqrt{x} = 10.$$

$$x = 25.$$

NOTE. — The given equation is satisfied also by $x = 0$. The root $x = 0$ was removed by dividing both members by \sqrt{x} .

20.

Squaring,

Canceling $4x^2 = 4x^2$,

Solving,

$$\sqrt{4x^2 + 6x - 10} = 2x + 4.$$

$$4x^2 + 6x - 10 = 4x^2 + 16x + 16.$$

$$6x - 10 = 16x + 16.$$

$$x = -\frac{13}{5}.$$

$x = -\frac{13}{5}$ satisfies the equation $-\sqrt{4x^2 + 6x - 10} = 2x + 4$, but not the given equation, which is an *impossible* equation.

21.

Transposing,

Squaring,

Canceling $x^2 = x^2$,

Solving,

$$\sqrt{x^2 - 5x + 7} + 2 = x.$$

$$\sqrt{x^2 - 5x + 7} = x - 2.$$

$$x^2 - 5x + 7 = x^2 - 4x + 4.$$

$$-5x + 7 = -4x + 4.$$

$$x = 3.$$

22. $4 - \sqrt{4 - 8x + 9x^2} = 3x.$
 Transposing, $-\sqrt{4 - 8x + 9x^2} = 3x - 4.$
 Squaring, § 218, Prin. 1, $+(4 - 8x + 9x^2) = 9x^2 - 24x + 16.$
 Canceling $9x^2 = 9x^2$, $4 - 8x = -24x + 16.$
 Solving, $x = \frac{3}{4}.$

23. $\sqrt{2(1-x)(3-2x)} - 1 = 2x.$
 Transposing, $\sqrt{2(1-x)(3-2x)} = 2x + 1.$
 Squaring, $2(1-x)(3-2x) = 4x^2 + 4x + 1.$
 Expanding, $6 - 10x + 4x^2 = 4x^2 + 4x + 1.$
 Canceling $4x^2 = 4x^2$, $6 - 10x = 4x + 1.$
 Solving, $x = \frac{5}{14}.$

24. $\sqrt{2x-1} + \sqrt{2x+4} = 5.$
 Transposing, $\sqrt{2x-1} = 5 - \sqrt{2x+4}.$
 Squaring, $2x-1 = 25 - 10\sqrt{2x+4} + (2x+4).$
 Canceling $2x = 2x$, transposing and uniting terms,
 $10\sqrt{2x+4} = 30.$
 Dividing by 10 and squaring, $2x+4 = 9.$
 Solving, $x = \frac{5}{2}.$

25. $\sqrt{3x-5} + \sqrt{3x+7} = 6.$
 Transposing, $\sqrt{3x+7} = 6 - \sqrt{3x-5}.$
 Squaring, $3x+7 = 36 - 12\sqrt{3x-5} + (3x-5).$
 Canceling $3x = 3x$, transposing and uniting terms,
 $12\sqrt{3x-5} = 24.$
 Dividing by 12 and squaring, $3x-5 = 4.$
 Solving, $x = 3.$

26. $\sqrt{16x+3} + \sqrt{16x+8} = 5.$
 Transposing, $\sqrt{16x+8} = 5 - \sqrt{16x+3}.$
 Squaring, $16x+8 = 25 - 10\sqrt{16x+3} + (16x+3)$
 Canceling $16x = 16x$, transposing and uniting terms,
 $10\sqrt{16x+3} = 20.$
 Dividing by 10 and squaring, $16x+3 = 4.$
 Solving, $x = \frac{1}{16}.$

27. $\sqrt{9x+8} + \sqrt{9x-4} = 0.$
 Transposing, $\sqrt{9x-4} = -\sqrt{9x+8}.$
 Squaring, § 218, Prin. 1, $9x-4 = + (9x+8).$
 Canceling $9x = 9x$, $-4 = 8.$
 Dividing by -8 , transposing, etc., $\sqrt{9x} = 1.$
 Squaring, $9x = 1.$
 $\therefore x = \frac{1}{9}.$

28.

$$\sqrt{1 + x\sqrt{x^2 + 12}} = 1 + x.$$

Squaring,

$$1 + x\sqrt{x^2 + 12} = 1 + 2x + x^2.$$

Canceling $1 = 1$ and dividing by x , $\sqrt{x^2 + 12} = 2 + x.$

Squaring,

$$x^2 + 12 = 4 + 4x + x^2.$$

Canceling $x^2 = x^2$,

$$12 = 4 + 4x.$$

Solving,

$$x = 2.$$

NOTE. — The given equation is satisfied also for $x = 0$.

29.

$$\sqrt{7 + 3\sqrt{5x - 16}} - 4 = 0.$$

Transposing,

$$\sqrt{7 + 3\sqrt{5x - 16}} = 4.$$

Squaring,

$$7 + 3\sqrt{5x - 16} = 16.$$

Transposing, uniting terms, and dividing by 3,

$$\sqrt{5x - 16} = 3.$$

Squaring,

$$5x - 16 = 9.$$

Solving,

$$x = 5.$$

30.

$$2x + \sqrt{4x^2 - \sqrt{16x^2 - 7}} = 1.$$

Transposing,

$$\sqrt{4x^2 - \sqrt{16x^2 - 7}} = 1 - 2x.$$

Squaring,

$$4x^2 - \sqrt{16x^2 - 7} = 1 - 4x + 4x^2.$$

Canceling $4x^2 = 4x^2$,

$$-\sqrt{16x^2 - 7} = 1 - 4x.$$

Squaring, § 218, Prin. 1,

$$+ (16x^2 - 7) = 1 - 8x + 16x^2.$$

Canceling $16x^2 = 16x^2$,

$$-7 = 1 - 8x.$$

Solving,

$$x = 1.$$

$x = 1$ satisfies the equation $2x - \sqrt{4x^2 - \sqrt{16x^2 - 7}} = 1$, but not the given equation, which is an *impossible* equation.

31.

$$\sqrt{7 + \sqrt{1 + \sqrt{4 + \sqrt{1 + 2\sqrt{x}}}}} = 3.$$

Squaring,

$$7 + \sqrt{1 + \sqrt{4 + \sqrt{1 + 2\sqrt{x}}}} = 9.$$

Transposing, etc.,

$$\sqrt{1 + \sqrt{4 + \sqrt{1 + 2\sqrt{x}}}} = 2.$$

Squaring,

$$1 + \sqrt{4 + \sqrt{1 + 2\sqrt{x}}} = 4.$$

Transposing, etc.,

$$\sqrt{4 + \sqrt{1 + 2\sqrt{x}}} = 3.$$

Squaring,

$$4 + \sqrt{1 + 2\sqrt{x}} = 9.$$

Transposing, etc.,

$$\sqrt{1 + 2\sqrt{x}} = 5.$$

Squaring,

$$1 + 2\sqrt{x} = 25.$$

Transposing, etc.,

$$2\sqrt{x} = 24.$$

Dividing by 2 and squaring,

$$x = 144.$$

32.

Multiplying by $\sqrt{3x+2}$,
 Transposing, etc.,
 Squaring,
 Canceling $9x^2 = 9x^2$,
 Solving,

$$\begin{aligned}\frac{5}{\sqrt{3x+2}} &= \sqrt{3x+2} + \sqrt{3x-1}. \\ 5 &= 3x+2 + \sqrt{9x^2+3x-2}. \\ 3-3x &= \sqrt{9x^2+3x-2}. \\ 9-18x+9x^2 &= 9x^2+3x-2. \\ 9-18x &= 3x-2. \\ x &= \frac{11}{21}.\end{aligned}$$

33.

Reducing to mixed numbers,

Canceling $1 = 1$ and dividing by 16,

Clearing of fractions, etc.,

Squaring, etc.,

$$\begin{aligned}\frac{\sqrt{2x+9}}{\sqrt{2x-7}} &= \frac{\sqrt{2x+20}}{\sqrt{2x-12}}. \\ 1 + \frac{16}{\sqrt{2x-7}} &= 1 + \frac{32}{\sqrt{2x-12}}. \\ \frac{1}{\sqrt{2x-7}} &= \frac{2}{\sqrt{2x-12}}. \\ \sqrt{2x-7} &= \sqrt{2x-12}. \\ \sqrt{2x-2}\sqrt{2x} &= 12-14. \\ -\sqrt{2x} &= -2. \\ x &= 2.\end{aligned}$$

34.

Canceling $1 = 1$ and dividing by 16,

Clearing of fractions, etc.,

Squaring,

$$\begin{aligned}\frac{\sqrt{x+18}}{\sqrt{x+2}} &= \frac{32}{\sqrt{x+6}} + 1. \\ 1 + \frac{16}{\sqrt{x+2}} &= \frac{32}{\sqrt{x+6}} + 1. \\ \frac{1}{\sqrt{x+2}} &= \frac{2}{\sqrt{x+6}}. \\ \sqrt{x+2} &= \sqrt{x+6}. \\ \sqrt{x-2}\sqrt{x} &= -6+4. \\ -\sqrt{x} &= -2. \\ x &= 4.\end{aligned}$$

35.

Clearing of fractions,

Squaring,

Canceling $x^2 = x^2$,

Solving,

$$\begin{aligned}\frac{\sqrt{x-1}}{\sqrt{x+5}} &= \frac{\sqrt{x-3}}{\sqrt{x-1}}. \\ x-1 &= \sqrt{x^2+2x-15}. \\ x^2-2x+1 &= x^2+2x-15. \\ -2x+1 &= 2x-15. \\ x &= 4.\end{aligned}$$

36.

Reducing to mixed numbers,

Canceling $1 = 1$, changing signs,

Clearing of fractions, etc.,

Dividing by 2 and squaring,

$$\begin{aligned}\frac{\sqrt{x-6}}{\sqrt{x-1}} &= \frac{\sqrt{x-8}}{\sqrt{x-5}}. \\ 1 - \frac{5}{\sqrt{x-1}} &= 1 - \frac{3}{\sqrt{x-5}}. \\ \frac{5}{\sqrt{x-1}} &= \frac{3}{\sqrt{x-5}}. \\ 5\sqrt{x-1} &= 3\sqrt{x-5}. \\ 2\sqrt{x} &= 22. \\ x &= 121.\end{aligned}$$

37.

Clearing of fractions,

Squaring,

Canceling $x^2 = x^2$,

Solving,

$$\frac{\sqrt{x-3}}{\sqrt{x+1}} = \frac{\sqrt{x-4}}{\sqrt{x-2}}$$

$$\sqrt{x^2-5x+6} = \sqrt{x^2-3x-4}$$

$$x^2-5x+6 = x^2-3x-4$$

$$-5x+6 = -3x-4$$

$$x = 5.$$

38.

Reducing to mixed numbers,

Canceling $1 = 1$,

Clearing of fractions, etc.,

Squaring, etc.,

$$\frac{\sqrt{2x+6}}{\sqrt{2x+4}} = \frac{\sqrt{2x+2}}{\sqrt{2x+1}}$$

$$1 + \frac{2}{\sqrt{2x+4}} = 1 + \frac{1}{\sqrt{2x+1}}$$

$$\frac{2}{\sqrt{2x+4}} = \frac{1}{\sqrt{2x+1}}$$

$$2\sqrt{2x} - \sqrt{2x} = -2 + 4$$

$$\sqrt{2x} = 2$$

$$x = 2.$$

39.

Clearing of fractions,

Transposing, etc.,

Dividing by $\sqrt{11}$,

Squaring,

Solving,

$$\frac{\sqrt{11x} + \sqrt{2x+3}}{\sqrt{11x} - \sqrt{2x+3}} = \frac{8}{3}$$

$$3\sqrt{11x} + 3\sqrt{2x+3} = 8\sqrt{11x} - 8\sqrt{2x+3}$$

$$11\sqrt{2x+3} = 5\sqrt{11x}$$

$$\sqrt{11}\sqrt{2x+3} = 5\sqrt{x}$$

$$22x + 33 = 25x$$

$$x = 11.$$

40.

Reducing to lowest terms,

Reducing to mixed numbers,

Canceling and dividing by 2,

Clearing of fractions, etc.,

Transposing and squaring,

Solving,

$$\frac{2\sqrt{2x+4}}{2\sqrt{2x}-4} = \frac{3\sqrt{x+1}+9}{3\sqrt{x+1}-9}$$

$$\frac{\sqrt{2x}+2}{\sqrt{2x}-2} = \frac{\sqrt{x+1}+3}{\sqrt{x+1}-3}$$

$$1 + \frac{4}{\sqrt{2x}-2} = 1 + \frac{6}{\sqrt{x+1}-3}$$

$$\frac{4}{\sqrt{2x}-2} = \frac{6}{\sqrt{x+1}-3}$$

$$2\sqrt{x+1} - 3\sqrt{2x} = 6 - 6 = 0$$

$$18x = 4x + 4$$

$$x = \frac{2}{7}.$$

41. See next page.

42.

Clearing of fractions, etc.,

Dividing by 4 and squaring,

Solving,

$$\frac{\sqrt{4x+3} + 2\sqrt{x-1}}{\sqrt{4x+3} - 2\sqrt{x-1}} = 5$$

$$12\sqrt{x-1} = 4\sqrt{4x+3}$$

$$9(x-1) = 4x+3$$

$$x = \frac{13}{5}.$$

$$41. \quad \frac{\sqrt{\sqrt{5x-9}}}{\sqrt{\sqrt{5x+11}}} = \frac{\sqrt{\sqrt{5x-21}}}{\sqrt{\sqrt{5x-16}}}$$

$$\text{Squaring,} \quad \frac{\sqrt{5x-9}}{\sqrt{5x+11}} = \frac{\sqrt{5x-21}}{\sqrt{5x-16}}$$

$$\text{Reducing to mixed numbers,} \quad 1 - \frac{20}{\sqrt{5x+11}} = 1 - \frac{5}{\sqrt{5x-16}}$$

$$\text{Canceling } 1 = 1 \text{ and dividing by } -5, \quad \frac{4}{\sqrt{5x+11}} = \frac{1}{\sqrt{5x-16}}$$

$$\text{Clearing of fractions, etc.,} \quad 4\sqrt{5x} - \sqrt{5x} = 64 + 11.$$

$$\text{Squaring, etc.,} \quad \begin{aligned} \sqrt{5x} &= 25. \\ x &= 125. \end{aligned}$$

$$43. \quad \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} = \frac{1}{2}$$

$$\text{Clearing of fractions, etc.,} \quad \sqrt{x+1} = 3\sqrt{x-1}.$$

$$\text{Squaring,} \quad x+1 = 9(x-1).$$

$$\text{Solving,} \quad x = \frac{5}{4}.$$

$$44. \quad \frac{x-3}{\sqrt{x}-\sqrt{3}} = \frac{\sqrt{x}+\sqrt{3}}{2} + 2\sqrt{3}.$$

$$\text{Reducing the first member,} \quad \sqrt{x} + \sqrt{3} = \frac{\sqrt{x} + \sqrt{3}}{2} + 2\sqrt{3}.$$

$$\text{Clearing of fractions,} \quad 2\sqrt{x} + 2\sqrt{3} = \sqrt{x} + \sqrt{3} + 4\sqrt{3}.$$

$$\text{Transposing, etc.,} \quad \sqrt{x} = 3\sqrt{3}.$$

$$\text{Squaring,} \quad x = 27.$$

$$45. \quad \frac{\sqrt{19x} + \sqrt{2x+11}}{\sqrt{19x} - \sqrt{2x+11}} = 2\frac{1}{4}.$$

$$\text{Clearing of fractions,} \quad 6\sqrt{19x} + 6\sqrt{2x+11} = 13\sqrt{19x} - 13\sqrt{2x+11}$$

$$\text{Transposing, etc.,} \quad 19\sqrt{2x+11} = 7\sqrt{19x}.$$

$$\text{Dividing by } \sqrt{19}, \quad \sqrt{19}\sqrt{2x+11} = 7\sqrt{x}.$$

$$\text{Squaring,} \quad 19(2x+11) = 49x.$$

$$\text{Solving,} \quad \begin{aligned} 38x + 209 &= 49x. \\ x &= 19. \end{aligned}$$

$$46. \quad 2\sqrt{x} - \sqrt{4x-22} - \sqrt{2} = 0.$$

$$\text{Dividing by } \sqrt{2}, \quad \sqrt{2x} - \sqrt{2x-11} - 1 = 0.$$

$$\text{Transposing,} \quad -\sqrt{2x-11} = 1 - \sqrt{2x}.$$

$$\text{Squaring,} \quad + (2x-11) = 1 - 2\sqrt{2x} + 2x.$$

$$\text{Canceling } 2x = 2x, \text{ etc.,} \quad -12 = -2\sqrt{2x}.$$

$$\text{Dividing by } -2, \quad 6 = \sqrt{2x}.$$

$$\text{Squaring, etc.,} \quad x = 18.$$

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47. $1 + \sqrt{(3-5x)^2 + 16} = 2(3-x).$

Transposing, etc., $\sqrt{(3-5x)^2 + 16} = 5 - 2x.$

Squaring, etc., $9 - 30x + 25x^2 + 16 = 25 - 20x + 4x^2.$

Canceling $9 + 16 = 25$, $25x^2 - 30x = 4x^2 - 20x.$

Transposing, dividing by x , etc., $21x = 10.$

$\therefore x = \frac{10}{21}.$

NOTE. — The given equation is satisfied also for $x = 0$.

48. $\sqrt{x} + \sqrt{x - \sqrt{a^2 - x}} = \sqrt{a}.$

Transposing, $\sqrt{x - \sqrt{a^2 - x}} = \sqrt{a} - \sqrt{x}.$

Squaring, $x - \sqrt{a^2 - x} = a - 2\sqrt{ax} + x.$

Canceling $x = x$ and squaring, $+(a^2 - x) = a^2 - 4a\sqrt{ax} + 4ax.$

Canceling $a^2 = a^2$, $-x = -4a\sqrt{ax} + 4ax.$

Dividing by $-\sqrt{x}$ and transposing, $4a\sqrt{x} + \sqrt{x} = 4a\sqrt{a}.$

Squaring, $(4a + 1)^2 x = 16a^3.$

$\therefore x = \frac{16a^3}{(4a + 1)^2}.$

49. $\sqrt{x} + \sqrt{x - (a-b)^2} = a + b.$

Transposing, $\sqrt{x - (a-b)^2} = a + b - \sqrt{x}.$

Squaring, etc., $x - a^2 + 2ab - b^2 = a^2 + 2ab + b^2 - 2(a+b)\sqrt{x} + x.$

Canceling, etc., $(a+b)\sqrt{x} = a^2 + b^2.$

Dividing by $a + b$ and squaring, $x = \left(\frac{a^2 + b^2}{a + b}\right)^2.$

50. $\sqrt{mn - x} - \sqrt{x}\sqrt{mn - 1} = \sqrt{mn}\sqrt{1 - x}.$

Squaring,

$mn - x - 2\sqrt{x(mn - x)(mn - 1)} + mn - x = mn - mn x.$

Transposing and uniting terms, and dividing by 2,

$mn x - x = \sqrt{x(mn - x)(mn - 1)}$

Dividing by $\sqrt{mn x - x}$,

$\sqrt{mn x - x} = \sqrt{mn - x}.$

Squaring,

$mn x - x = mn - x.$

$\therefore x = 1.$

NOTE. — The given equation is satisfied also for $x = 0$.

51. $a\sqrt{x} - b\sqrt{x} = a^2 + b^2 - 2ab.$

Dividing by $a - b$,

$\sqrt{x} = a - b.$

Squaring,

$x = (a - b)^2.$

52. $\sqrt{5ax - 9a^2} + a = \sqrt{5ax}.$

Transposing, $\sqrt{5ax - 9a^2} = \sqrt{5ax} - a.$

Squaring, $5ax - 9a^2 = 5ax - 2a\sqrt{5ax} + a^2.$

Canceling, etc., $-10a^2 = -2a\sqrt{5ax}.$

Dividing by $-2a$ and squaring, $25a^2 = 5ax.$

$\therefore x = 5a.$

53.

$$\sqrt{x+3a} = \frac{6a}{\sqrt{x+3a}} - \sqrt{x}.$$

Clearing of fractions,

$$x+3a = 6a - \sqrt{x(x+3a)}.$$

Transposing, etc.,

$$x-3a = -\sqrt{x^2+3ax}.$$

Squaring,

$$x^2 - 6ax + 9a^2 = + (x^2 + 3ax).$$

Canceling $x^2 = x^2$,

$$-6ax + 9a^2 = 3ax.$$

Solving,

$$x = a.$$

54. See next page.

55.

$$\sqrt{2x} + \sqrt{10x+1} = \sqrt{2x} + 1.$$

Squaring,

$$2x + \sqrt{10x+1} = 2x + 2\sqrt{2x} + 1.$$

Canceling $2x = 2x$,

$$\sqrt{10x+1} = 2\sqrt{2x} + 1.$$

Squaring,

$$10x+1 = 8x + 4\sqrt{2x} + 1.$$

Transposing, etc.,

$$2x = 4\sqrt{2x}.$$

Dividing by $\sqrt{2x}$,

$$\sqrt{2x} = 4.$$

Squaring, etc.,

$$x = 8.$$

NOTE. — The given equation is satisfied also for $x = 0$.

56.

$$\frac{\sqrt{x+a} + \sqrt{x-a}}{\sqrt{x+a} - \sqrt{x-a}} = 2 + \frac{\sqrt{x^2-a^2}}{a}.$$

Rationalizing the denominator of the first member,

$$\frac{x+a+2\sqrt{x^2-a^2}+x-a}{x+a-(x-a)} = 2 + \frac{\sqrt{x^2-a^2}}{a}.$$

$$\frac{x+\sqrt{x^2-a^2}}{a} = \frac{2a+\sqrt{x^2-a^2}}{a}.$$

$$\therefore x = 2a.$$

58.

$$\sqrt{2x} + \sqrt{3x} + \sqrt{5x} = \sqrt{m}.$$

Factoring,

$$\sqrt{x}(\sqrt{2} + \sqrt{3} + \sqrt{5}) = \sqrt{m}.$$

Multiplying by $\sqrt{2} + \sqrt{3} - \sqrt{5}$,

$$\sqrt{x}(2 + 2\sqrt{6} + 3 - 5) = \sqrt{m}(\sqrt{2} + \sqrt{3} - \sqrt{5}).$$

$$\sqrt{x} \cdot 2\sqrt{6} = \sqrt{m}(\sqrt{2} + \sqrt{3} - \sqrt{5}).$$

Squaring,

$$24x = m(\sqrt{2} + \sqrt{3} - \sqrt{5})^2.$$

$$\therefore x = \frac{m(\sqrt{2} + \sqrt{3} - \sqrt{5})^2}{24}.$$

59.

$$\sqrt{2x} + \sqrt{3x} - \sqrt{5x} = \sqrt{c}.$$

Factoring,

$$\sqrt{x}(\sqrt{2} + \sqrt{3} - \sqrt{5}) = \sqrt{c}.$$

Multiplying by $\sqrt{2} + \sqrt{3} + \sqrt{5}$,

$$\sqrt{x}(2 + 2\sqrt{6} + 3 - 5) = \sqrt{c}(\sqrt{2} + \sqrt{3} + \sqrt{5}).$$

$$\sqrt{x} \cdot 2\sqrt{6} = \sqrt{c}(\sqrt{2} + \sqrt{3} + \sqrt{5}).$$

Squaring,

$$24x = c(\sqrt{2} + \sqrt{3} + \sqrt{5})^2.$$

$$\therefore x = \frac{c(\sqrt{2} + \sqrt{3} + \sqrt{5})^2}{24}.$$

54.

$$\sqrt{2x} - \sqrt{2x-7} = \frac{3}{\sqrt{2x-7}}.$$

Clearing of fractions, $\sqrt{4x^2 - 14x} - (2x - 7) = 3$.

Transposing, etc.,

$$\sqrt{4x^2 - 14x} = 2x - 4.$$

Squaring,

$$4x^2 - 14x = 4x^2 - 16x + 16.$$

Canceling $4x^2 = 4x^2$,

$$-14x = -16x + 16.$$

Solving,

$$x = 8.$$

60.

$$\sqrt{x-a} + \sqrt{2(x-a)} = \sqrt{3x+a\sqrt{2}}.$$

Factoring,

$$\sqrt{x-a}(1+\sqrt{2}) = \sqrt{3x+a\sqrt{2}}.$$

Squaring,

$$(x-a)(3+2\sqrt{2}) = 3x+a\sqrt{2}.$$

$$3x+2x\sqrt{2}-3a-2a\sqrt{2}=3x+a\sqrt{2}.$$

Transposing, etc.,

$$2x\sqrt{2}=3a+3a\sqrt{2}.$$

$$\therefore x = \frac{3a(1+\sqrt{2})}{2\sqrt{2}} = \frac{3a}{4}(\sqrt{2}+2).$$

61.

$$\text{Transposing, } \sqrt{x-1} + \sqrt{2x-2} = \sqrt{3x-3} + \sqrt{2}.$$

$$\text{Factoring, } \sqrt{x-1} + \sqrt{2x-2} - \sqrt{3x-3} = \sqrt{2}.$$

Multiplying by $1 + \sqrt{2} + \sqrt{3}$,

$$\sqrt{x-1}(1+2\sqrt{2}+2-3) = \sqrt{2}(1+\sqrt{2}+\sqrt{3}).$$

$$2\sqrt{2}\sqrt{x-1} = \sqrt{2}(1+\sqrt{2}+\sqrt{3}).$$

Dividing by $\sqrt{2}$ and squaring,

$$4(x-1) = (1+\sqrt{2}+\sqrt{3})^2,$$

$$\therefore x-1 = \frac{(1+\sqrt{2}+\sqrt{3})^2}{4},$$

whence,

$$x = 1 + \frac{(1+\sqrt{2}+\sqrt{3})^2}{4} \\ = \frac{5+\sqrt{2}+\sqrt{3}+\sqrt{6}}{2}.$$

62.

$$\sqrt{2x-3} + \sqrt{4x-6} = \sqrt{2x} + \sqrt{x}.$$

Factoring,

$$\sqrt{2x-3}(1+\sqrt{2}) = \sqrt{x}(\sqrt{2}+1).$$

Dividing by $(1+\sqrt{2})$,

$$\sqrt{2x-3} = \sqrt{x}.$$

Squaring,

$$2x-3=x.$$

$$\therefore x=3.$$

REVIEW

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$$1. \frac{6x^3 - 7x^2 - 5x}{9x^3 - 25x} = \frac{x(2x+1)(3x-5)}{x(3x+5)(3x-5)} = \frac{2x+1}{3x+5}.$$

$$2. \frac{8x^2 + 18x - 5}{12x^2 + 5x - 2} = \frac{(4x-1)(2x+5)}{(4x-1)(3x+2)} = \frac{2x+5}{3x+2}.$$

$$3. \frac{a^2x^2 - a\sqrt{x} + x}{\sqrt{x}} = a^2x\sqrt{x} - a + \sqrt{x}.$$

$$4. \frac{a^2 - 2a\sqrt{b} + b}{a - \sqrt{b}} = \frac{(a - \sqrt{b})^2}{a - \sqrt{b}} = a - \sqrt{b}.$$

$$\begin{aligned} 5. \frac{\sqrt{2} - \sqrt{3} - \sqrt{5}}{\sqrt{2} + \sqrt{3} + \sqrt{5}} &= \frac{(\sqrt{2} - \sqrt{5}) - \sqrt{3}}{(\sqrt{2} + \sqrt{3}) + \sqrt{5}} \times \frac{(\sqrt{2} - \sqrt{5}) + \sqrt{3}}{(\sqrt{2} + \sqrt{3}) - \sqrt{5}} \\ &= \frac{2 - 2\sqrt{10} + 5 - 3}{2 + 2\sqrt{6} + 3 - 5} = \frac{4 - 2\sqrt{10}}{2\sqrt{6}} \\ &= \frac{2 - \sqrt{10}}{\sqrt{6}} = \frac{2\sqrt{6} - 2\sqrt{15}}{6} = \frac{\sqrt{6} - \sqrt{15}}{3}. \end{aligned}$$

$$\begin{aligned} 6. \frac{2 - \sqrt{5}}{2 + \sqrt{5}} + \frac{2\sqrt{3}}{\sqrt{243}} &= \frac{(2 - \sqrt{5})(2 - \sqrt{5})}{(2 + \sqrt{5})(2 - \sqrt{5})} + \frac{2\sqrt{3}}{9\sqrt{3}} \\ &= \frac{4 - 4\sqrt{5} + 5}{4 - 5} + \frac{2}{9} = -9 + 4\sqrt{5} + \frac{2}{9} \\ &= 4\sqrt{5} - 8\frac{7}{9}. \end{aligned}$$

$$\begin{aligned} 7. \frac{x - y}{x + y} - \frac{y + x}{y - x} - \frac{4x^2y^2}{x^4 - y^4} &= \frac{x - y}{x + y} + \frac{x + y}{x - y} - \frac{4x^2y^2}{x^4 - y^4} \\ &= \frac{2(x^2 + y^2)}{x^2 - y^2} - \frac{4x^2y^2}{x^4 - y^4} \\ &= \frac{2x^4 + 4x^2y^2 + 2y^4 - 4x^2y^2}{x^4 - y^4} = \frac{2(x^4 + y^4)}{x^4 - y^4}. \end{aligned}$$

$$\begin{aligned} 8. \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}} - \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}} &= \frac{(\sqrt{2} - \sqrt{3})^2 - (\sqrt{2} + \sqrt{3})^2}{(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})} \\ &= \frac{2 - 2\sqrt{6} + 3 - 2 - 2\sqrt{6} - 3}{2 - 3} = 4\sqrt{6}. \end{aligned}$$

9.

$$\begin{aligned} \frac{x + \sqrt{xy} + y}{\sqrt{x} + \sqrt{y}} + \frac{x\sqrt{x} + y\sqrt{y}}{x + y} &= \frac{(x + \sqrt{xy} + y)(\sqrt{x} - \sqrt{y})}{(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})} + \frac{x\sqrt{x} + y\sqrt{y}}{x + y} \\ &= \frac{x\sqrt{x} - y\sqrt{y}}{x - y} + \frac{x\sqrt{x} + y\sqrt{y}}{x + y} \\ &= \frac{x^2\sqrt{x} - xy\sqrt{y} + xy\sqrt{x} - y^2\sqrt{y} + x^2\sqrt{x} + xy\sqrt{y} - xy\sqrt{x} - y^2\sqrt{y}}{x^2 - y^2} \\ &= \frac{2(x^2\sqrt{x} - y^2\sqrt{y})}{x^2 - y^2}. \end{aligned}$$

$$\begin{aligned} 10. \frac{1}{\sqrt{a} + \sqrt{b}} - 1 + \frac{1}{\sqrt{a} - \sqrt{b}} &= \frac{\sqrt{a} - \sqrt{b} - (a - b) + \sqrt{a} + \sqrt{b}}{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})} \\ &= \frac{2\sqrt{a} - a + b}{a - b}. \end{aligned}$$

$$\begin{aligned}
 11. \quad \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} &= \frac{(\sqrt{1+x^2} - \sqrt{1-x^2})(\sqrt{1+x^2} - \sqrt{1-x^2})}{(\sqrt{1+x^2} + \sqrt{1-x^2})(\sqrt{1+x^2} - \sqrt{1-x^2})} \\
 &= \frac{1+x^2 - 2\sqrt{1-x^4} + 1-x^2}{1+x^2 - (1-x^2)} = \frac{2 - 2\sqrt{1-x^4}}{2x^2} = \frac{1 - \sqrt{1-x^4}}{x^2}.
 \end{aligned}$$

$$12. \quad \frac{1}{1 - \sqrt{2x}} + \frac{1}{1 + \sqrt{2x}} + \frac{1}{1 - 2x} = \frac{1 + \sqrt{2x} + 1 - \sqrt{2x} + 1}{1 - 2x} = \frac{3}{1 - 2x}.$$

$$\begin{aligned}
 13. \quad \frac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} - \frac{x - \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}} &= \frac{(x + \sqrt{x^2 - a^2})^2 - (x - \sqrt{x^2 - a^2})^2}{x^2 - (x^2 - a^2)} \\
 &= \frac{4x\sqrt{x^2 - a^2}}{a^2}.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \frac{\sqrt{a+1} + \sqrt{a-1}}{\sqrt{a+1} - \sqrt{a-1}} - \frac{\sqrt{a+1} - \sqrt{a-1}}{\sqrt{a+1} + \sqrt{a-1}} \\
 = \frac{(\sqrt{a+1} + \sqrt{a-1})^2 - (\sqrt{a+1} - \sqrt{a-1})^2}{a+1 - (a-1)} = \frac{4\sqrt{a^2-1}}{2} = 2\sqrt{a^2-1}.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \frac{a^2 - 2ax - 3x^2}{3a^2 + 5ax + 2x^2} \times \frac{6a^2 + 7ax + 2x^2}{a^2 - 4ax + 3x^2} \\
 = \frac{(a-3x)(a+x)}{(3a+2x)(a+x)} \times \frac{(3a+2x)(2a+x)}{(a-x)(a-3x)} = \frac{2a+x}{a-x}.
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \frac{a^2 - b}{a^2 - 2a\sqrt{b} + b} \times \frac{a^2 - 4a\sqrt{b} + 4b}{a^2 + 2a\sqrt{b} + b} \\
 = \frac{(a + \sqrt{b})(a - \sqrt{b})}{(a - \sqrt{b})(a - \sqrt{b})} \times \frac{(a - 2\sqrt{b})(a - 2\sqrt{b})}{(a + \sqrt{b})(a + \sqrt{b})} = \frac{(a - 2\sqrt{b})^2}{a^2 - b}.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \left(1 - \frac{a}{b}\right) \div \left(1 + \sqrt{\frac{a}{b}}\right) &= \left[1^2 - \left(\sqrt{\frac{a}{b}}\right)^2\right] \div \left[1 + \sqrt{\frac{a}{b}}\right] = 1 - \sqrt{\frac{a}{b}} \\
 &= 1 - \frac{\sqrt{ab}}{b}.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \frac{1+a+a^2}{1+\sqrt{a+a}} &= \frac{1+a+a^2}{1+\sqrt{a+a}} \times \frac{1-\sqrt{a}}{1-\sqrt{a+a}} \\
 &= \frac{(1+\sqrt{a+a})(1-\sqrt{a+a})}{1+\sqrt{a+a}} \times \frac{1-\sqrt{a}}{1-\sqrt{a+a}} = 1 - \sqrt{a}.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad 1 \div \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} \right) + \frac{\sqrt{ab}}{a+b} &= 1 \div \frac{a-b}{\sqrt{ab}} + \frac{\sqrt{ab}}{a+b} \\
 &= \frac{\sqrt{ab}}{a-b} + \frac{\sqrt{ab}}{a+b} \\
 &= \frac{(a+b+a-b)\sqrt{ab}}{a^2-b^2} = \frac{2a\sqrt{ab}}{a^2-b^2}.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \frac{\left(\frac{a}{\sqrt{x}} + \frac{\sqrt{x}}{a} \right) \left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a} \right)}{\left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a} \right) \left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a} \right)} &= \frac{\frac{a}{\sqrt{x}} + \frac{\sqrt{x}}{a}}{\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a}} = \frac{\frac{a^2+x}{a\sqrt{x}}}{\frac{a^2-x}{a\sqrt{x}}} = \frac{a^2+x}{a^2-x}.
 \end{aligned}$$

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$$\begin{aligned}
 21. \quad (a^3 - b^2)^3 &= (a^3)^3 - 3(a^3)^2(b^2) + 3(a^3)(b^2)^2 - (b^2)^3 \\
 &= a^9 - 3a^6b^2 + 3a^3b^4 - b^6.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad (2a - 3b)^4 &= (2a)^4 - 4(2a)^3(3b) + 6(2a)^2(3b)^2 - 4(2a)(3b)^3 + (3b)^4 \\
 &= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \left(\frac{a}{3} - \frac{b}{2} \right)^3 &= \left(\frac{a}{3} \right)^3 - 3 \left(\frac{a}{3} \right)^2 \left(\frac{b}{2} \right) + 3 \left(\frac{a}{3} \right) \left(\frac{b}{2} \right)^2 - \left(\frac{b}{2} \right)^3 \\
 &= \frac{a^3}{27} - \frac{a^2b}{6} + \frac{ab^2}{4} - \frac{b^3}{8}.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \left(ax + \frac{1}{a} \right)^5 &= (ax)^5 + 5(ax)^4 \left(\frac{1}{a} \right) + 10(ax)^3 \left(\frac{1}{a} \right)^2 + 10(ax)^2 \left(\frac{1}{a} \right)^3 + 5(ax) \left(\frac{1}{a} \right)^4 + \left(\frac{1}{a} \right)^5 \\
 &= a^5x^5 + 5a^4x^4 + 10a^3x^3 + \frac{10x^2}{a} + \frac{5x}{a^2} + \frac{1}{a^5}.
 \end{aligned}$$

$$25. \quad (a^{-2} + a^{-1})^2 = (a^{-2})^2 + 2(a^{-2})(a^{-1}) + (a^{-1})^2 = a^{-4} + 2a^{-3} + a^{-2}.$$

$$\begin{aligned}
 26. \quad (a^{-1} + b)^4 &= (a^{-1})^4 + 4(a^{-1})^3b + 6(a^{-1})^2b^2 + 4(a^{-1})b^3 + b^4 \\
 &= a^{-4} + 4a^{-3}b + 6a^{-2}b^2 + 4a^{-1}b^3 + b^4.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad (a^{\frac{1}{2}} - b^{\frac{1}{2}})^6 &= (a^{\frac{1}{2}})^6 - 6(a^{\frac{1}{2}})^5(b^{\frac{1}{2}}) + 15(a^{\frac{1}{2}})^4(b^{\frac{1}{2}})^2 - 20(a^{\frac{1}{2}})^3(b^{\frac{1}{2}})^3 \\
 &\quad + 15(a^{\frac{1}{2}})^2(b^{\frac{1}{2}})^4 - 6(a^{\frac{1}{2}})(b^{\frac{1}{2}})^5 + (b^{\frac{1}{2}})^6 \\
 &= a^3 - 6a^{\frac{5}{2}}b^{\frac{1}{2}} + 15a^2b - 20a^{\frac{3}{2}}b^{\frac{3}{2}} + 15ab^2 - 6a^{\frac{1}{2}}b^{\frac{5}{2}} + b^3.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad (a^{\frac{1}{2}} - b^{-\frac{1}{2}})^4 &= (a^{\frac{1}{2}})^4 - 4(a^{\frac{1}{2}})^3(b^{-\frac{1}{2}}) + 6(a^{\frac{1}{2}})^2(b^{-\frac{1}{2}})^2 - 4(a^{\frac{1}{2}})(b^{-\frac{1}{2}})^3 + (b^{-\frac{1}{2}})^4 \\
 &= a^2 - 4a^{\frac{3}{2}}b^{-\frac{1}{2}} + 6ab^{-1} - 4a^{\frac{1}{2}}b^{-\frac{3}{2}} + b^{-2}.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad (a^{-\frac{1}{2}} - b^{-\frac{1}{2}})^6 &= (a^{-\frac{1}{2}})^6 - 6(a^{-\frac{1}{2}})^5(b^{-\frac{1}{2}}) + 15(a^{-\frac{1}{2}})^4(b^{-\frac{1}{2}})^2 - 20(a^{-\frac{1}{2}})^3(b^{-\frac{1}{2}})^3 \\
 &\quad + 15(a^{-\frac{1}{2}})^2(b^{-\frac{1}{2}})^4 - 6(a^{-\frac{1}{2}})(b^{-\frac{1}{2}})^5 + (b^{-\frac{1}{2}})^6 \\
 &= a^{-3} - 6a^{-\frac{5}{2}}b^{-\frac{1}{2}} + 15a^{-2}b^{-1} - 20a^{-\frac{3}{2}}b^{-\frac{3}{2}} + 15a^{-1}b^{-2} - 6a^{-\frac{1}{2}}b^{-\frac{5}{2}} + b^{-3}.
 \end{aligned}$$

30. See next page.

$$\begin{aligned} 31. \quad (a - \sqrt{b})^4 &= a^4 - 4a^3\sqrt{b} + 6a^2(\sqrt{b})^2 - 4a(\sqrt{b})^3 + (\sqrt{b})^4 \\ &= a^4 - 4a^3\sqrt{b} + 6a^2b - 4ab\sqrt{b} + b^2. \end{aligned}$$

$$\begin{aligned} 32. \quad (\sqrt{x} + \sqrt{y})^6 &= (\sqrt{x})^6 + 6(\sqrt{x})^5\sqrt{y} + 15(\sqrt{x})^4(\sqrt{y})^2 + 20(\sqrt{x})^3(\sqrt{y})^3 \\ &\quad + 15(\sqrt{x})^2(\sqrt{y})^4 + 6\sqrt{x}(\sqrt{y})^5 + (\sqrt{y})^6 \\ &= x^3 + 6x^2\sqrt{xy} + 15x^2y + 20xy\sqrt{xy} + 15xy^2 + 6y^2\sqrt{xy} + y^3. \end{aligned}$$

$$\begin{aligned} 33. \quad (\sqrt{2} - \sqrt{3})^4 &= (\sqrt{2})^4 - 4(\sqrt{2})^3\sqrt{3} + 6(\sqrt{2})^2(\sqrt{3})^2 - 4\sqrt{2}(\sqrt{3})^3 + (\sqrt{3})^4 \\ &= 2^2 - 4 \cdot 2\sqrt{2} \cdot \sqrt{3} + 6 \cdot 2 \cdot 3 - 4 \cdot 3\sqrt{2} \cdot \sqrt{3} + 3^2 \\ &= 4 - 8\sqrt{6} + 36 - 12\sqrt{6} + 9 = 49 - 20\sqrt{6}. \end{aligned}$$

$$\begin{aligned} 34. \quad (\sqrt{5} - 2)^6 &= (\sqrt{5})^6 - 6(\sqrt{5})^5 \cdot 2 + 15(\sqrt{5})^4 \cdot 2^2 - 20(\sqrt{5})^3 \cdot 2^3 \\ &\quad + 15(\sqrt{5})^2 \cdot 2^4 - 6\sqrt{5} \cdot 2^5 + 2^6 \\ &= 5^3 - 6 \cdot 5^2\sqrt{5} \cdot 2 + 15 \cdot 5^2 \cdot 2^2 - 20 \cdot 5\sqrt{5} \cdot 2^3 \\ &\quad + 15 \cdot 5 \cdot 2^4 - 6\sqrt{5} \cdot 2^5 + 2^6 \\ &= 125 - 300\sqrt{5} + 1500 - 800\sqrt{5} + 1200 - 192\sqrt{5} + 64 \\ &= 2889 - 1292\sqrt{5}. \end{aligned}$$

$$\begin{aligned} 35. \quad (\sqrt[3]{4} - \sqrt[3]{2})^3 &= (\sqrt[3]{4})^3 - 3(\sqrt[3]{4})^2(\sqrt[3]{2}) + 3(\sqrt[3]{4})(\sqrt[3]{2})^2 - (\sqrt[3]{2})^3 \\ &= 4 - 3 \cdot 2\sqrt[3]{4} + 3 \cdot 2\sqrt[3]{2} - 2 = 2 - 6\sqrt[3]{4} + 6\sqrt[3]{2}. \end{aligned}$$

$$\begin{aligned} 36. \quad (\sqrt{2} - \sqrt[3]{2})^6 &= (\sqrt{2})^6 - 6(\sqrt{2})^5(\sqrt[3]{2}) + 15(\sqrt{2})^4(\sqrt[3]{2})^2 - 20(\sqrt{2})^3(\sqrt[3]{2})^3 \\ &\quad + 15(\sqrt{2})^2(\sqrt[3]{2})^4 - 6\sqrt{2}(\sqrt[3]{2})^5 + (\sqrt[3]{2})^6 \\ &= 2^3 - 6 \cdot 2^2\sqrt{2} \cdot \sqrt[3]{2} + 15 \cdot 2^2 \cdot \sqrt[3]{4} - 20 \cdot 2\sqrt{2} \cdot 2 \\ &\quad + 15 \cdot 2 \cdot 2\sqrt[3]{2} - 6\sqrt{2} \cdot 2\sqrt[3]{4} + 2^3 \\ &= 8 - 24\sqrt[6]{32} + 60\sqrt[3]{4} - 80\sqrt{2} + 60\sqrt[3]{2} - 24\sqrt[6]{2} + 4 \\ &= 12 - 24\sqrt[6]{32} + 60\sqrt[3]{4} - 80\sqrt{2} + 60\sqrt[3]{2} - 24\sqrt[6]{2}. \end{aligned}$$

$$\begin{aligned} 37. \quad &\frac{9x^4}{4} + 3x^3 - x^2 - \frac{4x}{3} + \frac{4}{9} \left| \frac{3x^2}{2} + x - \frac{2}{3} \right. \\ &\frac{9x^4}{4} \\ &\frac{3x^2}{3x^2 + x} \left| \frac{3x^3}{3x^3 + x^2} \right. \\ &\frac{3x^2 + 2x}{3x^2 + 2x} \left| -2x^2 \right. \\ &\frac{3x^2 + 2x - \frac{2}{3}}{3} \left| -2x^2 - \frac{4x}{3} + \frac{4}{9} \right. \end{aligned}$$

$$30. \quad (a^{\frac{1}{3}} + b^{\frac{1}{3}})^6 = (a^{\frac{1}{3}})^6 + 6(a^{\frac{1}{3}})^5(b^{\frac{1}{3}}) + 15(a^{\frac{1}{3}})^4(b^{\frac{1}{3}})^2 + 20(a^{\frac{1}{3}})^3(b^{\frac{1}{3}})^3 \\ + 15(a^{\frac{1}{3}})^2(b^{\frac{1}{3}})^4 + 6(a^{\frac{1}{3}})(b^{\frac{1}{3}})^5 + (b^{\frac{1}{3}})^6 \\ = a^2 + 6a^{\frac{5}{3}}b^{\frac{1}{3}} + 15a^{\frac{4}{3}}b^{\frac{2}{3}} + 20ab^{\frac{3}{2}} + 15a^{\frac{2}{3}}b^2 + 6a^{\frac{1}{3}}b^{\frac{5}{3}} + b^3.$$

$$38. \quad \frac{x^2}{4} - 2xy + \frac{xz}{4} + 4y^2 - yz + \frac{z^2}{16} \left| \frac{x}{2} - 2y + \frac{z}{4} \right.$$

| | | |
|------------------------|----------------|-------------------------|
| x | $-2xy$ | |
| $x - 2y$ | $-2xy$ | $+ 4y^2$ |
| $x - 4y$ | $\frac{xz}{4}$ | $- yz + \frac{z^2}{16}$ |
| $x - 4y + \frac{z}{4}$ | $\frac{xz}{4}$ | $- yz + \frac{z^2}{16}$ |

$$39. \quad \frac{a^2 + 12a^{\frac{3}{2}}b^{\frac{1}{2}} + 54ab + 108a^{\frac{3}{2}}b^{\frac{3}{2}} + 81b^2}{a^2} \left| a + 6a^{\frac{1}{2}}b^{\frac{1}{2}} + 9b \right.$$

| | |
|--|--|
| $2a$ | $12a^{\frac{3}{2}}b^{\frac{1}{2}}$ |
| $2a + 6a^{\frac{1}{2}}b^{\frac{1}{2}}$ | $12a^{\frac{3}{2}}b^{\frac{1}{2}} + 36ab$ |
| $2a + 12a^{\frac{1}{2}}b^{\frac{1}{2}}$ | $18ab$ |
| $2a + 12a^{\frac{1}{2}}b^{\frac{1}{2}} + 9b$ | $18ab + 108a^{\frac{3}{2}}b^{\frac{3}{2}} + 81b^2$ |

$$40. \quad \frac{1 + 2\sqrt{x} - x - 2x\sqrt{x} + x^2}{1} \left| 1 + \sqrt{x} - x \right.$$

| | |
|---------------------|--------------------------|
| 2 | $2\sqrt{x}$ |
| $2 + \sqrt{x}$ | $2\sqrt{x} + x$ |
| $2 + 2\sqrt{x}$ | $-2x$ |
| $2 + 2\sqrt{x} - x$ | $-2x - 2x\sqrt{x} + x^2$ |

$$41. \quad \frac{a - 4\sqrt{ab} + 4b + 6\sqrt{ac} - 12\sqrt{bc} + 9c}{a} \left| \sqrt{a} - 2\sqrt{b} + 3\sqrt{c} \right.$$

| | |
|-------------------------------------|---------------------------------|
| $2\sqrt{a}$ | $-4\sqrt{ab} + 4b$ |
| $2\sqrt{a} - 2\sqrt{b}$ | $-4\sqrt{ab} + 4b$ |
| $2\sqrt{a} - 4\sqrt{b}$ | $6\sqrt{ac} - 12\sqrt{bc} + 9c$ |
| $2\sqrt{a} - 4\sqrt{b} + 3\sqrt{c}$ | $6\sqrt{ac} - 12\sqrt{bc} + 9c$ |

$$42. \quad \frac{x^2 - 4x\sqrt{xy} + 6xy - 4y\sqrt{xy} + y^2}{x^2} \left| x - 2\sqrt{xy} + y \right.$$

| | |
|-----------------------|---------------------------|
| $2x$ | $-4x\sqrt{xy}$ |
| $2x - 2\sqrt{xy}$ | $-4x\sqrt{xy} + 4xy$ |
| $2x - 4\sqrt{xy}$ | $2xy$ |
| $2x - 4\sqrt{xy} + y$ | $2xy - 4y\sqrt{xy} + y^2$ |

43. See next page.

$$44. \quad \begin{array}{r} 81 \cdot 23 \cdot 41 \cdot 69 \\ 81 \end{array} \quad | \quad 9013$$

$$\begin{array}{r} 900 \times 2 = 1800 \quad | \quad 23 \quad 41 \\ 1800 + 1 = 1801 \quad | \quad 18 \quad 01 \\ 9010 \times 2 = 18020 \quad | \quad 5 \quad 40 \quad 69 \\ 18020 + 3 = 18023 \quad | \quad 5 \quad 40 \quad 69 \end{array}$$

$$46. \quad \begin{array}{r} .00 \cdot 02 \cdot 28 \cdot 01 \\ 1 \end{array} \quad | \quad .0151$$

$$\begin{array}{r} 10 \times 2 = 20 \quad | \quad 1 \quad 28 \\ 20 + 5 = 25 \quad | \quad 1 \quad 25 \\ 150 \times 2 = 300 \quad | \quad 3 \quad 01 \\ 300 + 1 = 301 \quad | \quad 3 \quad 01 \end{array}$$

$$45. \quad \begin{array}{r} 64 \cdot 06 \cdot 40 \cdot 16 \\ 64 \end{array} \quad | \quad 8004$$

$$\begin{array}{r} 8000 \times 2 = 16000 \quad | \quad 6 \quad 40 \quad 16 \\ 16000 + 4 = 16004 \quad | \quad 6 \quad 40 \quad 16 \end{array}$$

$$47. \quad \begin{array}{r} 10 \cdot 00 \cdot 00 \cdot 00 \\ 9 \end{array} \quad | \quad .3162$$

$$\begin{array}{r} 30 \times 2 = 60 \quad | \quad 1 \quad 00 \\ 60 + 1 = 61 \quad | \quad 61 \\ 310 \times 2 = 620 \quad | \quad 39 \quad 00 \\ 620 + 6 = 626 \quad | \quad 37 \quad 56 \\ 3160 \times 2 = 6320 \quad | \quad 1 \quad 44 \quad 00 \\ 6320 + 2 = 6322 \quad | \quad 1 \quad 26 \quad 44 \end{array}$$

$$48. \quad \sqrt{56 + 14\sqrt{15}} = \sqrt{56 + 2\sqrt{7 \times 7 \times 3 \times 5}} \\ = \sqrt{35 + 21 + 2\sqrt{35 \times 21}} = \sqrt{35} + \sqrt{21}.$$

$$49. \quad \sqrt{47 - 12\sqrt{15}} = \sqrt{47 - 2\sqrt{2 \times 3 \times 2 \times 3 \times 3 \times 5}} \\ = \sqrt{27 + 20 - 2\sqrt{27 \times 20}} = \sqrt{27} - \sqrt{20} = 3\sqrt{3} - 2\sqrt{5}.$$

$$50. \quad \sqrt{62 + 20\sqrt{6}} = \sqrt{62 + 2\sqrt{600}} = \sqrt{50 + 12 + 2\sqrt{50 \times 12}} \\ = \sqrt{50} + \sqrt{12} = 5\sqrt{2} + 2\sqrt{3}.$$

$$51. \quad \sqrt{51 - 36\sqrt{2}} = \sqrt{51 - 2\sqrt{3 \times 3 \times 2 \times 3 \times 3 \times 2 \times 2}} \\ = \sqrt{27 + 24 - 2\sqrt{27 \times 24}} = \sqrt{27} - \sqrt{24} = 3\sqrt{3} - 2\sqrt{6}.$$

$$52. \quad \begin{array}{r} x^3 - 9x + 27x^{-1} - 27x^{-3} \\ x^3 \end{array} \quad | \quad x - 3x^{-1} \\ \begin{array}{r} 3x^2 \\ 3x^2 - 9 + 9x^{-2} \end{array} \quad | \quad \begin{array}{r} -9x \\ -9x + 27x^{-1} - 27x^{-3} \end{array}$$

$$53. \quad \begin{array}{r} 27x^3 + 27x^2 - 5 + \frac{1}{3x^2} - \frac{1}{27x^3} \\ 27x^3 \end{array} \quad | \quad \begin{array}{r} 1 \\ 3x + 1 - \frac{1}{3x} \end{array} \\ \begin{array}{r} 27x^2 \\ 27x^2 + 9x + 1 \end{array} \quad | \quad \begin{array}{r} 27x^2 \\ 27x^2 + 9x + 1 \end{array} \\ \begin{array}{r} 27x^2 + 18x + 3 \\ 27x^2 + 18x \end{array} \quad | \quad \begin{array}{r} -9x - 6 + \frac{1}{3x^2} - \frac{1}{27x^3} \\ -\frac{1}{x} + \frac{1}{9x^2} \end{array} \\ \begin{array}{r} 27x^2 + 18x \\ 27x^2 + 18x \end{array} \quad | \quad \begin{array}{r} -9x - 6 + \frac{1}{3x^2} - \frac{1}{27x^3} \\ -9x - 6 + \frac{1}{3x^2} - \frac{1}{27x^3} \end{array}$$

$$54. \quad \begin{array}{r} x^3 + 3x^2\sqrt{x} - 5x\sqrt{x} + 3\sqrt{x} - 1 \\ x^3 \end{array} \quad | \quad x + \sqrt{x} - 1 \\ \begin{array}{r} 3x^2 \\ 3x^2 + 3x\sqrt{x} + x \end{array} \quad | \quad \begin{array}{r} 3x^2\sqrt{x} \\ 3x^2\sqrt{x} + 3x^2 + x\sqrt{x} \end{array} \\ \begin{array}{r} 3x^2 + 6x\sqrt{x} + 3x \\ 3x^2 + 6x\sqrt{x} \end{array} \quad | \quad \begin{array}{r} -3x^2 - 6x\sqrt{x} \\ -3\sqrt{x} + 1 \end{array} \\ \begin{array}{r} 3x^2 + 6x\sqrt{x} \\ 3x^2 + 6x\sqrt{x} \end{array} \quad | \quad \begin{array}{r} -3x^2 - 6x\sqrt{x} \\ -3x^2 - 6x\sqrt{x} + 3\sqrt{x} - 1 \end{array}$$

$$43. \quad \frac{x^2 - 12x^3y^{\frac{1}{2}} + 60x^3y - 160xy^{\frac{3}{2}} + 240x^3y^2 - 192x^3y^{\frac{3}{2}} + 64y^2}{x^2} \left[x - 6x^3y^{\frac{1}{2}} + 12x^3y - 8y^{\frac{3}{2}} \right]$$

$$\begin{array}{r|l} 2x & -12x^3y^{\frac{1}{2}} \\ 2x - 6x^3y^{\frac{1}{2}} & -12x^3y^{\frac{1}{2}} + 36x^3y \\ \hline 2x - 12x^3y^{\frac{1}{2}} & 24x^3y \\ 2x - 12x^3y^{\frac{1}{2}} + 12x^3y & 24x^3y - 144xy^{\frac{3}{2}} + 144x^3y^2 \\ \hline 2x - 12x^3y^{\frac{1}{2}} + 24x^3y & -16xy^{\frac{3}{2}} + 96x^3y^2 \\ 2x - 12x^3y^{\frac{1}{2}} + 24x^3y - 8y^{\frac{3}{2}} & -16xy^{\frac{3}{2}} + 96x^3y^2 - 192x^3y^{\frac{3}{2}} + 64y^2 \end{array}$$

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$$55. \quad \frac{2\sqrt{2} - 6\sqrt[3]{2} + 3\sqrt[3]{2^3/4} - 2}{2\sqrt{2}} \left[\sqrt{2} - \sqrt[3]{2} \right]$$

$$6 \quad \frac{6 - 3\sqrt[3]{2^3/2} + \sqrt[3]{4}}{-6\sqrt[3]{2} + 3\sqrt[3]{2^3/4} - 2}$$

56. Expanding and arranging, the polynomial becomes

$$\frac{a\sqrt{a} - 6a\sqrt{b} + 12b\sqrt{a} - 8b\sqrt{b} - 3a\sqrt{c} + 12\sqrt{abc} - 12b\sqrt{c} + 3c\sqrt{a} - 6c\sqrt{b} - c\sqrt{c}}{a\sqrt{a}} \left[\sqrt{a} - 2\sqrt{b} - \sqrt{c} \right]$$

$$\begin{array}{r|l} 3a & -6a\sqrt{b} \\ 3a - 6\sqrt{ab} + 4b & -6a\sqrt{b} + 12b\sqrt{a} - 8b\sqrt{b} \\ \hline 3a - 12\sqrt{ab} + 12b & -3a\sqrt{c} \\ 3a - 12\sqrt{ab} + 12b - 3\sqrt{ac} + 6\sqrt{bc} + c & -3a\sqrt{c} + 12\sqrt{abc} - 12b\sqrt{c} + 3c\sqrt{a} - 6c\sqrt{b} - c\sqrt{c} \end{array}$$

57.

$$\begin{array}{r|l}
 510 \cdot 082 \cdot 399 & \underline{799} \\
 343 & \\
 \hline
 70^2 \times 3 & = 14700 \\
 70 \times 9 \times 3 & = 1890 \\
 9^2 & = 81 \\
 \hline
 & 16671 \\
 & 150 \ 039 \\
 \hline
 790^2 \times 3 & = 1872300 \\
 790 \times 9 \times 3 & = 21330 \\
 9^2 & = 81 \\
 \hline
 & 1893711 \\
 & 17 \cdot 043 \ 399
 \end{array}$$

58.

$$\begin{array}{r|l}
 1 \cdot 042 \cdot 590 \cdot 744 & \underline{1014} \\
 1 \ 000 & \\
 \hline
 100^2 \times 3 & = 30000 \\
 100 \times 3 & = 300 \\
 1^2 & = 1 \\
 \hline
 & 30301 \\
 & 30 \ 301 \\
 \hline
 1010^2 \times 3 & = 3060300 \\
 1010 \times 4 \times 3 & = 12120 \\
 4^2 & = 16 \\
 \hline
 & 3072436 \\
 & 12 \ 289 \ 744
 \end{array}$$

59. See page 167, solution of Ex. 21.

60.

$$\begin{array}{r|l}
 1 + x - x^2 & \\
 1 & \\
 \hline
 2 & \begin{array}{l} x \\ x + \frac{x^2}{4} \end{array} \\
 2 + \frac{x}{2} & \\
 \hline
 2 + x & \begin{array}{l} -\frac{5x^2}{4} \\ -\frac{5x^2}{4} - \frac{5x^3}{8} + \frac{25x^4}{64} \end{array} \\
 2 + x - \frac{5x^2}{8} & \\
 \hline
 2 + x - \frac{5x^2}{4} & \begin{array}{l} \frac{5x^3}{8} - \frac{25x^4}{64} \end{array}
 \end{array}
 \quad \begin{array}{l} 1 + \frac{x}{2} - \frac{5x^2}{8} + \frac{5x^3}{16} \end{array}$$

61. See next page.

62.

$$\begin{array}{r|l}
 a^6 - 4a^4\sqrt{ab^{-1}} + 6a^3b^{-1} - 4ab^{-1}\sqrt{ab^{-1}} + b^{-2} & \underline{a^3 - 2a\sqrt{ab^{-1}} + b^{-1}} \\
 a^6 & \\
 \hline
 2a^3 & \begin{array}{l} -4a^4\sqrt{ab^{-1}} \\ -4a^4\sqrt{ab^{-1}} + 4a^3b^{-1} \end{array} \\
 2a^3 - 2a\sqrt{ab^{-1}} & \\
 \hline
 2a^3 - 4a\sqrt{ab^{-1}} & \begin{array}{l} 2a^3b^{-1} \\ 2a^3b^{-1} - 4ab^{-1}\sqrt{ab^{-1}} + b^{-2} \end{array} \\
 2a^3 - 4a\sqrt{ab^{-1}} + b^{-1} &
 \end{array}$$

Since the square root of the given polynomial is $a^3 - 2a\sqrt{ab^{-1}} + b^{-1}$ and the square root of $a^3 - 2a\sqrt{ab^{-1}} + b^{-1}$ is $a\sqrt{a} - \sqrt{b^{-1}}$, the fourth root of the given polynomial is $a\sqrt{a} - \sqrt{b^{-1}}$.

61.

$$\begin{array}{r|l}
 1 + x^3 & 1 + \frac{x^3}{3} - \frac{x^6}{9} \\
 \hline
 1 & \\
 \hline
 3 & x^3 \\
 3 + x^3 + \frac{x^6}{9} & x^3 + \frac{x^6}{3} + \frac{x^9}{27} \\
 \hline
 3 + 2x^3 + \frac{x^6}{3} & -\frac{x^6}{3} - \frac{x^9}{27}
 \end{array}$$

$$63. \quad 8 - 48\sqrt{a} + 120a - 160a\sqrt{a} + 120a^2 - 48a^2\sqrt{a} + 8a^3$$

$$= 8(1 - 6\sqrt{a} + 15a - 20a\sqrt{a} + 15a^2 - 6a^2\sqrt{a} + a^3).$$

$$\begin{array}{r|l}
 1 & 1 - 3\sqrt{a} + 3a - a\sqrt{a} \\
 \hline
 2 & -6\sqrt{a} \\
 2 - 3\sqrt{a} & -6\sqrt{a} + 9a \\
 \hline
 2 - 6\sqrt{a} & 6a \\
 2 - 6\sqrt{a} + 3a & 6a - 18a\sqrt{a} + 9a^2 \\
 \hline
 2 - 6\sqrt{a} + 6a & -2a\sqrt{a} + 6a^2 \\
 2 - 6\sqrt{a} + 6a - a\sqrt{a} & -2a\sqrt{a} + 6a^2 - 6a^2\sqrt{a} + a^3 \\
 \hline
 & 1 - 3\sqrt{a} + 3a - a\sqrt{a} \quad | \quad 1 - \sqrt{a} \\
 & \hline
 & 1 \\
 & \hline
 3 & -3\sqrt{a} \\
 3 - 3\sqrt{a} + a & -3\sqrt{a} + 3a - a\sqrt{a}
 \end{array}$$

Since the sixth root of 8 is equal to the square root of 2, or to $\sqrt{2}$, and the sixth root of $1 - 6\sqrt{a} + 15a - 20a\sqrt{a} + 15a^2 - 6a^2\sqrt{a} + a^3$ is equal to the cube root of the square root of this polynomial factor, or to $1 - \sqrt{a}$, the sixth root of the given polynomial is equal to $\sqrt{2}(1 - \sqrt{a})$.

$$64. \quad a^m \times a^n = a^{m+n} \text{ for all values of } m \text{ and } n. \quad (1)$$

$$\text{If } n = 0, \quad a^m \times a^0 = a^{m+0} = a^m. \quad (2)$$

$$\text{Dividing by } a^m, \quad a^0 = 1. \quad (3)$$

Since (1) is true for all values of m and n , let $m = -2$ and $n = 2$.

$$\text{Then,} \quad a^{-2} \times a^2 = a^{-2+2} = a^0;$$

$$\text{Therefore, by (3), Ax. 1,} \quad a^{-2} \times a^2 = 1.$$

$$\text{Dividing by } a^2, \text{ Ax. 5,} \quad a^{-2} = \frac{1}{a^2}.$$

$$65. \quad a^m \times a^n = a^{m+n} \text{ for all values of } m \text{ and } n. \quad (1)$$

$$\text{Let } m = \frac{3}{2} \text{ and } n = \frac{3}{2}.$$

$$\text{Then, by (1),} \quad a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^{\frac{3}{2} + \frac{3}{2}} = a^3. \quad (2)$$

Taking the square root of both members of (2),

$$\text{Ax. 7, § 26,} \quad a^{\frac{3}{2}} = \sqrt{a^3}. \quad (3)$$

$$\text{Again, let } m = \frac{1}{2} \text{ and } n = \frac{1}{2}.$$

$$\text{Then, by (1),} \quad a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1. \quad (4)$$

$$\text{If } m = 1 \text{ and } n = \frac{1}{2}, \quad a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 \times a^{\frac{1}{2}} = a^{1 + \frac{1}{2}} = a^{\frac{3}{2}}. \quad (5)$$

$$\text{Hence, § 24,} \quad a^{\frac{3}{2}} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3. \quad (6)$$

$$\text{From (3) and (6), Ax. 1,} \quad a^{\frac{3}{2}} = \sqrt{a^3} = (\sqrt{a})^3.$$

66. $a^m \times a^n = a^{m+n}$ for all values of m and n . (1)

Let $m = -\frac{1}{3}$ and $n = 1$.

Then, by (1), $a^{-\frac{1}{3}} \times a^1 = a^{-\frac{1}{3}+1} = a^{\frac{2}{3}}$
by Ex. 65, (3), $\frac{2}{a} = \sqrt[3]{a^2}$. (2)

Hence, Ax. 4 and 5, $\frac{2}{a} (a^{-\frac{1}{3}} \times a^1) = \frac{2}{a} \sqrt[3]{a^2}$;

that is, $2 a^{-\frac{1}{3}} = \frac{2 \sqrt[3]{a^2}}{a}$.

67. $a^m \times a^n = a^{m+n}$ for all values of m and n . (1)

Then, $(ab)^m \times (ab)^n = (ab)^{m+n}$ for all values of m and n , (2)

since a in (1) represents any number.

Let $n = 0$.

Then, in (2), $(ab)^m \times (ab)^0 = (ab)^{m+0} = (ab)^m$. (3)

Dividing both members of (3) by $(ab)^m$, Ax. 5,

$$(ab)^0 = 1.$$

68. $(abc)^3 = abc \cdot abc \cdot abc$

82, $= aad \cdot bbb \cdot ccc$

24, $= a^3 b^3 c^3$.

69. $\left(\frac{a}{b}\right)^3 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b}$ (1)

$$b^3 = b \cdot b \cdot b. \quad (2)$$

Multiplying (1) by (2), Ax. 4, $\left(\frac{a}{b}\right)^3 \cdot b^3 = \left(\frac{a}{b} \times b\right) \left(\frac{a}{b} \times b\right) \left(\frac{a}{b} \times b\right)$

§§ 102, 24, $= a \cdot a \cdot a = a^3$.

Dividing by b^3 , Ax. 5, $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$.

70. $16^{\frac{3}{2}} = (\sqrt[4]{16})^3 = 2^3 = 8$.

71. $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$.

72. $8^{-\frac{2}{3}} = 1 \div 8^{\frac{2}{3}} = 1 \div (\sqrt[3]{8})^2 = 1 \div 2^2 = \frac{1}{4}$.

73. $(a^4 x^4)^{\frac{3}{2}} = a^{4(\frac{3}{2})} x^{4(\frac{3}{2})} = a^6 x^6$.

74. $(b^2 y^4)^{-\frac{3}{2}} = b^{2(-\frac{3}{2})} y^{4(-\frac{3}{2})} = b^{-3} y^{-6} = \frac{1}{b^3 y^6}$.

75. $(a^n b^n)^{-\frac{1}{n}} = a^{n(-\frac{1}{n})} b^{n(-\frac{1}{n})} = a^{-1} b^{-1} = \frac{1}{ab}$.

76. $\left(\frac{3 \cdot 2}{2 \cdot 4 \cdot 3}\right)^{-\frac{3}{2}} = 1 \div \left(\frac{3 \cdot 2}{2 \cdot 4 \cdot 3}\right)^{\frac{3}{2}} = (1 \div \frac{3 \cdot 2}{2 \cdot 4 \cdot 3})^{\frac{3}{2}} = \left(\frac{2 \cdot 4 \cdot 3}{3 \cdot 2}\right)^{\frac{3}{2}} = (\sqrt[2]{\frac{2 \cdot 4 \cdot 3}{3 \cdot 2}})^3 = \left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$.

77. $\left(\frac{3 \cdot 6}{4 \cdot 9}\right)^{-\frac{3}{2}} = 1 \div \left(\frac{3 \cdot 6}{4 \cdot 9}\right)^{\frac{3}{2}} = (1 \div \frac{3 \cdot 6}{4 \cdot 9})^{\frac{3}{2}} = \left(\frac{4 \cdot 9}{3 \cdot 6}\right)^{\frac{3}{2}} = (\sqrt[2]{\frac{4 \cdot 9}{3 \cdot 6}})^3 = \left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$.

78. $\left(-\frac{8}{27}\right)^{-\frac{2}{3}} = 1 \div \left(-\frac{8}{27}\right)^{\frac{2}{3}} = 1 \div (\sqrt[3]{-\frac{8}{27}})^2 = 1 \div \left(-\frac{2}{3}\right)^2 = 1 \div \frac{4}{9} = \frac{9}{4}$.

79. Since $b-a = -1(a-b)$, $(b-a)^n = [-1(a-b)]^n = (-1)^n (a-b)^n$.

If n is even, $(-1)^n = 1$ and $(-1)^n (a-b)^n$, or $(b-a)^n = (a-b)^n$.

80. $(36 a^{-3} \div 25 a^{-2})^{-\frac{1}{2}} = \left(\frac{3 \cdot 6}{2 \cdot 5} a^{-1}\right)^{-\frac{1}{2}} = \left(\frac{3 \cdot 6}{2 \cdot 5}\right)^{-\frac{1}{2}} a^{\frac{1}{2}} = a^{\frac{1}{2}} \div \left(\frac{3 \cdot 6}{2 \cdot 5}\right)^{\frac{1}{2}}$
 $= a^{\frac{1}{2}} \div \left(\frac{6}{5}\right) = \frac{5}{6} a^{\frac{1}{2}}$.

$$81. (8a^3x^6 \times 64a^{-4}x^{-5})^{-\frac{1}{2}} = (2^3 \cdot 2^6 \cdot a^{-1}x)^{-\frac{1}{2}} = 2^{-\frac{9}{2}}a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{8x^{\frac{1}{2}}}.$$

$$82. (a^{\frac{1}{2}}b^{\frac{1}{2}})^{\frac{2}{3}} \div (a^{\frac{1}{2}}b^{\frac{1}{2}})^2 = a^{\frac{2}{3}}b^{\frac{2}{3}} \div a^{\frac{1}{2}}b^{\frac{1}{2}} = a^{\frac{1}{6}}b^{-\frac{1}{6}} = \frac{a^{\frac{1}{6}}}{b^{\frac{1}{6}}}.$$

$$83. (\sqrt{a^3x^{-3}} \div \sqrt[3]{a^2x^{-2}})^{\frac{2}{3}} = (a^{\frac{3}{2}}x^{-\frac{3}{2}} \div a^{\frac{2}{3}}x^{-\frac{2}{3}})^{\frac{2}{3}} = (a^{\frac{3}{2}-\frac{2}{3}}x^{-\frac{3}{2}+\frac{2}{3}})^{\frac{2}{3}} \\ = (a^{\frac{5}{6}}x^{-\frac{5}{6}})^{\frac{2}{3}} = a^{\frac{1}{3}}x^{-\frac{1}{3}} = \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

$$84. (\sqrt{a^{-1}b^4} \div \sqrt{a^2b})^{-\frac{1}{2}} = (a^{-\frac{1}{2}}b^2 \div ab^{\frac{1}{2}})^{-\frac{1}{2}} = (a^{-\frac{1}{2}-1}b^{2-\frac{1}{2}})^{-\frac{1}{2}} \\ = (a^{-\frac{3}{2}}b^{\frac{3}{2}})^{-\frac{1}{2}} = a^{\frac{1}{4}}b^{-\frac{1}{4}} = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}.$$

$$85. (\sqrt{a} \div \sqrt[3]{a}) \div \sqrt[4]{a} = (a^{\frac{1}{2}} \div a^{\frac{1}{3}}) \div a^{\frac{1}{4}} = a^{\frac{1}{2}} \div a^{\frac{1}{4}} = a^{-\frac{1}{4}} = \frac{1}{a^{\frac{1}{4}}}.$$

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$$86. \frac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}} - \frac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}} + \frac{2ab}{a^{\frac{3}{2}}-b^{\frac{3}{2}}} \\ = (a^{\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+b^{\frac{1}{2}}) - (a^{\frac{1}{2}}-a^{\frac{1}{2}}b^{\frac{1}{2}}+b^{\frac{1}{2}}) + \frac{2ab^{\frac{1}{2}}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}} \\ = 2a^{\frac{1}{2}}b^{\frac{1}{2}} + \frac{2ab^{\frac{1}{2}}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}} = \frac{2ab^{\frac{1}{2}}-2ab^{\frac{1}{2}}+2ab^{\frac{1}{2}}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}} = \frac{2ab^{\frac{1}{2}}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}}.$$

$$87. \frac{1+a^{-1}b}{1-a^{-1}b} \div \left(\frac{1+ab^{-1}+a^2b^{-2}}{1-ab^{-1}+a^2b^{-2}} \times \frac{1+a^{-3}b^3}{1-a^{-3}b^3} \right)$$

Multiplying both terms of the first fraction by a , of the second fraction by b^2 , and of the third by a^3 ,

$$= \frac{a+b}{a-b} \div \left(\frac{b^2+ab+a^2}{b^2-ab+a^2} \times \frac{a^3+b^3}{a^3-b^3} \right) \\ = \frac{a+b}{a-b} \div \frac{a+b}{a-b} = 1.$$

$$88. \frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{3-5x}{1-x^2}.$$

Multiplying by x^2-1 ,

$$x^2+2x+1-(x^2-2x+1) = -(3-5x). \\ 4x = 5x-3. \\ \therefore x = 3.$$

$$89. \frac{7-2x}{10} - \frac{2x-1}{5} + \frac{1}{2}x = \frac{5x-6\frac{1}{2}}{2x} - \frac{17+3x}{30}. \\ \frac{7}{10} - \frac{x}{5} - \frac{2x}{5} + \frac{1}{5} + \frac{x}{2} = \frac{5}{2} - \frac{31}{10x} - \frac{17}{30} - \frac{x}{10}.$$

Transposing, etc.,

$$\frac{31}{10x} = \frac{31}{30}. \\ \therefore x = 3.$$

$$90. \quad \frac{4x-17}{9} - \frac{3\frac{2}{3}-22x}{33} = x - \frac{6}{x} \left(1 - \frac{x^2}{54}\right).$$

$$\frac{4x}{9} - \frac{17}{9} - \frac{1}{9} + \frac{2x}{3} = x - \frac{6}{x} + \frac{x}{9}.$$

$$\text{Canceling,} \quad -2 = -\frac{6}{x}.$$

$$\therefore x = 3.$$

$$91. \quad \begin{cases} \frac{3x-5y}{3} - \frac{2x-8y-9}{12} = \frac{y}{2} + \frac{7}{12}, & (1) \end{cases}$$

$$\begin{cases} \frac{7}{2} \left(\frac{x}{7} + \frac{y}{4} + 1\frac{1}{3} \right) - \frac{10}{3} \left(4x - \frac{y}{8} - 24 \right) = 0. & (2) \end{cases}$$

$$\text{Reducing (1),} \quad 5x - 9y = -1. \quad (3)$$

$$\text{Expanding (2),} \quad \frac{x}{2} + \frac{7y}{8} + \frac{14}{3} - \frac{40x}{3} + \frac{5y}{12} + 80 = 0;$$

$$\therefore 308x - 31y = 2032. \quad (4)$$

$$\text{Multiplying (3) by 31,} \quad 155x - 279y = -31. \quad (5)$$

$$\text{Multiplying (4) by 9,} \quad 2772x - 279y = 18288. \quad (6)$$

$$\text{Subtracting (5) from (6),} \quad 2617x = 18319. \quad (7)$$

$$\therefore x = 7.$$

$$\text{Substituting (7) in (3),} \quad y = 4.$$

$$92. \quad \begin{cases} 3x + 1 = 2y, & (1) \\ (x + 5)(y + 7) = (x + 1)(y - 9) + 112. & (2) \end{cases}$$

$$\text{Reducing (2),} \quad 4x + y = 17. \quad (3)$$

$$\text{Multiplying (3) by 2,} \quad 8x + 2y = 34. \quad (4)$$

$$\text{Adding (1) and (4),} \quad 11x + 2y + 1 = 2y + 34. \quad (5)$$

$$\therefore x = 3.$$

$$\text{Substituting (5) in (1),} \quad y = 5.$$

$$93. \quad \begin{cases} x - y = 3, & (1) \\ (x + 1)(x + 2) - (x - 2)(x + 1) = 11y + 2. & (2) \end{cases}$$

$$\text{Reducing (2),} \quad 4x - 11y = -2. \quad (3)$$

$$\text{Multiplying (1) by 4,} \quad 4x - 4y = 12. \quad (4)$$

$$\text{Subtracting (3) from (4),} \quad 7y = 14. \quad (5)$$

$$\therefore y = 2.$$

$$\text{Substituting (5) in (1),} \quad x = 5.$$

$$94. \quad \frac{\left(\frac{a}{27} \div \frac{a^{-2}}{8}\right)^{-\frac{3}{2}} - x^2}{\frac{3a^{-1} + 2x}{2}} = \frac{\left(\frac{a}{27} \times 8a^2\right)^{-\frac{3}{2}} - x^2}{\frac{3 + 2ax}{2a}} = \frac{\frac{9}{4a^2} - x^2}{\frac{3 + 2ax}{2a}} \\ = \frac{9 - 4a^2x^2}{2a(3 + 2ax)} = \frac{3 - 2ax}{2a}.$$

$$95. \quad \{a^{-2}[a^{\frac{2}{3}}(a^{\frac{3}{5}})^{\frac{1}{5}}]^{\frac{5}{2}}\}^{\frac{2}{5}} = \{a^{-2}[a^{\frac{2}{3}}a^{\frac{1}{5}}]^{\frac{5}{2}}\}^{\frac{2}{5}} \\ = \{a^{-2} \cdot a^{\frac{1}{2}}\}^{\frac{2}{5}} = (a^2)^{\frac{1}{5}} = a^{\frac{2}{5}}.$$

96.
$$\left[\frac{\left(\frac{a^3 b}{x^2 y} \right)^{\frac{1}{2}}}{\left(\frac{a^3 b^2}{x y^2} \right)^{\frac{1}{2}}} \right]^6 - \frac{(ax^{-1})^3}{b} = \left[\frac{\left(\frac{a^3 b^3}{x^6 y^3} \right)^{\frac{1}{2}}}{\left(\frac{a^6 b^4}{x^2 y^4} \right)^{\frac{1}{2}}} \right]^6 - \frac{a^3}{bx^3}$$

$$= \frac{a^3 b^3}{x^6 y^3} \times \frac{x^2 y^4}{a^6 b^4} - \frac{a^3}{bx^3}$$

$$= \frac{a^3 y}{bx^4} - \frac{a^3 x}{bx^4} = \frac{a^3 (y - x)}{bx^4}.$$
97.
$$\left[\frac{x^{-\frac{1}{2}} y^{-\frac{2}{3}}}{x^{-\frac{1}{2}} y^{-1}} \cdot \frac{x^{-2} y^2}{(xy)^{-3}} \right]^{-3} = (x^{-\frac{1}{2}} y^{\frac{1}{3}} \div xy^5)^{-3} = (x^{-\frac{1}{2}} y^{-\frac{14}{3}})^{-3} = x^4 y^{14}.$$
98.
$$\{(a^{\frac{1}{2}} b^{\frac{5}{2}})^{\frac{1}{2}} \div (a^{-\frac{1}{2}} b)^{-2}\}^{\frac{1}{2}} = \{a^{\frac{1}{4}} b^{\frac{1}{2}} \div ab^{-2}\}^{\frac{1}{2}} = (a^{-\frac{3}{4}} b^{\frac{5}{2}})^{\frac{1}{2}}$$

$$= a^{-\frac{3}{8}} b^{\frac{5}{4}} = \frac{b^{\frac{5}{4}}}{a^{\frac{3}{8}}}.$$
99.
$$\frac{a+b}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} - \frac{a-b}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} = \frac{a^{\frac{3}{2}} + ab^{\frac{1}{2}} + a^{\frac{1}{2}}b + b^{\frac{3}{2}} - (a^{\frac{3}{2}} - ab^{\frac{1}{2}} - a^{\frac{1}{2}}b + b^{\frac{3}{2}})}{(a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}})}$$

$$= \frac{2ab^{\frac{1}{2}} + 2a^{\frac{1}{2}}b}{a^{\frac{3}{2}} - b^{\frac{3}{2}}} = \frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}})}{a^{\frac{3}{2}} - b^{\frac{3}{2}}}.$$
100.
$$\frac{\left[-\frac{a^{-1} + b^{-1}}{a^{-1} - b^{-1}} \times (a^2 - b^2) \right]^{\frac{1}{2}}}{\frac{b+a}{ab}} = \frac{\left[-\frac{b+a}{b-a} \times (a^2 - b^2) \right]^{\frac{1}{2}}}{\frac{b+a}{ab}}$$

$$= \left[\frac{a+b}{a-b} \times (a+b)(a-b) \right]^{\frac{1}{2}} \times \frac{ab}{a+b}$$

$$= (a+b) \times \frac{ab}{a+b} = ab.$$

QUADRATIC EQUATIONS

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3. Transposing, etc., $7x^2 - 25 = 5x^2 + 73.$
 Dividing by 2, $2x^2 = 98.$
 Extracting the square root, $x^2 = 49.$
 $x = \pm 7.$
4. Expanding, $(x+4)^2 = 8x + 25.$
 Canceling, etc., $x^2 + 8x + 16 = 8x + 25.$
 Extracting the square root, $x^2 = 9.$
 $x = \pm 3.$
5. Expanding, $(a-x)^2 = (3x+a)(x-a).$
 Canceling, etc., $a^2 - 2ax + x^2 = 3x^2 - 2ax - a^2.$
 $2x^2 = 2a^2.$
 $x^2 = a^2.$
 Extracting the square root, $x = \pm a.$

6. $ax^2 = (a - b)(a^2 - b^2) - bx^2$
 Transposing, etc., $(a + b)x^2 = (a - b)(a^2 - b^2)$
 Dividing by $a + b$, $x^2 = (a - b)(a - b)$
 Extracting the square root, $x = \pm (a - b)$

7. $a^2x^2 + 2ax^2 = (a^2 - 1)^2 - x^2$
 Transposing, etc., $(a^2 + 2a + 1)x^2 = (a^2 - 1)^2$
 Extracting the square root, $(a + 1)x = \pm (a^2 - 1)$
 Dividing by $a + 1$, $x = \pm (a - 1)$

8. $(x + 2)^2 - 4(x + 2) = 4$
 Expanding, $x^2 + 4x + 4 - 4x - 8 = 4$
 Canceling, etc., $x^2 = 8$
 Extracting the square root, $x = \pm 2\sqrt{2}$

9. $\frac{x - 8}{6} = \frac{6}{x + 8}$
 Clearing of fractions, $x^2 - 64 = 36$
 Transposing, etc., $x^2 = 100$
 Extracting the square root, $x = \pm 10$

10. $\frac{1}{1 - x} + \frac{1}{1 + x} = 2\frac{2}{3}$
 Uniting terms, $\frac{2}{1 - x^2} = \frac{8}{3}$
 Dividing by 2 and clearing of fractions, $3 = 4 - 4x^2$
 Transposing, etc., $x^2 = \frac{1}{4}$
 Extracting the square root, $x = \pm \frac{1}{2}$

11. $\frac{x}{12} + \frac{x^2 - 15}{5x} = \frac{x}{5}$
 $\frac{x}{12} + \frac{x}{5} - \frac{3}{x} = \frac{x}{5}$
 Canceling and clearing of fractions, $x^2 - 36 = 0$
 Transposing and extracting the square root, $x = \pm 6$

12. $\frac{x + 3}{x - 3} + \frac{x - 3}{x + 3} = 4$
 Clearing of fractions, $(x + 3)^2 + (x - 3)^2 = 4x^2 - 36$
 Expanding, etc., $2x^2 + 18 = 4x^2 - 36$
 Transposing, etc., $x^2 = 27$
 Extracting the square root, $x = \pm 3\sqrt{3}$

13. $\frac{x - 2}{x + 1} + \frac{x + 2}{x - 1} = -1$
 Clearing of fractions, $x^2 - 3x + 2 + x^2 + 3x + 2 = -x^2 + 1$
 $3x^2 = -3$
 $x^2 = -1$
 Extracting the square root, $x = \pm \sqrt{-1}$

14. $\frac{x}{a+b} - \frac{a-b}{x} = 0.$

Clearing of fractions,

$$x^2 - (a^2 - b^2) = 0.$$

Transposing,

$$x^2 = a^2 - b^2.$$

Extracting the square root,

$$x = \pm \sqrt{a^2 - b^2}.$$

15. $\frac{x-3}{x-2} + \frac{x+3}{x+2} = 1\frac{1}{2}.$

Reducing to mixed numbers and subtracting 2 from each side,

$$-\frac{1}{x-2} + \frac{1}{x+2} = -\frac{1}{2}.$$

Clearing of fractions, $-8x - 16 + 8x - 16 = -x^2 + 4.$

Transposing, etc.,

$$x^2 = 36.$$

Extracting the square root,

$$x = \pm 6.$$

16. $\frac{a}{x} + \frac{x}{a} = \frac{ab}{x}.$

Transposing,

$$\frac{x}{a} = \frac{ab}{x} - \frac{a}{x} = \frac{a(b-1)}{x}.$$

Clearing of fractions,

$$x^2 = a^2(b-1).$$

Extracting the square root,

$$x = \pm a\sqrt{b-1}.$$

17. $\sqrt{x^2+8} - \frac{6}{\sqrt{x^2+8}} = x.$

Clearing of fractions,

$$x^2 + 8 - 6 = x\sqrt{x^2+8}.$$

Uniting terms,

$$x^2 + 2 = x\sqrt{x^2+8}.$$

Squaring,

$$x^4 + 4x^2 + 4 = x^4 + 8x^2.$$

Canceling, etc.,

$$4 = 4x^2.$$

$$\therefore x = \pm 1$$

18. $x + \sqrt{x^2 + m^2} = \frac{2m^2}{\sqrt{x^2 + m^2}}.$

Clearing of fractions, $x\sqrt{x^2 + m^2} + x^2 + m^2 = 2m^2.$

Transposing, etc.,

$$x\sqrt{x^2 + m^2} = m^2 - x^2.$$

Squaring,

$$x^4 + m^2x^2 = m^4 - 2m^2x^2 + x^4.$$

Canceling, etc.,

$$3m^2x^2 = m^4.$$

$$x^2 = \frac{m^2}{3}.$$

Extracting the square root,

$$x = \pm \frac{m}{3}\sqrt{3}.$$

19. $\frac{x+a}{x+b} + \frac{x-a}{x-b} = \frac{a^2+b^2}{x^2-b^2}.$

Clearing of fractions,

$$x^2 + ax - bx - ab + x^2 - ax + bx - ab = a^2 + b^2.$$

Uniting terms, etc.,

$$2x^2 = a^2 + 2ab + b^2.$$

$$x^2 = \frac{1}{2}(a+b)^2.$$

Extracting the square root,

$$x = \pm \frac{1}{2}(a+b)\sqrt{2}.$$

$$20. \quad x + \sqrt{x^2 + 3} = \frac{6}{\sqrt{x^2 + 3}}.$$

$$\text{Clearing of fractions, } x\sqrt{x^2 + 3} + x^2 + 3 = 6.$$

$$\text{Transposing, etc., } x\sqrt{x^2 + 3} = 3 - x^2.$$

$$\text{Squaring, } x^4 + 3x^2 = 9 - 6x^2 + x^4.$$

$$\text{Canceling, etc., } 9x^2 = 9.$$

$$\therefore x = \pm 1.$$

$$21. \quad \frac{24}{\sqrt{x^2 + 12}} - \sqrt{x^2 + 12} = x.$$

$$\text{Clearing of fractions, } 24 - (x^2 + 12) = x\sqrt{x^2 + 12}.$$

$$\text{Uniting terms, } 12 - x^2 = x\sqrt{x^2 + 12}.$$

$$\text{Squaring, } 144 - 24x^2 + x^4 = x^4 + 12x^2.$$

$$\text{Canceling, etc., } -36x^2 = -144.$$

$$x^2 = 4.$$

$$\text{Extracting the square root, } x = \pm 2.$$

$$22. \quad \frac{x+a}{x-a} + \frac{x-a}{x+a} = \frac{2a}{1-a}.$$

$$\text{Uniting terms, } \frac{2(x^2 + a^2)}{x^2 - a^2} = \frac{2a}{1-a}.$$

$$\text{Dividing by 2 and clearing of fractions,}$$

$$x^2 + a^2 - ax^2 - a^3 = ax^2 - a^3.$$

$$\text{Transposing, etc., } x^2(1 - 2a) = -a^2.$$

$$\text{Dividing by } 1 - 2a, \quad x^2 = \frac{a^2}{2a - 1}.$$

$$\text{Extracting the square root, } x = \pm \frac{a}{2a - 1} \sqrt{2a - 1}.$$

$$23. \quad \frac{x+7}{x^2-7x} - \frac{x-7}{x^2+7x} = \frac{7}{x^2-73}.$$

$$\text{Uniting terms, etc., } \frac{4}{x^2 - 49} = \frac{1}{x^2 - 73}.$$

$$\text{Clearing of fractions, } 4x^2 - 292 = x^2 - 49.$$

$$\text{Transposing, etc., } x^2 = 81.$$

$$\text{Extracting the square root, } x = \pm 9.$$

$$24. \quad x + \sqrt{x^2 - a^2} = \frac{a^2}{\sqrt{x^2 - a^2}}.$$

$$\text{Clearing of fractions, } x\sqrt{x^2 - a^2} + x^2 - a^2 = a^2.$$

$$\text{Transposing, etc., } x\sqrt{x^2 - a^2} = 2a^2 - x^2.$$

$$\text{Squaring, } x^4 - a^2x^2 = 4a^4 - 4a^2x^2 + x^4.$$

$$\text{Canceling, etc., } 3a^2x^2 = 4a^4.$$

$$\text{Dividing by } 3a^2, \quad x^2 = \frac{4a^2}{3}.$$

$$\text{Extracting the square root, } x = \pm \frac{2}{3}a\sqrt{3}.$$

25. See next page.

$$26. \quad \frac{2x + \sqrt{4x^2 - 1}}{2x - \sqrt{4x^2 - 1}} = 4.$$

Clearing of fractions, $2x + \sqrt{4x^2 - 1} = 8x - 4\sqrt{4x^2 - 1}.$

Transposing, etc., $5\sqrt{4x^2 - 1} = 6x.$

Squaring, $100x^2 - 25 = 36x^2.$

$$\therefore x^2 = \frac{25}{64}.$$

Extracting the square root, $x = \pm \frac{5}{8}.$

$$27. \quad \frac{\sqrt{1+x}}{1+\sqrt{1+x}} + \frac{\sqrt{1-x}}{1+\sqrt{1-x}} = 0.$$

Rationalizing the denominators,

$$\frac{\sqrt{1+x} - (1+x)}{1 - (1+x)} + \frac{\sqrt{1-x} - (1-x)}{1 - (1-x)} = 0;$$

that is,

$$\frac{\sqrt{1+x} - 1 - x}{-x} + \frac{-\sqrt{1-x} + 1 - x}{-x} = 0.$$

$$\therefore \sqrt{1+x} - 1 - x - \sqrt{1-x} + 1 - x = 0.$$

Transposing, etc., $\sqrt{1+x} - 2x = \sqrt{1-x}.$

Squaring, $1 + x - 4x\sqrt{1+x} + 4x^2 = 1 - x.$

Simplifying, dividing by $2x$, $1 + 2x = 2\sqrt{1+x}.$

Squaring, $1 + 4x + 4x^2 = 4 + 4x.$

Canceling, $4x^2 = 3.$

Extracting the square root, etc., $x = \pm \frac{1}{2}\sqrt{3}.$

$$28. \quad \frac{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} = \frac{1}{2}.$$

Clearing of fractions, $2\sqrt{x^2 + 1} - 2\sqrt{x^2 - 1} = \sqrt{x^2 + 1} + \sqrt{x^2 - 1}.$

Transposing, etc., $\sqrt{x^2 + 1} = 3\sqrt{x^2 - 1}.$

Squaring, $x^2 + 1 = 9x^2 - 9.$

$$\therefore x^2 = \frac{5}{4}.$$

Extracting the square root, $x = \pm \frac{1}{2}\sqrt{5}.$

$$29. \quad \frac{1}{1 + \sqrt{1-x}} - \frac{1}{\sqrt{1+x} + 1} = \frac{1}{x}.$$

Rationalizing the denominators,

$$\frac{1 - \sqrt{1-x}}{1 - (1-x)} - \frac{\sqrt{1+x} - 1}{(1+x) - 1} = \frac{1}{x};$$

that is,

$$\frac{1 - \sqrt{1-x}}{x} - \frac{\sqrt{1+x} - 1}{x} = \frac{1}{x}.$$

$$\therefore 1 - \sqrt{1-x} - \sqrt{1+x} + 1 = 1.$$

Transposing, etc., $1 - \sqrt{1+x} = \sqrt{1-x}.$

Squaring, $1 - 2\sqrt{1+x} + 1 + x = 1 - x.$

Transposing, etc., $1 + 2x = 2\sqrt{1+x}.$

Squaring, $1 + 4x + 4x^2 = 4 + 4x.$

Canceling, $4x^2 = 3.$

Extracting the square root, etc., $x = \pm \frac{1}{2}\sqrt{3}.$

25.

$$\sqrt{25 - 6x} + \sqrt{25 + 6x} = 8.$$

$$\text{Squaring, } 25 - 6x + 2\sqrt{625 - 36x^2} + 25 + 6x = 64.$$

$$\text{Canceling, etc., } \sqrt{625 - 36x^2} = 7.$$

$$\text{Squaring, } 625 - 36x^2 = 49.$$

$$\therefore x^2 = 16.$$

$$\text{Extracting the square root, } x = \pm 4.$$

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30.

$$\frac{2}{x + \sqrt{2 - x^2}} + \frac{2}{x - \sqrt{2 - x^2}} = x.$$

Clearing of fractions,

$$2x - 2\sqrt{2 - x^2} + 2x + 2\sqrt{2 - x^2} = 2x^3 - 2x.$$

Canceling, etc.,

$$6x = 2x^3.$$

Dividing by $2x$,

$$3 = x^2.$$

$$\therefore x = \pm\sqrt{3}.$$

NOTE.—The given equation is satisfied also for $x = 0$.

31.

$$\frac{\sqrt{x+2a} - \sqrt{x-2a}}{\sqrt{x-2a} + \sqrt{x+2a}} = \frac{x}{2a}.$$

Multiplying both terms of the first fraction by $\sqrt{x+2a} - \sqrt{x-2a}$,

$$\frac{x + 2a - 2\sqrt{x^2 - 4a^2} + x - 2a}{(x + 2a) - (x - 2a)} = \frac{x}{2a}.$$

Uniting terms,

$$\frac{2x - 2\sqrt{x^2 - 4a^2}}{4a} = \frac{x}{2a},$$

that is,

$$\frac{x - \sqrt{x^2 - 4a^2}}{2a} = \frac{x}{2a}.$$

$$\therefore x - \sqrt{x^2 - 4a^2} = x.$$

Canceling,

$$-\sqrt{x^2 - 4a^2} = 0.$$

Squaring,

$$+ (x^2 - 4a^2) = 0.$$

Transposing,

$$x^2 = 4a^2.$$

Extracting the square root,

$$x = \pm 2a.$$

32.

$$\sqrt{\frac{x-a}{x+a}} + \sqrt{\frac{x+a}{x-a}} = a^2.$$

Multiplying by $\sqrt{x+a}\sqrt{x-a}$, or $\sqrt{x^2 - a^2}$,

$$x - a + x + a = a^2\sqrt{x^2 - a^2}.$$

$$2x = a^2\sqrt{x^2 - a^2}.$$

Squaring,

$$4x^2 = a^4x^2 - a^6.$$

Transposing, etc.,

$$x^2(a^4 - 4) = a^6.$$

$$\therefore x^2 = \frac{a^6}{a^4 - 4}.$$

Extracting the square root,

$$x = \pm \frac{a^3}{a^4 - 4} \sqrt{a^4 - 4}.$$

1. Let

Then,

$$\begin{aligned}
 x &= \text{the number.} \\
 x^2 + 25 &= 13^2. \\
 x^2 &= 169 - 25 = 144. \\
 \therefore x &= \pm 12.
 \end{aligned}$$

Hence, the number is 12 or - 12.

2. Let

Then,

$$\begin{aligned}
 x &= \text{the number.} \\
 x^2 &= 25^2 - 20^2 \\
 &= 625 - 400 = 225. \\
 \therefore x &= \pm 15.
 \end{aligned}$$

Hence, the number is 15 or - 15.

3. Let

Then,

$$\begin{aligned}
 x &= \text{the number.} \\
 (x + 5)(x - 5) &= 75. \\
 x^2 - 25 &= 75. \\
 x^2 &= 100. \\
 \therefore x &= \pm 10.
 \end{aligned}$$

Hence, the number is 10 or - 10.

4. Let

and

Then,

$$\begin{aligned}
 3x &= \text{first number,} \\
 4x &= \text{second number.} \\
 9x^2 + 16x^2 &= 15^2 = 225. \\
 x^2 &= 9. \\
 \therefore x &= \pm 3,
 \end{aligned}$$

whence $3x = 9$ or -9 , and $4x = 12$ or -12 .Hence, the numbers are 9 and 12, or -9 and -12 .

5. Let

and

Then,

$$\begin{aligned}
 4x &= \text{first number,} \\
 3x &= \text{second number.} \\
 16x^2 + 9x^2 &= 400. \\
 x^2 &= 16. \\
 \therefore x &= \pm 4,
 \end{aligned}$$

whence, $4x = 16$ or -16 , and $3x = 12$ or -12 .Hence, the numbers are 16 and 12, or -16 and -12 .

6. Let

and

Then,

$$\begin{aligned}
 2x &= \text{number of yards in length of smaller room,} \\
 3x &= \text{number of yards in length of larger room.} \\
 9x^2 - 2 \cdot 4x^2 &= 9. \\
 \therefore x &= \pm 3,
 \end{aligned}$$

whence, $2x = 6$ or -6 , and $3x = 9$ or -9 .

Hence, the negative values being inadmissible, the smaller room is 6 yards square and the larger is 9 yards square.

7. See next page.

8. Let

and

Then,

$$\begin{aligned}
 3x &= \text{number of rods in length of field,} \\
 2x &= \text{number of rods in width.} \\
 2x &= \text{number of rods in length and in width of the} \\
 &\quad \text{square field;} \\
 \therefore 4x^2 &= 160 \times 10 = 1600.
 \end{aligned}$$

Solving,

$$x = \pm 20,$$

whence, $3x = 60$ or -60 , and $2x = 40$ or -40 .

Rejecting the negative values, the dimensions of the original field were 60 rods by 40 rods.

7. Let x = number of rods in side of the part sold.
 Then, $4x^2 = 80^2 = 6400$.
 $\therefore x = \pm 40$.

Hence, the negative value being inadmissible, the field sold is 40 rods square.

9. Let x = number of rods in each side of garden.
 Then, $4x$ = number of rods of fence,
 and x^2 = number of square rods in area of garden;
 $\therefore x^2 = 160 \times 2\frac{1}{2} = 400$.
 Solving, $x = \pm 20$,

whence, rejecting the negative value, $4x = 80$, the number of rods of fence required.

10. Let $5 + x$ = one number,
 and $5 - x$ = the other number.
 Then, $(5 + x)(5 - x) = 21$.
 $25 - x^2 = 21$.

Solving, $x = \pm 2$,
 whence, $5 + x = 7$ or 3 , and $5 - x = 3$ or 7 .
 Hence, the numbers are 7 and 3.

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11. Let $8 + x$ = one number,
 and $8 - x$ = the other number.
 Then, $(8 + x)(8 - x) = 55$.
 $64 - x^2 = 55$.

Solving, $x = \pm 3$,
 whence, $8 + x = 11$ or 5 , and $8 - x = 5$ or 11 .
 Hence, the numbers are 11 and 5.

12. Let $13 + x$ = one number,
 and $13 - x$ = the other number.
 Then, $(13 + x)(13 - x) = 69$.
 $169 - x^2 = 69$.

Solving, $x = \pm 10$,
 whence, $13 + x = 23$ or 3 , and $13 - x = 3$ or 23 .
 Hence, the numbers are 23 and 3.

13. Let $\frac{5}{2} + x$ = one number,
 and $\frac{5}{2} - x$ = the other number.
 Then, $(\frac{5}{2} + x)(\frac{5}{2} - x) = -14$.
 $\frac{25}{4} - x^2 = -14$.

Solving, $x = \pm \frac{9}{2}$,
 whence, $\frac{5}{2} + x = 7$ or -2 , and $\frac{5}{2} - x = -2$ or 7 .
 Hence, the numbers are 7 and -2 .

14. Let $\frac{1}{2}x + x$ and $\frac{1}{2}x - x$ represent the two factors of 60 whose algebraic sum is 17.

Then, $(\frac{1}{2}x + x)(\frac{1}{2}x - x) = 60$.
 $\frac{25}{4}x^2 - x^2 = 60$.

Solving, $x = \pm \frac{7}{2}$,
 whence, $\frac{1}{2}x + x = 12$ or 5 , and $\frac{1}{2}x - x = 5$ or 12 .

Since the two factors of 60, whose algebraic sum is 17, are 12 and 5,
 $x^2 + 17x + 60 = (x + 12)(x + 5)$.

15. Let $1 + x$ and $1 - x$, whose sum is 2, be the two factors of -2 .

Then,
$$(1 + x)(1 - x) = -2.$$
$$1 - x^2 = -2.$$

Solving,
$$x = \pm \sqrt{3},$$

whence, $1 + x = 1 + \sqrt{3}$ or $1 - \sqrt{3}$ and $1 - x = 1 - \sqrt{3}$ or $1 + \sqrt{3}$.

Since the two factors of -2 whose algebraic sum is 2 are $1 + \sqrt{3}$ and $1 - \sqrt{3}$,

$$a^2 + 2a - 2 = (a + 1 + \sqrt{3})(a + 1 - \sqrt{3}).$$

16. Let $-1 + x$ and $-1 - x$, whose sum is -2 , be the two factors of -1 .

Then,
$$(-1 + x)(-1 - x) = -1.$$
$$1 - x^2 = -1.$$

Solving,
$$x = \pm \sqrt{2},$$

whence, $-1 + x = -1 \pm \sqrt{2}$ and $-1 - x = -1 \mp \sqrt{2}$.

Since the two factors of -1 whose algebraic sum is -2 are $-1 + \sqrt{2}$ and $-1 - \sqrt{2}$,

$$x^2 - 2x - 1 = (x - 1 + \sqrt{2})(x - 1 - \sqrt{2}).$$

17. Let
and

$$12 + x = \text{first part,}$$

$$12 - x = \text{second part.}$$

Then,
$$(12 + x)(12 - x) = 143.$$
$$144 - x^2 = 143.$$

Solving,
$$x = \pm 1,$$

whence, $12 + x = 13$ or 11 , and $12 - x = 11$ or 13 .

Hence, the parts are 13 and 11.

18. Let

$$x = \text{number of rods in width.}$$

Then,

$$4x = \text{number of rods in length;}$$

$$\therefore 4x \cdot x = 160 \cdot 10.$$

Solving,

$$x = \pm 20,$$

whence,

$$4x = \pm 80.$$

Hence, rejecting negative values, the length of the field was 80 rods, and its width was 20 rods.

19. Let

$$x = \text{one number.}$$

Then,

$$x^2 = \text{its square,}$$

and

$$x^2 + 56 = \text{square of the other number,}$$

whence,

$$\sqrt{x^2 + 56} = \text{the other number;}$$

$$\therefore x^2 + x^2 + 56 = 394.$$

Solving,

$$x = \pm 13,$$

whence,

$$\sqrt{x^2 + 56} = \pm 15.$$

Hence, the numbers are 13 and 15, 13 and -15 , -13 and 15, or -13 and -15 .

20. Let

$$50 + x = \text{number of rods in a side of one,}$$

and

$$50 - x = \text{number of rods in a side of the other.}$$

Then,

$$(50 + x)^2 + (50 - x)^2 = 160 \times \frac{295}{4}.$$

$$5000 + 2x^2 = 8200.$$

Solving,

$$x = \pm 40,$$

whence, $50 + x = 90$ or 10 , and $50 - x = 10$ or 90 .

Hence, the larger field is 90 rods square, and the smaller is 10 rods square.

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18. Let

$$(30 - p)(30 + p) = 891.$$

$$900 - p^2 = 891.$$

$$\therefore p = 3,$$

whence,

$$30 - p = 27 \text{ and } 30 + p = 33.$$

$$\therefore x^2 + 60x + 891 = (x + 27)(x + 33) = 0,$$

whence,

$$x = -27 \text{ or } -33.$$

19. Let

$$(-22 + p)(-22 - p) = 403.$$

$$484 - p^2 = 403.$$

$$\therefore p = 9,$$

whence,

$$-22 + p = -13, \text{ and } -22 - p = -31.$$

$$\therefore x^2 - 44x + 403 = (x - 13)(x - 31) = 0,$$

whence,

$$x = 13 \text{ or } 31.$$

20. Let

$$(10 - p)(10 + p) = -629.$$

$$100 - p^2 = -629.$$

$$\therefore p = 27,$$

whence,

$$10 - p = -17, \text{ and } 10 + p = 37.$$

$$\therefore x^2 + 20x - 629 = (x - 17)(x + 37) = 0,$$

whence,

$$x = 17 \text{ or } -37.$$

21. Let

$$(-15 - p)(-15 + p) = -2275.$$

$$225 - p^2 = -2275.$$

$$\therefore p = 50,$$

whence,

$$-15 - p = -65 \text{ and } -15 + p = 35.$$

$$\therefore x^2 - 30x - 2275 = (x - 65)(x + 35) = 0,$$

whence,

$$x = 65 \text{ or } -35.$$

22. Let

$$(12 - p)(12 + p) = 119.$$

$$144 - p^2 = 119.$$

$$\therefore p = 5,$$

whence,

$$12 - p = 7 \text{ and } 12 + p = 17.$$

$$\therefore x^2 + 24x + 119 = (x + 7)(x + 17) = 0,$$

whence,

$$x = -7 \text{ or } -17.$$

23. Let

$$(1 - p)(1 + p) = -323.$$

$$1 - p^2 = -323.$$

$$\therefore p = 18,$$

whence,

$$1 - p = -17 \text{ and } 1 + p = 19.$$

$$\therefore x^2 + 2x - 323 = (x - 17)(x + 19) = 0,$$

whence,

$$x = 17 \text{ or } -19.$$

24. Let

$$(-3 - p)(-3 + p) = -475.$$

$$9 - p^2 = -475.$$

$$\therefore p = 22,$$

whence,

$$-3 - p = -25, \text{ and } -3 + p = 19.$$

$$\therefore x^2 - 6x - 475 = (x - 25)(x + 19) = 0,$$

whence,

$$x = 25 \text{ or } -19.$$

25. Let

$$(4 - p)(4 + p) = -768.$$

$$16 - p^2 = -768.$$

$$\therefore p = 28,$$

whence,

$$4 - p = -24, \text{ and } 4 + p = 32.$$

$$\therefore x^2 + 8x - 768 = (x - 24)(x + 32) = 0,$$

whence,

$$x = 24 \text{ or } -32.$$

26. Let

$$\begin{aligned} \left(\frac{3}{2} - p\right) \left(\frac{3}{2} + p\right) &= -418. \\ \frac{9}{4} - p^2 &= -\frac{1672}{4}. \\ \therefore p &= \frac{41}{2}, \end{aligned}$$

whence,

$$\frac{3}{2} - p = -19, \text{ and } \frac{3}{2} + p = 22.$$

$$\therefore x^2 + 3x - 418 = (x - 19)(x + 22) = 0,$$

whence,

$$x = 19 \text{ or } -22.$$

27. Let

$$\begin{aligned} \left(\frac{5}{2} - p\right) \left(\frac{5}{2} + p\right) &= -546. \\ \frac{25}{4} - p^2 &= -\frac{2184}{4}. \\ \therefore p &= \frac{47}{2}, \end{aligned}$$

whence,

$$\frac{5}{2} - p = -21, \text{ and } \frac{5}{2} + p = 26.$$

$$\therefore x^2 + 5x - 546 = (x - 21)(x + 26) = 0,$$

whence,

$$x = 21 \text{ or } -26.$$

28. Let

$$\begin{aligned} \left(\frac{1}{2} - p\right) \left(\frac{1}{2} + p\right) &= -756. \\ \frac{1}{4} - p^2 &= -\frac{3024}{4}. \\ \therefore p &= \frac{55}{2}, \end{aligned}$$

whence,

$$\frac{1}{2} - p = -27, \text{ and } \frac{1}{2} + p = 28.$$

$$\therefore x^2 + x - 756 = (x - 27)(x + 28) = 0,$$

whence,

$$x = 27 \text{ or } -28.$$

29. Let

$$\begin{aligned} \left(-\frac{1}{2} - p\right) \left(-\frac{1}{2} + p\right) &= -506. \\ \frac{1}{4} - p^2 &= -\frac{2024}{4}. \\ \therefore p &= \frac{45}{2}, \end{aligned}$$

whence,

$$-\frac{1}{2} - p = -23, \text{ and } -\frac{1}{2} + p = 22.$$

$$\therefore x^2 - x - 506 = (x - 23)(x + 22) = 0,$$

whence,

$$x = 23 \text{ or } -22.$$

30.

Factoring,

$$\begin{aligned} x^2 + 2x - 168 &= 0. \\ (x - 12)(x + 14) &= 0. \\ \therefore x &= 12 \text{ or } -14. \end{aligned}$$

31.

Factoring,

$$\begin{aligned} x^2 + 6x - 135 &= 0. \\ (x - 9)(x + 15) &= 0. \\ \therefore x &= 9 \text{ or } -15. \end{aligned}$$

32.

Factoring,

$$\begin{aligned} x^2 + 3x - 154 &= 0. \\ (x - 11)(x + 14) &= 0. \\ \therefore x &= 11 \text{ or } -14. \end{aligned}$$

33. Let

$$\begin{aligned} \left(\frac{5}{2} - p\right) \left(\frac{5}{2} + p\right) &= 2. \\ \frac{25}{4} - p^2 &= \frac{8}{4}. \\ \therefore p &= \frac{1}{2}\sqrt{17}, \end{aligned}$$

whence,

$$\frac{5}{2} - p = \frac{5}{2} - \frac{1}{2}\sqrt{17}, \text{ and } \frac{5}{2} + p = \frac{5}{2} + \frac{1}{2}\sqrt{17}.$$

$$\therefore x^2 + 5x + 2 = \left(x + \frac{5}{2} - \frac{1}{2}\sqrt{17}\right) \left(x + \frac{5}{2} + \frac{1}{2}\sqrt{17}\right) = 0,$$

whence,

$$x = -\frac{5}{2} + \frac{1}{2}\sqrt{17} \text{ or } -\frac{5}{2} - \frac{1}{2}\sqrt{17},$$

that is,

$$x = -\frac{5}{2} \pm \frac{1}{2}\sqrt{17}.$$

34. Let

$$\begin{aligned} \left(\frac{1}{2} - p\right) \left(\frac{1}{2} + p\right) &= -10. \\ \frac{1}{4} - p^2 &= -\frac{40}{4}. \\ \therefore p &= \frac{1}{2}\sqrt{41}, \end{aligned}$$

whence,

$$\frac{1}{2} - p = \frac{1}{2} - \frac{1}{2}\sqrt{41}, \text{ and } \frac{1}{2} + p = \frac{1}{2} + \frac{1}{2}\sqrt{41}.$$

$$\therefore x^2 + x - 10 = \left(x + \frac{1}{2} - \frac{1}{2}\sqrt{41}\right) \left(x + \frac{1}{2} + \frac{1}{2}\sqrt{41}\right) = 0,$$

whence,

$$x = -\frac{1}{2} + \frac{1}{2}\sqrt{41} \text{ or } -\frac{1}{2} - \frac{1}{2}\sqrt{41},$$

that is,

$$x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{41}.$$

35. Let $(-\frac{1}{2} - p)(-\frac{1}{2} + p) = -1$,
 $\frac{1}{4} - p^2 = -1$.

$$\therefore p = \frac{1}{2}\sqrt{5}.$$

whence, $-\frac{1}{2} - p = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$, and $-\frac{1}{2} + p = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

$$\therefore x^2 - x - 1 = (x - \frac{1}{2} - \frac{1}{2}\sqrt{5})(x - \frac{1}{2} + \frac{1}{2}\sqrt{5}) = 0,$$

whence, $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1}{2} - \frac{1}{2}\sqrt{5}$,
 that is, $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

36. Let $(-1 - p)(-1 + p) = -4$,
 $1 - p^2 = -4$.

$$\therefore p = \sqrt{5},$$

whence, $-1 - p = -1 - \sqrt{5}$, and $-1 + p = -1 + \sqrt{5}$.

$$\therefore x^2 - 2x - 4 = (x - 1 - \sqrt{5})(x - 1 + \sqrt{5}) = 0,$$

whence, $x = 1 + \sqrt{5}$ or $1 - \sqrt{5}$,
 that is, $x = 1 \pm \sqrt{5}$.

37. Let $(-\frac{3}{2} - p)(-\frac{3}{2} + p) = -9$,
 $\frac{9}{4} - p^2 = -\frac{36}{4}$.

$$\therefore p = \frac{3}{2}\sqrt{5},$$

whence, $-\frac{3}{2} - p = -\frac{3}{2} - \frac{3}{2}\sqrt{5}$, and $-\frac{3}{2} + p = -\frac{3}{2} + \frac{3}{2}\sqrt{5}$.

$$\therefore x^2 - 3x - 9 = (x - \frac{3}{2} - \frac{3}{2}\sqrt{5})(x - \frac{3}{2} + \frac{3}{2}\sqrt{5}) = 0,$$

whence, $x = \frac{3}{2} + \frac{3}{2}\sqrt{5}$ or $\frac{3}{2} - \frac{3}{2}\sqrt{5}$,
 that is, $x = \frac{3}{2} \pm \frac{3}{2}\sqrt{5}$.

38. Let $(2 - p)(2 + p) = 8$,
 $4 - p^2 = 8$.

$$\therefore p = \sqrt{-4} = 2\sqrt{-1},$$

whence, $2 - p = 2 - 2\sqrt{-1}$, and $2 + p = 2 + 2\sqrt{-1}$.

$$\therefore x^2 + 4x + 8 = (x + 2 - 2\sqrt{-1})(x + 2 + 2\sqrt{-1}),$$

whence, $x = -2 + 2\sqrt{-1}$ or $-2 - 2\sqrt{-1}$,
 that is, $x = -2 \pm 2\sqrt{-1}$.

39. Let $(3 - p)(3 + p) = 14$,
 $9 - p^2 = 14$.

$$\therefore p = \sqrt{-5},$$

whence, $3 - p = 3 - \sqrt{-5}$, and $3 + p = 3 + \sqrt{-5}$.

$$\therefore x^2 + 6x + 14 = (x + 3 - \sqrt{-5})(x + 3 + \sqrt{-5}) = 0,$$

whence, $x = -3 + \sqrt{-5}$ or $-3 - \sqrt{-5}$,
 that is, $x = -3 \pm \sqrt{-5}$.

40. $x^2 + 8x = -25$.

Transposing,

$$x^2 + 8x + 25 = 0.$$

Let

$$(4 - p)(4 + p) = 25.$$

$$16 - p^2 = 25.$$

$$\therefore p = \sqrt{-9} = 3\sqrt{-1},$$

whence, $4 - p = 4 - 3\sqrt{-1}$, and $4 + p = 4 + 3\sqrt{-1}$.

$$\therefore x^2 + 8x + 25 = (x + 4 - 3\sqrt{-1})(x + 4 + 3\sqrt{-1}) = 0,$$

whence, $x = -4 + 3\sqrt{-1}$ or $-4 - 3\sqrt{-1}$,
 that is, $x = -4 \pm 3\sqrt{-1}$.

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15.

Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}x^2 + 3x &= 10. \\x^2 + 3x + \left(\frac{3}{2}\right)^2 &= 10 + \frac{9}{4} = \frac{49}{4}. \\x + \frac{3}{2} &= \pm \frac{7}{2}. \\x &= -\frac{3}{2} + \frac{7}{2} = 2. \\x &= -\frac{3}{2} - \frac{7}{2} = -5.\end{aligned}$$

16.

Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}x^2 - 3x &= 180. \\x^2 - 3x + \left(\frac{3}{2}\right)^2 &= 180 + \frac{9}{4} = \frac{729}{4}. \\x - \frac{3}{2} &= \pm \frac{27}{2}. \\x &= \frac{3}{2} + \frac{27}{2} = 15. \\x &= \frac{3}{2} - \frac{27}{2} = -12.\end{aligned}$$

17.

Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}x^2 + 15x &= 54. \\x^2 + 15x + \left(\frac{15}{2}\right)^2 &= 54 + \frac{225}{4} = \frac{441}{4}. \\x + \frac{15}{2} &= \pm \frac{21}{2}. \\x &= -\frac{15}{2} + \frac{21}{2} = 3. \\x &= -\frac{15}{2} - \frac{21}{2} = -18.\end{aligned}$$

18.

Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}x^2 - x &= 930. \\x^2 - x + \left(\frac{1}{2}\right)^2 &= 930 + \frac{1}{4} = \frac{3721}{4}. \\x - \frac{1}{2} &= \pm \frac{61}{2}. \\x &= \frac{1}{2} + \frac{61}{2} = 31. \\x &= \frac{1}{2} - \frac{61}{2} = -30.\end{aligned}$$

19.

Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}x^2 + 13x &= 140. \\x^2 + 13x + \left(\frac{13}{2}\right)^2 &= 140 + \frac{169}{4} = \frac{749}{4}. \\x + \frac{13}{2} &= \pm \frac{27}{2}. \\x &= -\frac{13}{2} + \frac{27}{2} = 7. \\x &= -\frac{13}{2} - \frac{27}{2} = -20.\end{aligned}$$

20.

Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}x^2 - 11x + 28 &= 0. \\x^2 - 11x + \left(\frac{11}{2}\right)^2 &= -28 + \frac{121}{4} = \frac{3}{4}. \\x - \frac{11}{2} &= \pm \frac{3}{2}. \\x &= \frac{11}{2} + \frac{3}{2} = 7. \\x &= \frac{11}{2} - \frac{3}{2} = 4.\end{aligned}$$

21.

Dividing by 5, etc.,
 Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}5x^2 - 3x - 2 &= 0. \\x^2 - \frac{3}{5}x &= \frac{2}{5}. \\x^2 - \frac{3}{5}x + \left(\frac{3}{10}\right)^2 &= \frac{2}{5} + \frac{9}{100} = \frac{49}{100}. \\x - \frac{3}{10} &= \pm \frac{7}{10}. \\x &= \frac{3}{10} + \frac{7}{10} = 1. \\x &= \frac{3}{10} - \frac{7}{10} = -\frac{2}{5}.\end{aligned}$$

22.

Dividing by 6, etc.,
 Completing the square,
 Extracting the square root,
 Taking the upper sign,
 Taking the lower sign,

$$\begin{aligned}6x^2 - 5x - 6 &= 0. \\x^2 - \frac{5}{6}x &= 1. \\x^2 - \frac{5}{6}x + \left(\frac{5}{12}\right)^2 &= 1 + \frac{25}{144} = \frac{169}{144}. \\x - \frac{5}{12} &= \pm \frac{13}{12}. \\x &= \frac{5}{12} + \frac{13}{12} = \frac{3}{2}. \\x &= \frac{5}{12} - \frac{13}{12} = -\frac{2}{3}.\end{aligned}$$

23.

Dividing by 2,

Completing the square,

Extracting the square root,

Taking the upper sign,

Taking the lower sign,

$$2x^2 + 9x = 35.$$

$$x^2 + \frac{9}{2}x = \frac{35}{2}.$$

$$x^2 + \frac{9}{2}x + \left(\frac{9}{4}\right)^2 = \frac{35}{2} + \frac{81}{4} = \frac{361}{4}.$$

$$x + \frac{9}{4} = \pm \frac{19}{4}.$$

$$x = -\frac{9}{4} + \frac{19}{4} = \frac{5}{2}.$$

$$x = -\frac{9}{4} - \frac{19}{4} = -\frac{7}{2}.$$

24.

Dividing by 3,

Completing the square,

Extracting the square root,

Taking the upper sign,

Taking the lower sign,

$$3x^2 - 7x = 10.$$

$$x^2 - \frac{7}{3}x = \frac{10}{3}.$$

$$x^2 - \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = \frac{10}{3} + \frac{49}{36} = \frac{169}{36}.$$

$$x - \frac{7}{6} = \pm \frac{13}{6}.$$

$$x = \frac{7}{6} + \frac{13}{6} = \frac{10}{3}.$$

$$x = \frac{7}{6} - \frac{13}{6} = -\frac{1}{3}.$$

25.

Dividing by 4,

Completing the square,

Extracting the square root,

Taking the upper sign,

Taking the lower sign,

$$4x^2 - 19x = 5.$$

$$x^2 - \frac{19}{4}x = \frac{5}{4}.$$

$$x^2 - \frac{19}{4}x + \left(\frac{19}{8}\right)^2 = \frac{5}{4} + \frac{361}{64} = \frac{441}{64}.$$

$$x - \frac{19}{8} = \pm \frac{21}{8}.$$

$$x = \frac{19}{8} + \frac{21}{8} = 5.$$

$$x = \frac{19}{8} - \frac{21}{8} = -\frac{1}{4}.$$

26.

Clearing of fractions,

Transposing, etc.,

Dividing by 10,

Completing the square,

Extracting the square root,

Taking the upper sign,

Taking the lower sign,

$$\frac{1}{x+1} + \frac{3}{x-1} = \frac{10}{3}.$$

$$3x - 3 + 9x + 9 = 10x^2 - 10.$$

$$10x^2 - 12x = 16.$$

$$x^2 - \frac{6}{5}x = \frac{8}{5}.$$

$$x^2 - \frac{6}{5}x + \left(\frac{3}{5}\right)^2 = \frac{8}{5} + \frac{9}{25} = \frac{49}{25}.$$

$$x - \frac{3}{5} = \pm \frac{7}{5}.$$

$$x = \frac{3}{5} + \frac{7}{5} = 2.$$

$$x = \frac{3}{5} - \frac{7}{5} = -\frac{4}{5}.$$

27.

Clearing of fractions,

Transposing, etc.,

Dividing by -7,

Completing the square,

Extracting the square root,

Taking the upper sign,

Taking the lower sign,

$$\frac{x^2}{x-2} - \frac{3x-5}{2} = \frac{x+2}{5}.$$

$$10x^2 - 15x^2 + 55x - 50 = 2x^2 - 8.$$

$$-7x^2 + 55x = 42.$$

$$x^2 - \frac{55}{7}x = -6.$$

$$x^2 - \frac{55}{7}x + \left(\frac{55}{14}\right)^2 = -6 + \frac{3025}{196} = \frac{1849}{196}.$$

$$x - \frac{55}{14} = \pm \frac{43}{14}.$$

$$x = \frac{55}{14} + \frac{43}{14} = 7.$$

$$x = \frac{55}{14} - \frac{43}{14} = \frac{6}{7}.$$

28.

Clearing of fractions,

Transposing, etc.,

Completing the square,

Extracting the square root,

Taking the upper sign,

Taking the lower sign,

$$\frac{1}{x+2} + \frac{x-2}{x} = \frac{x-7}{2x}.$$

$$2x + 2x^2 - 8 = x^2 - 5x - 14.$$

$$x^2 + 7x = -6.$$

$$x^2 + 7x + \left(\frac{7}{2}\right)^2 = -6 + \frac{49}{4} = \frac{25}{4}.$$

$$x + \frac{7}{2} = \pm \frac{5}{2}.$$

$$x = -\frac{7}{2} + \frac{5}{2} = -1.$$

$$x = -\frac{7}{2} - \frac{5}{2} = -6.$$

29. $x^2 + (m + n)(m - n) = 2mx.$
 Transposing, etc., $x^2 - 2mx = -m^2 + n^2.$
 Completing the square, $x^2 - 2mx + m^2 = -m^2 + n^2 + m^2 = n^2.$
 Extracting the square root, $x - m = \pm n.$
 $\therefore x = m \pm n.$

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3. $2x^2 - 5x = 42.$
 Multiplying by 2, $4x^2 - 10x = 84.$
 Completing the square, $4x^2 - 10x + (\frac{5}{2})^2 = 84 + \frac{25}{4} = \frac{361}{4}.$
 Extracting the square root, $2x - \frac{5}{2} = \pm \frac{19}{2}.$
 $2x = \frac{5}{2} \pm \frac{19}{2} = 12 \text{ or } -7.$
 $\therefore x = 6 \text{ or } -\frac{7}{2}.$

4. $6x^2 - 5x + 1 = 0.$
 Multiplying by 6, etc., $36x^2 - 30x = -6.$
 Completing the square, $36x^2 - 30x + (\frac{5}{2})^2 = -6 + \frac{25}{4} = \frac{1}{4}.$
 Extracting the square root, $6x - \frac{5}{2} = \pm \frac{1}{2}.$
 $6x = \frac{5}{2} \pm \frac{1}{2} = 3 \text{ or } 2.$
 $\therefore x = \frac{1}{2} \text{ or } \frac{1}{3}.$

5. $4x^2 - 12x = 27.$
 Completing the square, $4x^2 - 12x + 3^2 = 27 + 9 = 36.$
 Extracting the square root, $2x - 3 = \pm 6.$
 $2x = 3 \pm 6 = 9 \text{ or } -3.$
 $\therefore x = \frac{9}{2} \text{ or } -\frac{3}{2}.$

6. $8x^2 + 20x = 48.$
 Multiplying by 2, $16x^2 + 40x = 96.$
 Completing the square, $16x^2 + 40x + 5^2 = 96 + 25 = 121.$
 Extracting the square root, $4x + 5 = \pm 11.$
 $4x = -5 \pm 11 = 6 \text{ or } -16.$
 $\therefore x = \frac{3}{2} \text{ or } -4.$

7. $18x^2 + 6x = 4.$
 Multiplying by 2, $36x^2 + 12x = 8.$
 Completing the square, $36x^2 + 12x + 1^2 = 8 + 1 = 9.$
 Extracting the square root, $6x + 1 = \pm 3.$
 $6x = -1 \pm 3 = 2 \text{ or } -4.$
 $\therefore x = \frac{1}{3} \text{ or } -\frac{2}{3}.$

8. $3x^2 + 4x = 95.$
 Multiplying by 3, $9x^2 + 12x = 285.$
 Completing the square, $9x^2 + 12x + 2^2 = 285 + 4 = 289.$
 Extracting the square root, $3x + 2 = \pm 17.$
 $3x = -2 \pm 17 = 15 \text{ or } -19.$
 $\therefore x = 5 \text{ or } -\frac{19}{3}.$

9. $7x^2 + 2x = 32.$
 Multiplying by 7, $49x^2 + 14x = 224.$
 Completing the square, $49x^2 + 14x + 1^2 = 224 + 1 = 225.$
 Extracting the square root, $7x + 1 = \pm 15.$
 $7x = -1 \pm 15 = 14 \text{ or } -16.$
 $\therefore x = 2 \text{ or } -\frac{16}{7}.$

10. $8x^2 - 18x = 5$.
 Multiplying by 2, $16x^2 - 36x = 10$.
 Completing the square, $16x^2 - 36x + (\frac{9}{2})^2 = 10 + \frac{81}{4} = \frac{121}{4}$.
 Extracting the square root, $4x - \frac{9}{2} = \pm \frac{11}{2}$.
 $4x = \frac{9}{2} \pm \frac{11}{2} = 10 \text{ or } -1$.
 $\therefore x = \frac{5}{2} \text{ or } -\frac{1}{2}$.
11. $6x^2 + 5x = 4$.
 Multiplying by 6, $36x^2 + 30x = 24$.
 Completing the square, $36x^2 + 30x + (\frac{5}{2})^2 = 24 + \frac{25}{4} = \frac{121}{4}$.
 Extracting the square root, $6x + \frac{5}{2} = \pm \frac{11}{2}$.
 $6x = -\frac{5}{2} \pm \frac{11}{2} = 3 \text{ or } -8$.
 $\therefore x = \frac{1}{2} \text{ or } -\frac{4}{3}$.
12. $5x^2 + 6x = 8$.
 Multiplying by 5, $25x^2 + 30x = 40$.
 Completing the square, $25x^2 + 30x + 3^2 = 40 + 9 = 49$.
 Extracting the square root, $5x + 3 = \pm 7$.
 $5x = -3 \pm 7 = 4 \text{ or } -10$.
 $\therefore x = \frac{4}{5} \text{ or } -2$.
14. $2x^2 + 3x = 27$.
 Multiplying by 8 and adding 9 to each member,
 $16x^2 + 24x + 9 = 216 + 9 = 225$.
 Extracting the square root, $4x + 3 = \pm 15$.
 $4x = -3 \pm 15 = 12 \text{ or } -18$.
 $\therefore x = 3 \text{ or } -\frac{3}{2}$.
15. $2x^2 + 5x = 7$.
 Multiplying by 8 and adding 25 to each member,
 $16x^2 + 40x + 25 = 56 + 25 = 81$.
 Extracting the square root, $4x + 5 = \pm 9$.
 $4x = -5 \pm 9 = 4 \text{ or } -14$.
 $\therefore x = 1 \text{ or } -\frac{7}{2}$.
16. $2x^2 + 7x = -6$.
 Multiplying by 8 and adding 49 to each member,
 $16x^2 + 56x + 49 = -48 + 49 = 1$.
 Extracting the square root, $4x + 7 = \pm 1$.
 $4x = -7 \pm 1 = -6 \text{ or } -8$.
 $\therefore x = -\frac{3}{2} \text{ or } -2$.
17. $3x^2 - 5x = 2$.
 Multiplying by 12 and adding 25 to each member,
 $36x^2 - 60x + 25 = 24 + 25 = 49$.
 Extracting the square root, $6x - 5 = \pm 7$.
 $6x = 5 \pm 7 = 12 \text{ or } -2$.
 $\therefore x = 2 \text{ or } -\frac{1}{3}$.
18. $4x^2 - 15x = 4$.
 Multiplying by 16 and adding 225 to each member,
 $64x^2 - 240x + 225 = 64 + 225 = 289$.
 Extracting the square root, $8x - 15 = \pm 17$.
 $8x = 15 \pm 17 = 32 \text{ or } -2$.
 $\therefore x = 4 \text{ or } -\frac{1}{4}$.

19. $5x^2 - 7x = -2$.
 Multiplying by 20 and adding 49 to each member,
 $100x^2 - 140x + 49 = -40 + 49 = 9$.
 Extracting the square root, $10x - 7 = \pm 3$.
 $10x = 7 \pm 3 = 10 \text{ or } 4$.
 $\therefore x = 1 \text{ or } \frac{2}{5}$.
20. $6x^2 + 5x = -1$.
 Multiplying by 24 and adding 25 to each member,
 $144x^2 + 120x + 25 = -24 + 25 = 1$.
 Extracting the square root, $12x + 5 = \pm 1$.
 $12x = -5 \pm 1 = -4 \text{ or } -6$.
 $\therefore x = -\frac{1}{3} \text{ or } -\frac{1}{2}$.
21. $4x^2 - x - 3 = 0$.
 Transposing, multiplying by 16, and adding 1 to each member,
 $64x^2 - 16x + 1 = 48 + 1 = 49$.
 Extracting the square root, $8x - 1 = \pm 7$.
 $8x = 1 \pm 7 = 8 \text{ or } -6$.
 $\therefore x = 1 \text{ or } -\frac{3}{4}$.
22. $5x^2 - 2x - 16 = 0$.
 Transposing, multiplying by 20, and adding 4 to each member,
 $100x^2 - 40x + 4 = 320 + 4 = 324$.
 Extracting the square root, $10x - 2 = \pm 18$.
 $10x = 2 \pm 18 = 20 \text{ or } -16$.
 $\therefore x = 2 \text{ or } -\frac{8}{5}$.
23. $3x^2 + 7x - 110 = 0$.
 Transposing, multiplying by 12, and adding 49 to each member,
 $36x^2 + 84x + 49 = 1320 + 49 = 1369$.
 Extracting the square root, $6x + 7 = \pm 37$.
 $6x = -7 \pm 37 = 30 \text{ or } -44$.
 $\therefore x = 5 \text{ or } -\frac{22}{3}$.
24. $2x^2 - 5x - 150 = 0$.
 Transposing, multiplying by 8, and adding 25 to each member,
 $16x^2 - 40x + 25 = 1200 + 25 = 1225$.
 Extracting the square root, $4x - 5 = \pm 35$.
 $4x = 5 \pm 35 = 40 \text{ or } -30$.
 $\therefore x = 10 \text{ or } -\frac{15}{2}$.
25. $3x^2 + x - 200 = 0$.
 Transposing, multiplying by 12, and adding 1 to each member,
 $36x^2 + 12x + 1 = 2400 + 1 = 2401$.
 Extracting the square root, $6x + 1 = \pm 49$.
 $6x = -1 \pm 49 = 48 \text{ or } -50$.
 $\therefore x = 8 \text{ or } -\frac{25}{3}$.
26. $15x^2 - 7x - 2 = 0$.
 Transposing, multiplying by 60, and adding 49 to each member,
 $900x^2 - 420x + 49 = 120 + 49 = 169$.
 Extracting the square root, $30x - 7 = \pm 13$.
 $30x = 7 \pm 13 = 20 \text{ or } -6$.
 $\therefore x = \frac{2}{3} \text{ or } -\frac{1}{5}$.

27.

$$7x^2 - 20x - 32 = 0.$$

Transposing, multiplying by 28, and adding 400 to each member,

$$196x^2 - 560x + 400 = 896 + 400 = 1296.$$

Extracting the square root,

$$14x - 20 = \pm 36.$$

$$14x = 20 \pm 36 = 56 \text{ or } -16.$$

$$\therefore x = 4 \text{ or } -\frac{4}{7}.$$

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$$2. \quad 2x^2 + 5x + 2 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-5 \pm \sqrt{25 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} \\ &= \frac{-5 \pm 3}{4} = -\frac{1}{2} \text{ or } -2. \end{aligned}$$

$$3. \quad 3x^2 + 11x + 6 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-11 \pm \sqrt{121 - 4 \cdot 3 \cdot 6}}{2 \cdot 3} \\ &= \frac{-11 \pm 7}{6} = -\frac{2}{3} \text{ or } -3. \end{aligned}$$

$$4. \quad 6x^2 - 7x + 2 = 0.$$

$$\begin{aligned} \therefore x &= \frac{7 \pm \sqrt{49 - 4 \cdot 6 \cdot 2}}{2 \cdot 6} \\ &= \frac{7 \pm 1}{12} = \frac{2}{3} \text{ or } \frac{1}{2}. \end{aligned}$$

$$5. \quad 4x^2 + 4x - 15 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot (-15)}}{2 \cdot 4} \\ &= \frac{-4 \pm 16}{8} = \frac{3}{2} \text{ or } -\frac{5}{2}. \end{aligned}$$

$$6. \quad 2x^2 + 3x - 9 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-3 \pm \sqrt{9 - 4 \cdot 2 \cdot (-9)}}{2 \cdot 2} \\ &= \frac{-3 \pm 9}{4} = \frac{3}{2} \text{ or } -3. \end{aligned}$$

$$7. \quad 2x^2 + 3x + 1 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-3 \pm \sqrt{9 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} \\ &= \frac{-3 \pm 1}{4} = -\frac{1}{2} \text{ or } -1. \end{aligned}$$

$$8. \quad 3x^2 - 13x = 10.$$

$$\begin{aligned} \therefore x &= \frac{13 \pm \sqrt{169 - 4 \cdot 3 \cdot (-10)}}{2 \cdot 3} \\ &= \frac{13 \pm 17}{6} = 5 \text{ or } -\frac{2}{3}. \end{aligned}$$

$$9. \quad 7x^2 + 9x = 10.$$

$$\begin{aligned} \therefore x &= \frac{-9 \pm \sqrt{81 - 4 \cdot 7 \cdot (-10)}}{2 \cdot 7} \\ &= \frac{-9 \pm 19}{14} = \frac{5}{7} \text{ or } -2. \end{aligned}$$

$$10. \quad 2x^2 + 3x - 1 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-3 \pm \sqrt{9 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} \\ &= \frac{-3 \pm \sqrt{17}}{4} = -\frac{3}{4} \pm \frac{1}{4}\sqrt{17}. \end{aligned}$$

$$11. \quad 3x^2 + 2x - 4 = 0.$$

$$\begin{aligned} \therefore x &= \frac{-2 \pm \sqrt{4 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3} \\ &= \frac{-2 \pm 2\sqrt{13}}{6} = -\frac{1}{3} \pm \frac{1}{3}\sqrt{13}. \end{aligned}$$

$$12. \quad x^2 - 5x = -3.$$

$$\begin{aligned} \therefore x &= \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \\ &= \frac{5 \pm \sqrt{13}}{2} = \frac{5}{2} \pm \frac{1}{2}\sqrt{13}. \end{aligned}$$

$$13. \quad 3x^2 - 6x = -2.$$

$$\begin{aligned} \therefore x &= \frac{6 \pm \sqrt{36 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} \\ &= \frac{6 \pm 2\sqrt{3}}{6} = 1 \pm \frac{1}{3}\sqrt{3}. \end{aligned}$$

$$14. \quad 4x^2 - 3x - 2 = 0.$$

$$\begin{aligned} \therefore x &= \frac{3 \pm \sqrt{9 - 4 \cdot 4 \cdot (-2)}}{2 \cdot 4} \\ &= \frac{3 \pm \sqrt{41}}{8} = \frac{3}{8} \pm \frac{1}{8}\sqrt{41}. \end{aligned}$$

$$15. \quad x^2 - 6x + 10 = 0.$$

$$\begin{aligned} \therefore x &= \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} \\ &= \frac{6 \pm 2\sqrt{-1}}{2} = 3 \pm \sqrt{-1}. \end{aligned}$$

$$16. \quad x^2 + 4x + 12 = 0.$$

$$\therefore x = \frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 12}}{2 \cdot 1}$$

$$= \frac{-4 \pm 4\sqrt{-2}}{2} = -2 \pm 2\sqrt{-2}.$$

$$17. \quad x^2 - 8x = -20.$$

$$\therefore x = \frac{8 \pm \sqrt{64 - 4 \cdot 1 \cdot 20}}{2 \cdot 1}$$

$$= \frac{8 \pm 4\sqrt{-1}}{2} = 4 \pm 2\sqrt{-1}.$$

$$18. \quad x^2 - 6x + 5 = 0.$$

$$(x-1)(x-5) = 0.$$

$$\therefore x = 1 \text{ or } 5.$$

$$19. \quad x^2 - 8x + 7 = 0.$$

$$(x-1)(x-7) = 0.$$

$$\therefore x = 1 \text{ or } 7.$$

$$20. \quad 2x^2 - 5x = 42.$$

$$16x^2 - 40x + 25 = 336 + 25 = 361.$$

$$4x - 5 = \pm 19.$$

$$4x = 24 \text{ or } -14.$$

$$\therefore x = 6 \text{ or } -\frac{7}{2}.$$

$$21. \quad 7x^2 + 2x = 32.$$

$$49x^2 + 14x + 1 = 224 + 1 = 225.$$

$$7x + 1 = \pm 15.$$

$$7x = 14 \text{ or } -16.$$

$$\therefore x = 2 \text{ or } -\frac{16}{7}.$$

$$22. \quad x^2 - 12x = 64.$$

$$x^2 - 12x - 64 = 0.$$

$$(x-16)(x+4) = 0.$$

$$\therefore x = 16 \text{ or } -4.$$

$$23. \quad 18x^2 + 6x = 4.$$

$$36x^2 + 12x = 8.$$

$$36x^2 + 12x + 1 = 9.$$

$$6x + 1 = \pm 3.$$

$$6x = 2 \text{ or } -4.$$

$$\therefore x = \frac{1}{3} \text{ or } -\frac{2}{3}.$$

$$24. \quad x^2 - x - 72 = 0.$$

$$(x-9)(x+8) = 0.$$

$$\therefore x = 9 \text{ or } -8.$$

$$25. \quad 4x^2 - 12x = 27.$$

$$4x^2 - 12x + 9 = 36.$$

$$2x - 3 = \pm 6.$$

$$2x = 9 \text{ or } -3.$$

$$\therefore x = \frac{9}{2} \text{ or } -\frac{3}{2}.$$

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$$26. \quad x^2 = 3x + 10.$$

$$x^2 - 3x - 10 = 0.$$

$$(x-5)(x+2) = 0.$$

$$\therefore x = 5 \text{ or } -2.$$

$$27. \quad x^2 - 30 = 13x.$$

$$x^2 - 13x - 30 = 0.$$

$$(x-15)(x+2) = 0.$$

$$\therefore x = 15 \text{ or } -2.$$

$$28. \quad x^2 - 12x = 28.$$

$$x^2 - 12x - 28 = 0.$$

$$(x-14)(x+2) = 0.$$

$$\therefore x = 14 \text{ or } -2.$$

$$29. \quad x^2 - 50x = 159.$$

$$x^2 - 50x - 159 = 0.$$

$$(x-53)(x+3) = 0.$$

$$\therefore x = 53 \text{ or } -3.$$

$$30. \quad x^2 + 8x = 84.$$

$$x^2 + 8x - 84 = 0.$$

$$(x-6)(x+14) = 0.$$

$$\therefore x = 6 \text{ or } -14.$$

$$31. \quad x + \frac{1}{x} - \frac{5}{2} = 0.$$

$$2x^2 - 5x + 2 = 0.$$

$$(x-2)(2x-1) = 0.$$

$$\therefore x = 2 \text{ or } \frac{1}{2}.$$

$$32. \quad \frac{x^2}{9} + \frac{x}{3} = \frac{35}{4}.$$

$$\frac{x^2}{9} + \frac{x}{3} + \frac{1}{4} = \frac{36}{4} = 9.$$

$$\frac{x}{3} + \frac{1}{2} = \pm 3.$$

$$\frac{x}{3} = \frac{5}{2} \text{ or } -\frac{7}{2}.$$

$$\therefore x = \frac{15}{2} \text{ or } -\frac{21}{2}.$$

$$33. \quad \frac{x}{9(x-1)} = \frac{x-2}{6}.$$

$$2x = 3x^2 - 9x + 6.$$

$$3x^2 - 11x = -6.$$

$$36x^2 - 132x + 121 = -72 + 121 = 49.$$

$$6x - 11 = \pm 7.$$

$$6x = 18 \text{ or } 4.$$

$$\therefore x = 3 \text{ or } \frac{2}{3}.$$

$$34. \frac{4}{x^2 - 2x + 1} = \frac{1}{4}.$$

Extracting the square root,

$$\frac{2}{x-1} = \pm \frac{1}{2}.$$

$$4 = \pm (x-1)$$

$$= x-1 \text{ or } -x+1.$$

$$\therefore x = 5 \text{ or } -3.$$

$$36. \frac{9}{2x+1} + \frac{3}{x-3} = 4.$$

$$9x - 27 + 6x + 3 = 8x^2 - 20x - 12.$$

$$8x^2 - 35x + 12 = 0.$$

$$\therefore x = \frac{35 \pm \sqrt{1225 - 4 \cdot 8 \cdot 12}}{2 \cdot 8}$$

$$= \frac{35 \pm 29}{16} = 4 \text{ or } \frac{3}{8}.$$

$$35. \frac{x^2}{4} - \frac{2x}{3} = 28.$$

$$\frac{x^2}{4} - \frac{2x}{3} + \frac{4}{9} = \frac{252}{9} + \frac{4}{9} = \frac{256}{9}.$$

$$\frac{x}{2} - \frac{2}{3} = \pm \frac{16}{3}.$$

$$\frac{x}{2} = 6 \text{ or } -\frac{14}{3}.$$

$$\therefore x = 12 \text{ or } -\frac{28}{3}.$$

$$37. \frac{3x-1}{x+2} = \frac{x+1}{x-2}.$$

$$3x^2 - 7x + 2 = x^2 + 3x + 2.$$

$$x^2 - 5x = 0.$$

$$x(x-5) = 0.$$

$$\therefore x = 0 \text{ or } 5.$$

$$38. \frac{1+x}{x-3} - \frac{x-1}{x-2} = \frac{4}{5}.$$

$$5x^2 - 5x - 10 - 5x^2 + 20x - 15 = 4x^2 - 20x + 24.$$

$$4x^2 - 35x + 49 = 0.$$

$$\therefore x = \frac{35 \pm \sqrt{1225 - 4 \cdot 4 \cdot 49}}{2 \cdot 4}$$

$$= \frac{35 \pm 21}{8} = 7 \text{ or } \frac{7}{4}.$$

$$39. \frac{x}{x-5} - \frac{x-5}{x} = \frac{3}{2}.$$

$$2x^2 - 2x^2 + 20x - 50 = 3x^2 - 15x.$$

$$3x^2 - 35x + 50 = 0.$$

$$(x-10)(3x-5) = 0.$$

$$\therefore x = 10 \text{ or } \frac{5}{3}.$$

$$40. \frac{x+7}{x+5} + \frac{x+12}{x+6} = 7.$$

$$x^2 + 13x + 42 + x^2 + 17x + 60 = 7x^2 + 77x + 210.$$

$$5x^2 + 47x + 108 = 0.$$

$$(x+4)(5x+27) = 0.$$

$$\therefore x = -4 \text{ or } -\frac{27}{5}.$$

$$41. \frac{x+4}{x-2} + 3 = \frac{x+3}{x-3}.$$

$$x^2 + x - 12 + 3x^2 - 15x + 18 = x^2 + x - 6.$$

$$3x^2 - 15x + 12 = 0.$$

$$x^2 - 5x + 4 = 0.$$

$$(x-4)(x-1) = 0.$$

$$\therefore x = 4 \text{ or } 1.$$

$$\begin{aligned}
 42. \quad & \frac{4x}{x-1} - \frac{x+3}{x} = 4. \\
 & 4x^2 - x^2 - 2x + 3 = 4x^2 - 4x. \\
 & x^2 - 2x - 3 = 0. \\
 & (x-3)(x+1) = 0. \\
 & \therefore x = 3 \text{ or } -1.
 \end{aligned}$$

$$\begin{aligned}
 43. \quad & \frac{x}{x+2} + \frac{1}{2} = \frac{x+2}{2x}. \\
 & 2x^2 + x^2 + 2x = x^2 + 4x + 4. \\
 & 2x^2 - 2x - 4 = 0. \\
 & x^2 - x - 2 = 0. \\
 & (x-2)(x+1) = 0. \\
 & \therefore x = 2 \text{ or } -1.
 \end{aligned}$$

$$\begin{aligned}
 44. \quad & \frac{5x}{x+7} + \frac{x+6}{x+3} = 3. \\
 & 5x^2 + 15x + x^2 + 13x + 42 = 3x^2 + 30x + 63. \\
 & 3x^2 - 2x - 21 = 0. \\
 & (x-3)(3x+7) = 0. \\
 & \therefore x = 3 \text{ or } -\frac{7}{3}.
 \end{aligned}$$

$$\begin{aligned}
 45. \quad & \frac{x+2}{x-7} - \frac{x+5}{x-5} = 1. \\
 & x^2 - 3x - 10 - x^2 + 2x + 35 = x^2 - 12x + 35. \\
 & x^2 - 11x + 10 = 0. \\
 & (x-1)(x-10) = 0. \\
 & \therefore x = 1 \text{ or } 10.
 \end{aligned}$$

$$\begin{aligned}
 46. \quad & \frac{x-3}{x+4} + \frac{x+2}{x-2} = \frac{23}{10}. \\
 & 10x^2 - 50x + 60 + 10x^2 + 60x + 80 = 23x^2 + 46x - 184. \\
 & 3x^2 + 36x - 324 = 0. \\
 & x^2 + 12x - 108 = 0. \\
 & (x-6)(x+18) = 0. \\
 & \therefore x = 6 \text{ or } -18.
 \end{aligned}$$

$$\begin{aligned}
 47. \quad & \frac{2x+1}{1-2x} - \frac{5}{7} = \frac{x-8}{2}. \\
 & 28x + 14 - 10 + 20x = -14x^2 + 119x - 56. \\
 & 14x^2 - 71x = -60. \\
 & 56 \cdot 14x^2 - 56 \cdot 71x + 71^2 = 56(-60) + 71^2 = 1681. \\
 & 28x - 71 = \pm 41. \\
 & 28x = 112 \text{ or } 30. \\
 & \therefore x = 4 \text{ or } \frac{15}{14}.
 \end{aligned}$$

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$$\begin{aligned}
 2. \quad & x^2 - ax = ab - bx. \\
 & x(x-a) = -b(x-a). \\
 & (x+b)(x-a) = 0. \\
 & \therefore x = -b \text{ or } a.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & x^2 + ax = ac + ex. \\
 & x(x+a) = c(x+a). \\
 & (x-c)(x+a) = 0. \\
 & \therefore x = c \text{ or } -a.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & x^2 = (m-n)x + mn. \\
 & x^2 - (m-n)x - mn = 0. \\
 & (x-m)(x+n) = 0. \\
 & \therefore x = m \text{ or } -n.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & x^2 - 3bx = 2ax - 6ab. \\
 & x(x-3b) = 2a(x-3b). \\
 & (x-2a)(x-3b) = 0. \\
 & \therefore x = 2a \text{ or } 3b.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & 5x - 2ax = x^2 - 10a. \\
 & x^2 - 5x + 2ax - 10a = 0. \\
 & (x-5)(x+2a) = 0. \\
 & \therefore x = 5 \text{ or } -2a.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & x^2 + 3bx = 5cx + 15bc. \\
 & x(x+3b) = 5c(x+3b). \\
 & (x-5c)(x+3b) = 0. \\
 & \therefore x = 5c \text{ or } -3b.
 \end{aligned}$$

$$\begin{aligned} 8. \quad 2abx - x^2 &= 14ab - 7x. \\ x(2ab - x) &= 7(2ab - x). \\ (x - 7)(2ab - x) &= 0. \\ \therefore x &= 7 \text{ or } 2ab. \end{aligned}$$

$$\begin{aligned} 9. \quad 6x^2 + 3ax &= 2bx + ab. \\ 3x(2x + a) &= b(2x + a). \\ (3x - b)(2x + a) &= 0. \\ 3x - b = 0 \text{ or } 2x + a = 0. \\ \therefore x &= \frac{b}{3} \text{ or } -\frac{a}{2}. \end{aligned}$$

$$\begin{aligned} 10. \quad acx^2 - bcx - bd + adx &= 0. \\ cx(ax - b) + d(-b + ax) &= 0. \\ (cx + d)(ax - b) &= 0. \\ ax - b = 0 \text{ or } cx + d = 0. \\ \therefore x &= \frac{b}{a} \text{ or } -\frac{d}{c}. \end{aligned}$$

$$\begin{aligned} 11. \quad x^2 + 4mx + 3nx + 12mn &= 0. \\ x(x + 4m) + 3n(x + 4m) &= 0. \\ (x + 3n)(x + 4m) &= 0. \\ \therefore x &= -3n \text{ or } -4m. \end{aligned}$$

$$\begin{aligned} 12. \quad x^2 - 2ax &= a^2. \\ x^2 - 2ax + a^2 &= 2a^2. \\ x - a &= \pm a\sqrt{2}. \\ \therefore x &= a \pm a\sqrt{2} = a(1 \pm \sqrt{2}). \end{aligned}$$

$$\begin{aligned} 13. \quad x^2 + 4bx &= b^2. \\ x^2 + 4bx + 4b^2 &= 5b^2. \\ x + 2b &= \pm b\sqrt{5}. \\ \therefore x &= -2b \pm b\sqrt{5} = b(-2 \pm \sqrt{5}). \end{aligned}$$

$$\begin{aligned} 14. \quad x^2 &= 4ax - 2a^2. \\ x^2 - 4ax + 4a^2 &= 4a^2 - 2a^2 = 2a^2. \\ x - 2a &= \pm a\sqrt{2}. \\ \therefore x &= 2a \pm a\sqrt{2} = a(2 \pm \sqrt{2}). \end{aligned}$$

$$\begin{aligned} 15. \quad x^2 - ax + a^2 &= 0. \\ \therefore x &= \frac{a \pm \sqrt{a^2 - 4 \cdot 1 \cdot a^2}}{2} \\ &= \frac{a \pm a\sqrt{-3}}{2} \\ &= \frac{a}{2}(1 \pm \sqrt{-3}). \end{aligned}$$

$$\begin{aligned} 16. \quad x^2 &= bx + b^2. \\ x^2 - bx - b^2 &= 0. \\ \therefore x &= \frac{b \pm \sqrt{b^2 - 4 \cdot 1 \cdot (-b^2)}}{2} \\ &= \frac{b \pm b\sqrt{5}}{2} = \frac{b}{2}(1 \pm \sqrt{5}). \end{aligned}$$

$$\begin{aligned} 17. \quad x^2 + px + q &= 0. \\ \therefore x &= \frac{-p \pm \sqrt{p^2 - 4 \cdot 1 \cdot q}}{2} \\ &= \frac{1}{2}(-p \pm \sqrt{p^2 - 4q}). \end{aligned}$$

$$\begin{aligned} 18. \quad x^2 - 2x + a &= 0. \\ \therefore x &= \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot a}}{2} \\ &= \frac{2 \pm 2\sqrt{1 - a}}{2} \\ &= 1 \pm \sqrt{1 - a}. \end{aligned}$$

$$\begin{aligned} 19. \quad 4ax - x^2 &= 3a^2. \\ x^2 - 4ax &= -3a^2. \\ x^2 - 4ax + 4a^2 &= a^2. \\ x - 2a &= \pm a. \\ \therefore x &= 3a \text{ or } a. \end{aligned}$$

$$\begin{aligned} 20. \quad x^2 - a &= 1 + ax. \\ x^2 - 1 &= ax + a. \\ (x - 1)(x + 1) &= a(x + 1). \\ (x - 1 - a)(x + 1) &= 0. \\ \therefore x &= 1 + a \text{ or } -1. \end{aligned}$$

$$\begin{aligned} 21. \quad x^2 - b^2 &= a^2 - bx. \\ x^2 + bx &= a^2 + b^2. \\ x^2 + bx + \left(\frac{b}{2}\right)^2 &= \frac{4a^2 + 4b^2}{4} + \frac{b^2}{4}. \\ &= \frac{4a^2 + 5b^2}{4}. \\ x + \frac{b}{2} &= \pm \frac{1}{2}\sqrt{4a^2 + 5b^2}. \\ \therefore x &= -\frac{b}{2} \pm \frac{1}{2}\sqrt{4a^2 + 5b^2}. \end{aligned}$$

$$\begin{aligned} 22. \quad 21b^2 - 4bx &= x^2. \\ x^2 + 4bx - 21b^2 &= 0. \\ (x - 3b)(x + 7b) &= 0. \\ \therefore x &= 3b \text{ or } -7b. \end{aligned}$$

$$\begin{aligned} 23. \quad 5ax + 6a^2 &= 6x^2. \\ 6x^2 - 5ax - 6a^2 &= 0. \\ (2x - 3a)(3x + 2a) &= 0. \\ 2x - 3a = 0 \text{ or } 3x + 2a = 0. \\ \therefore x &= \frac{3a}{2} \text{ or } -\frac{2a}{3}. \end{aligned}$$

$$\begin{aligned} 24. \quad x^2 - 1 &= 4ax - a^2. \\ x^2 - 4ax &= 1 - a^2. \\ x^2 - 4ax + 4a^2 &= 1 + 3a^2. \\ x - 2a &= \pm \sqrt{1 + 3a^2}. \\ \therefore x &= 2a \pm \sqrt{1 + 3a^2}. \end{aligned}$$

$$x^2 + b^2 = 4(a^2 + bx).$$

$$x^2 - 4bx = 4a^2 - b^2.$$

$$x^2 - 4bx + 4b^2 = 4a^2 + 3b^2.$$

$$x - 2b = \pm \sqrt{4a^2 + 3b^2}.$$

$$\therefore x = 2b \pm \sqrt{4a^2 + 3b^2}.$$

$$26. \quad \frac{x}{a} + \frac{a}{x} = \frac{5}{2}.$$

$$2x^2 + 2a^2 = 5ax.$$

$$2x^2 - 5ax + 2a^2 = 0.$$

$$(x - 2a)(2x - a) = 0.$$

$$\therefore x = 2a \text{ or } \frac{a}{2}.$$

$$27. \quad \frac{3c^2x^2}{4} + \frac{2cx}{3} = \frac{5}{3}.$$

$$9c^2x^2 + 8cx = 20.$$

$$81c^2x^2 + 72cx = 180.$$

$$81c^2x^2 + 72cx + 16 = 196.$$

$$9cx + 4 = \pm 14.$$

$$9cx = 10 \text{ or } -18.$$

$$\therefore x = \frac{10}{9c} \text{ or } -\frac{2}{c}.$$

$$28. \quad \frac{2x}{x-a} - \frac{3a}{x+a} = 3.$$

$$2x^2 + 2ax - 3ax + 3a^2 = 3x^2 - 3a^2.$$

$$x^2 + ax - 6a^2 = 0.$$

$$(x - 2a)(x + 3a) = 0.$$

$$\therefore x = 2a \text{ or } -3a.$$

$$29. \quad \frac{a}{x-a} - 2 = \frac{2a}{x}.$$

$$ax - 2x^2 + 2ax = 2ax - 2a^2.$$

$$2x^2 - ax = 2a^2.$$

$$16x^2 - 8ax + a^2 = 16a^2 + a^2 = 17a^2.$$

$$4x - a = \pm a\sqrt{17}.$$

$$\therefore x = \frac{a}{4}(1 \pm \sqrt{17}).$$

30.

$$\frac{1}{ax+4} = 1 - \frac{ax-4}{16}.$$

$$16 = 16ax + 64 - a^2x^2 + 16.$$

$$a^2x^2 - 16ax = 64.$$

$$a^2x^2 - 16ax + 64 = 64 + 64 = 128.$$

$$ax - 8 = \pm 8\sqrt{2}.$$

$$\therefore x = \frac{8}{a}(1 \pm \sqrt{2}).$$

$$31. \quad x^2 + \frac{a}{b}x = \frac{a+b}{b}.$$

$$x^2 + \frac{a}{b}x + \left(\frac{a}{2b}\right)^2 = \frac{4ab + 4b^2}{4b^2} + \frac{a^2}{4b^2}$$

$$= \frac{a^2 + 4ab + 4b^2}{4b^2}.$$

$$x + \frac{a}{2b} = \pm \frac{a+b}{2b}.$$

$$\therefore x = 1 \text{ or } -\frac{a+b}{b}.$$

$$32. \quad x^2 + 2 = \frac{(2a^2 + 1)x}{a}.$$

$$ax^2 + 2a = 2a^2x + x.$$

$$ax^2 - x = 2a^2x - 2a.$$

$$x(ax - 1) = 2a(ax - 1).$$

$$(x - 2a)(ax - 1) = 0.$$

$$\therefore x = 2a \text{ or } \frac{1}{a}.$$

$$33. \quad x^2 - \frac{2x}{ab} = \frac{4(ab-1)}{ab}.$$

$$x^2 - \frac{2x}{ab} + \left(\frac{1}{ab}\right)^2 = \frac{4ab(ab-1)}{a^2b^2} + \frac{1}{a^2b^2}$$

$$= \frac{4a^2b^2 - 4ab + 1}{a^2b^2}.$$

$$x - \frac{1}{ab} = \pm \frac{2ab-1}{ab}.$$

$$x = \frac{1 \pm (2ab-1)}{ab}$$

$$= 2 \text{ or } \frac{2(1-ab)}{ab}.$$

34. $x^2 - 2(a-b)x = 4ab$.
 $x^2 - 2(a-b)x + (a-b)^2 = (a-b)^2 + 4ab$
 $= a^2 + 2ab + b^2$.
 $x - (a-b) = \pm(a+b)$.
 $\therefore x = a-b + a+b$ or $a-b - a-b$
 $= 2a$ or $-2b$.
35. $x^2 + 2(a+8)x = -32a$.
 $x^2 + 2(a+8)x + (a+8)^2 = (a+8)^2 - 32a$
 $= a^2 - 16a + 64$.
 $x + a + 8 = \pm(a-8)$.
 $\therefore x = -a-8 + a-8$ or $-a-8 - a+8$
 $= -16$ or $-2a$.
36. $x^2 + x + bx + b = ax + a$.
 $x(x+1) + b(x+1) = a(x+1)$.
 $(x-a+b)(x+1) = 0$.
 $x-a+b=0$ or $x+1=0$.
 $\therefore x = a-b$ or -1 .
37. $2ax - a + 2bx - b = 2x^2 - x$.
 $a(2x-1) + b(2x-1) = x(2x-1)$.
 $(a+b-x)(2x-1) = 0$.
 $a+b-x=0$ or $2x-1=0$.
 $\therefore x = a+b$ or $\frac{1}{2}$.
38. $x^2 + 4(a-1)x = 8a - 4a^2$.
 $x^2 + 4(a-1)x + 4(a-1)^2 = 4(a-1)^2 + 8a - 4a^2 = 4$.
 $x + 2a - 2 = \pm 2$.
 $\therefore x = -2a + 2 + 2$ or $-2a + 2 - 2$
 $= -2a + 4$ or $-2a$.
39. $a(x-2a+b) + a(x+a-b) = x^2 - (a-b)^2$.
 $a(2x-a) = x^2 - a^2 + 2ab - b^2$.
 $2ax - a^2 = x^2 - a^2 + 2ab - b^2$.
 $x^2 - 2ax = -2ab + b^2$.
 $x^2 - 2ax + a^2 = a^2 - 2ab + b^2$.
 $x - a = a - b$ or $-a + b$.
 $\therefore x = 2a - b$ or b .

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40. See next page.

41.

$$\frac{1}{a-x} - \frac{1}{a+x} = \frac{3+x^2}{a^2-x^2}$$

$$a+x-a+x = 3+x^2$$

$$x^2-2x = -3$$

$$x^2-2x+1 = -2$$

$$x-1 = \pm\sqrt{-2}$$

$$\therefore x = 1 \pm \sqrt{-2}$$

42.

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{a-b}$$

$$ax + ab - bx - b^2 + ax + a^2 - bx - ab = x^2 + ax + bx + ab$$

$$x^2 - ax + 3bx = a^2 - ab - b^2$$

$$4x^2 - 4(a-3b)x + (a-3b)^2 = 4a^2 - 4ab - 4b^2 + a^2 - 6ab + 9b^2$$

$$= 5(a^2 - 2ab + b^2)$$

$$2x - (a-3b) = \pm(a-b)\sqrt{5}$$

$$\therefore x = \frac{a-3b \pm (a-b)\sqrt{5}}{2}$$

40.

$$\begin{aligned} \frac{2a+x}{2a-x} + \frac{a-2x}{a+2x} &= \frac{8}{3} \\ 1 + \frac{2x}{2a-x} + 1 - \frac{4x}{a+2x} &= 2\frac{2}{3} \\ \frac{x}{2a-x} - \frac{2x}{a+2x} &= \frac{1}{3} \\ \frac{2a-x}{3ax} - \frac{a+2x}{6x^2-12ax+6x^2} &= \frac{2a^2+3ax-2x^2}{14x^2-12ax-2a^2} = 0 \\ \frac{7x^2-6ax-a^2}{(x-a)(7x+a)} &= 0 \\ \therefore x &= a \text{ or } -\frac{a}{7} \end{aligned}$$

43.

$$\begin{aligned} \frac{x^2+1}{x} - \frac{a+b}{c} &= \frac{c}{a+b} \\ \frac{x^2+1}{x} - \frac{(a+b)^2+c^2}{c(a+b)} &= 0 \\ x^2+1 - \frac{(a+b)^2+c^2}{c(a+b)}x &= 0 \\ x^2 - \frac{(a+b)^2+c^2}{c(a+b)}x &= -1 \\ x^2 - \frac{(a+b)^2+c^2}{c(a+b)}x + \left[\frac{(a+b)^2+c^2}{2c(a+b)} \right]^2 &= \frac{(a+b)^4+2c^2(a+b)^2+c^4}{4c^2(a+b)^2} - \frac{4c^2(a+b)^2}{4c^2(a+b)^2} \\ &= \frac{(a+b)^4-4c^2(a+b)^2+c^4}{4c^2(a+b)^2} \\ x - \frac{(a+b)^2+c^2}{2c(a+b)} &= \pm \frac{(a+b)^2-c^2}{2c(a+b)} \\ \therefore x &= \frac{(a+b)^2+c^2+(a+b)^2-c^2}{2c(a+b)} \text{ or } \frac{(a+b)^2+c^2-(a+b)^2+c^2}{2c(a+b)} \\ &= \frac{a+b}{c} \text{ or } \frac{c}{a+b} \end{aligned}$$

44. See next page.

45.

$$\begin{aligned} \frac{bx}{a-x} + b &= \frac{a(x+2b)}{a+b} \\ abx + b^2x + a^2b - abx + ab^2 - b^2x &= a^2x + 2a^2b - ax^2 - 2abx \\ ax^2 - a^2x + 2abx &= a^2b - ab^2 \\ x^2 - ax + 2bx &= ab - b^2 \\ 4x^2 - 4x(a-2b) + (a-2b)^2 &= 4ab - 4b^2 + a^2 - 4ab + 4b^2 = a^2 \\ 2x - (a-2b) &= \pm a \\ \therefore x &= \frac{a-2b+a}{2} \text{ or } \frac{a-2b-a}{2} = a-b \text{ or } -b \end{aligned}$$

46.

$$\begin{aligned} \sqrt{a+x} - \sqrt{a-x} &= \sqrt{2x} \\ a+x - 2\sqrt{a^2-x^2} + a-x &= 2x \\ 2a-2x &= 2\sqrt{a^2-x^2} \\ a-x &= \sqrt{a^2-x^2} \\ a^2-2ax+x^2 &= a^2-x^2 \\ 2x^2-2ax &= 0 \\ x(x-a) &= 0 \\ \therefore x &= 0 \text{ or } a \end{aligned}$$

$$44. \quad \frac{2x-a}{b} + 3 = \frac{4a}{2x-b}.$$

$$4x^2 - 2ax - 2bx + ab + 6bx - 3b^2 = 4ab.$$

$$4x^2 - 2ax + 4bx = 3b^2 + 3ab.$$

$$4x^2 - 2x(a-2b) = 3b^2 + 3ab.$$

$$16x^2 - 8x(a-2b) + (a-2b)^2 = 12b^2 + 12ab + a^2 - 4ab + 4b^2$$

$$= a^2 + 8ab + 16b^2.$$

$$4x - (a-2b) = \pm(a+4b).$$

$$\therefore x = \frac{a-2b+a+4b}{4} \text{ or } \frac{a-2b-a-4b}{4}$$

$$= \frac{a+b}{2} \text{ or } \frac{-3b}{2}.$$

$$47. \quad \sqrt{x-a} + \sqrt{b-x} = \sqrt{b-a}.$$

$$x-a + 2\sqrt{(x-a)(b-x)} + b-x = b-a.$$

$$2\sqrt{(x-a)(b-x)} = 0.$$

$$\sqrt{(x-a)(b-x)} = 0.$$

$$(x-a)(b-x) = 0.$$

$$\therefore x = a \text{ or } b.$$

48.

$$\sqrt{x^2-b^2} = \sqrt{x+b}\sqrt{a+b}.$$

$$x^2-b^2 = (x+b)(a+b).$$

$$(x+b)(x-b) = (x+b)(a+b).$$

$$(x-b-a-b)(x+b) = 0.$$

$$(x-a-2b)(x+b) = 0.$$

$$\therefore x = a+2b \text{ or } -b.$$

49.

$$\sqrt{a-x} + \sqrt{b-x} = \sqrt{a+b-2x}.$$

$$a-x + 2\sqrt{(a-x)(b-x)} + b-x = a+b-2x.$$

$$2\sqrt{(a-x)(b-x)} = 0.$$

$$\sqrt{(a-x)(b-x)} = 0.$$

$$(a-x)(b-x) = 0.$$

$$\therefore x = a \text{ or } b.$$

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51.

$$8\sqrt{x} - 8x = \frac{3}{2}.$$

$$16\sqrt{x} = 16x + 3.$$

$$256x = 256x^2 + 96x + 9.$$

$$256x^2 - 160x + 25 = -9 + 25 = 16.$$

$$16x - 5 = \pm 4.$$

$$16x = 9 \text{ or } 1.$$

$$x = \frac{9}{16} \text{ or } \frac{1}{16}.$$

Substituting $\frac{9}{16}$ for x in the given equation,

$$8\sqrt{\frac{9}{16}} - \frac{9}{2} = \frac{3}{2},$$

$$\frac{6}{2} = \frac{3}{2}.$$

which reduces to

Hence, $\frac{9}{16}$ is a root of the equation.Substituting $\frac{1}{16}$ for x in the given equation,

$$8\sqrt{\frac{1}{16}} - \frac{1}{2} = \frac{3}{2},$$

$$\frac{2}{2} = \frac{3}{2}.$$

which reduces to

Hence, $\frac{1}{16}$ is a root of the equation.

52.

$$3x + \sqrt{x} = 5\sqrt{4x}.$$

Since \sqrt{x} is a factor of both members, $x = 0$ derived from $\sqrt{x} = 0$, satisfies the equation, and hence 0 is a root of the equation.

Dividing by \sqrt{x} ,

$$3\sqrt{x} + 1 = 5\sqrt{4}.$$

$$3\sqrt{x} = 10 - 1 = 9.$$

$$\sqrt{x} = 3.$$

$$\therefore x = 9.$$

Substituting 9 for x in the given equation,

$$27 + \sqrt{9} = 5\sqrt{36},$$

$$30 = 30.$$

which reduces to

Hence, 9 is a root of the equation.

53. See next page.

54.

$$x - 5 - \sqrt{x - 3} = 0.$$

$$x - 5 = \sqrt{x - 3}.$$

$$x^2 - 10x + 25 = x - 3.$$

$$x^2 - 11x + 28 = 0.$$

$$(x - 4)(x - 7) = 0.$$

$$\therefore x = 4 \text{ or } 7.$$

Substituting 4 for x in the given equation,

$$4 - 5 - \sqrt{4 - 3} = 0,$$

which is not true unless the negative square root of $4 - 3$ is taken.

Hence, 4 is not a root of the given equation, but of the equation

$$x - 5 + \sqrt{x - 3} = 0.$$

Substituting 7 for x in the given equation,

$$7 - 5 - \sqrt{7 - 3} = 0,$$

$$0 = 0.$$

which reduces to

Hence, 7 is a root of the equation.

55.

$$\sqrt{4x + 17} + \sqrt{x + 1} - 4 = 0.$$

$$\sqrt{4x + 17} = 4 - \sqrt{x + 1}.$$

$$4x + 17 = 16 - 8\sqrt{x + 1} + x + 1.$$

$$3x = -8\sqrt{x + 1}.$$

$$9x^2 = 64x + 64.$$

$$81x^2 - 576x = 576.$$

$$81x^2 - 576x + 1024 = 1600.$$

$$9x - 32 = \pm 40.$$

$$9x = 72 \text{ or } -8.$$

$$\therefore x = 8 \text{ or } -\frac{8}{9}.$$

Substituting 8 for x in the given equation,

$$\sqrt{49} + \sqrt{9} - 4 = 0,$$

which is not true unless the negative square root of 9 is taken.

Hence, 8 is not a root of the given equation, but of the equation

$$\sqrt{4x + 17} - \sqrt{x + 1} - 4 = 0.$$

Substituting $-\frac{8}{9}$ for x in the given equation,

$$\sqrt{\frac{121}{9}} + \sqrt{\frac{1}{9}} - 4 = 0,$$

$$0 = 0.$$

which reduces to

Hence, $-\frac{8}{9}$ is a root of the equation.

53.

$$x - 1 + \sqrt{x + 5} = 0.$$

$$x - 1 = -\sqrt{x + 5}.$$

$$x^2 - 2x + 1 = (x + 5).$$

$$x^2 - 3x - 4 = 0.$$

$$(x + 1)(x - 4) = 0.$$

$$\therefore x = -1 \text{ or } 4.$$

Substituting -1 for x in the given equation,

$$-1 - 1 + \sqrt{-1 + 5} = 0,$$

$$0 = 0.$$

which reduces to

Hence, -1 is a root of the equation.

Substituting 4 for x in the given equation,

$$4 - 1 + \sqrt{4 + 5} = 0,$$

which is not true unless the negative square root of $4 + 5$ is taken.

Hence, 4 is not a root of the given equation, but of the equation

$$x - 1 - \sqrt{x + 5} = 0.$$

56.

$$\sqrt{x - 1} + \sqrt{2x - 1} - \sqrt{5x} = 0.$$

$$\sqrt{x - 1} - \sqrt{5x} = -\sqrt{2x - 1}.$$

$$x - 1 - 2\sqrt{5x^2 - 5x} + 5x = (2x - 1).$$

$$\sqrt{5x^2 - 5x} = 2x.$$

$$5x^2 - 5x = 4x^2.$$

$$x(x - 5) = 0.$$

$$\therefore x = 0 \text{ or } 5.$$

Substituting 0 for x in the given equation,

$$\sqrt{-1} + \sqrt{-1} = 0,$$

from which it is evident that 0 is not a root of the given equation, but of the equation

$$\sqrt{x - 1} - \sqrt{2x - 1} - \sqrt{5x} = 0.$$

Substituting 5 for x in the given equation,

$$\sqrt{4} + \sqrt{9} - \sqrt{25} = 0,$$

$$0 = 0.$$

which reduces to

Hence, 5 is a root of the equation.

57.

$$\sqrt{2x - 7} - \sqrt{2x} + \sqrt{x - 7} = 0.$$

$$\sqrt{x - 7} - \sqrt{2x} = -\sqrt{2x - 7}.$$

$$x - 7 - 2\sqrt{2x^2 - 14x} + 2x = (2x - 7).$$

$$2\sqrt{2x^2 - 14x} = x.$$

$$8x^2 - 56x = x^2.$$

$$7x(x - 8) = 0.$$

$$\therefore x = 0 \text{ or } 8.$$

Substituting 0 for x in the given equation,

$$\sqrt{-7} + \sqrt{-7} = 0,$$

which is not true. It is evident that 0 is a root of the equation

$$\sqrt{2x - 7} - \sqrt{2x} - \sqrt{x - 7} = 0,$$

or of

$$-\sqrt{2x - 7} - \sqrt{2x} + \sqrt{x - 7} = 0.$$

Substituting 8 for x in the given equation,

$$\sqrt{9} - \sqrt{16} + \sqrt{1} = 0,$$

$$0 = 0.$$

which reduces to

Hence, 8 is a root of the equation.

$$\begin{aligned}
 58. \quad & \sqrt{x+3} + \sqrt{4x+1} - \sqrt{10x+4} = 0. \\
 & \sqrt{x+3} + \sqrt{4x+1} = \sqrt{10x+4}. \\
 & x+3 + 2\sqrt{4x^2+13x+3} + 4x+1 = 10x+4. \\
 & 2\sqrt{4x^2+13x+3} = 5x. \\
 & 16x^2 + 52x + 12 = 25x^2. \\
 & 9x^2 - 52x = 12. \\
 & 81x^2 - 468x + 26^2 = 108 + 676 = 784. \\
 & 9x - 26 = \pm 28. \\
 & \therefore x = 6 \text{ or } -\frac{2}{3}.
 \end{aligned}$$

Substituting 6 for x in the given equation,

$$\begin{aligned}
 \sqrt{9} + \sqrt{25} - \sqrt{64} &= 0, \\
 0 &= 0.
 \end{aligned}$$

which reduces to

Hence, 6 is a root of the equation.

Substituting $-\frac{2}{3}$ for x in the given equation,

$$\sqrt{\frac{2}{9}} + \sqrt{\frac{1}{9}} - \sqrt{\frac{16}{9}} = 0,$$

which is not true unless the second square root is taken with a sign of quality opposite to that taken for the first and third square roots.

Hence, $-\frac{2}{3}$ is not a root of the given equation, but of the equation

$$\sqrt{x+3} - \sqrt{4x+1} - \sqrt{10x+4} = 0, \text{ or } -\sqrt{x+3} + \sqrt{4x+1} + \sqrt{10x+4} = 0.$$

$$\begin{aligned}
 59. \quad & \sqrt{6+x} + \sqrt{x} - \sqrt{10-4x} = 0. \\
 & \sqrt{6+x} + \sqrt{x} = \sqrt{10-4x}. \\
 & 6+x + 2\sqrt{6x+x^2+x} = 10-4x. \\
 & \sqrt{6x+x^2} = 2-3x. \\
 & 6x+x^2 = 4-12x+9x^2. \\
 & 4x^2-9x+2=0. \\
 & (x-2)(4x-1)=0. \\
 & \therefore x = 2 \text{ or } \frac{1}{4}.
 \end{aligned}$$

Substituting 2 for x in the given equation,

$$\sqrt{8} + \sqrt{2} - \sqrt{2} = 0,$$

which is not true unless the second square root is taken with a sign of quality opposite to that taken for the first and third square roots.

Hence, 2 is not a root of the given equation, but of the equation

$$\sqrt{6+x} - \sqrt{x} - \sqrt{10-4x} = 0, \text{ or } -\sqrt{6-x} + \sqrt{x} + \sqrt{10-4x} = 0.$$

Substituting $\frac{1}{4}$ for x in the given equation,

$$\begin{aligned}
 \sqrt{\frac{25}{4}} + \sqrt{\frac{1}{4}} - \sqrt{9} &= 0, \\
 0 &= 0.
 \end{aligned}$$

which reduces to

Hence, $\frac{1}{4}$ is a root of the equation.

$$\begin{aligned}
 60. \quad & \sqrt{4x-3} - \sqrt{2x+2} = \sqrt{x-6}. \\
 & 4x-3 - 2\sqrt{8x^2+2x-6} + 2x+2 = x-6. \\
 & 2\sqrt{8x^2+2x-6} = 5x+5. \\
 & 32x^2+8x-24 = 25x^2+50x+25. \\
 & 7x^2-42x-49=0. \\
 & x^2-6x-7=0. \\
 & (x-7)(x+1)=0. \\
 & \therefore x = 7 \text{ or } -1.
 \end{aligned}$$

Substituting 7 for x in the given equation,

$$\sqrt{25} - \sqrt{16} = \sqrt{1},$$

$$1 = 1.$$

which reduces to

Hence, 7 is a root of the equation.

Substituting -1 for x in the given equation,

$$\sqrt{-7} - \sqrt{0} = \sqrt{-7},$$

Hence, -1 is also a root of the equation.

61. See next page.

62.

$$\begin{aligned} \sqrt{3x-5} + \sqrt{x-9} &= \sqrt{4x-4}. \\ 3x-5 + 2\sqrt{3x^2-32x+45} + x-9 &= 4x-4. \\ \sqrt{3x^2-32x+45} &= 5. \\ 3x^2-32x+45 &= 25. \\ 3x^2-32x &= -20. \\ 9x^2-96x+256 &= -60+256=196. \\ 3x-16 &= \pm 14. \\ \therefore x &= 10 \text{ or } \frac{2}{3}. \end{aligned}$$

Substituting 10 for x in the given equation,

$$\sqrt{25} + \sqrt{1} = \sqrt{36},$$

$$6 = 6.$$

which reduces to

Hence, 10 is a root of the equation.

Substituting $\frac{2}{3}$ for x in the given equation,

$$\sqrt{-3} + \sqrt{-\frac{75}{9}} = \sqrt{-\frac{12}{9}}, \text{ or } \sqrt{-3} + \sqrt{\frac{25}{9}} \sqrt{-3} = \sqrt{\frac{4}{9}} \sqrt{-3},$$

which is not true unless the first square root is taken with a sign of quality opposite to that of the second and third square roots.

Hence, $\frac{2}{3}$ is not a root of the given equation, but of the equation

$$-\sqrt{3x-5} + \sqrt{x-9} = \sqrt{4x-4}, \text{ or } \sqrt{3x-5} - \sqrt{x-9} = -\sqrt{4x-4}.$$

63.

$$\begin{aligned} \sqrt{x+a^2} - \sqrt{x-2a^2} &= \sqrt{2x-5a^2}. \\ x+a^2-2\sqrt{x^2-a^2x-2a^4}+x-2a^2 &= 2x-5a^2. \\ \sqrt{x^2-a^2x-2a^4} &= 2a^2. \\ x^2-a^2x-2a^4 &= 4a^4. \\ x^2-a^2x &= 6a^4. \\ 4x^2-4a^2x+a^4 &= 24a^4+a^4=25a^4. \\ 2x-a^2 &= \pm 5a^2. \\ 2x &= 6a^2 \text{ or } -4a^2. \\ \therefore x &= 3a^2 \text{ or } -2a^2. \end{aligned}$$

Substituting $3a^2$ for x in the given equation,

$$\sqrt{4a^2} - \sqrt{a^2} = \sqrt{a^2},$$

$$a = a.$$

which reduces to

Hence, $3a^2$ is a root of the equation.

Substituting $-2a^2$ for x in the given equation,

$$\sqrt{-a^2} - \sqrt{-4a^2} = \sqrt{-9a^2},$$

which is not true unless the first square root is taken with a sign of quality opposite to that of the second and third square roots.

Hence, $-2a^2$ is not a root of the given equation, but of the equation

$$\begin{aligned} \sqrt{x+a^2} - \sqrt{x-2a^2} &= \sqrt{2x-5a^2}, \\ \sqrt{x+a^2} + \sqrt{x-2a^2} &= -\sqrt{2x-5a^2}. \end{aligned}$$

or

61.

$$\begin{aligned}
 \sqrt{2x+3} - \sqrt{x+1} &= \sqrt{5x-14}. \\
 2x+3 - 2\sqrt{2x^2+5x+3} + x+1 &= 5x-14. \\
 -\sqrt{2x^2+5x+3} &= x-9. \\
 + (2x^2+5x+3) &= x^2-18x+81. \\
 x^2+23x-78 &= 0. \\
 (x-3)(x+26) &= 0. \\
 \therefore x &= 3 \text{ or } -26.
 \end{aligned}$$

Substituting 3 for x in the given equation,

$$\begin{aligned}
 \sqrt{9} - \sqrt{4} &= \sqrt{1}, \\
 1 &= 1.
 \end{aligned}$$

which reduces to

Hence, 3 is a root of the equation.

Substituting -26 for x in the given equation,

$$\sqrt{-49} - \sqrt{-25} = \sqrt{-144},$$

which is not true unless the second square root is taken with a sign of quality opposite to that taken for the first and third square roots.

Hence, -26 is not a root of the given equation, but of the equation

$$\sqrt{2x+3} + \sqrt{x+1} = \sqrt{5x-14}, \text{ or } -\sqrt{2x+3} - \sqrt{x+1} = -\sqrt{5x-14}.$$

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4. Let

 x = one part.

Then,

 $20 - x$ = the other part.

Since their product is 96,

$$20x - x^2 = 96.$$

Solving,

$$x = 12 \text{ or } 8,$$

whence,

$$20 - x = 8 \text{ or } 12.$$

Hence, the parts are 12 and 8.

5. Let

 x = one part.

Then,

 $14 - x$ = the other part.

Since their product is 45,

$$14x - x^2 = 45.$$

Solving,

$$x = 9 \text{ or } 5,$$

whence,

$$14 - x = 5 \text{ or } 9.$$

Hence, the parts are 9 and 5.

6. Let

 x = number of sheep.

Then,

$$\frac{75}{x} = \text{number of dollars paid for each,}$$

and

$$\frac{75}{x+3} = \text{number of dollars for each had there been three more sheep;}$$

$$\therefore \frac{75}{x+3} = \frac{75}{x} - \frac{5}{4}.$$

Solving,

$$x = 12 \text{ or } -15.$$

Hence, the man purchased 12 sheep.

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7. Let

 x = number of rods in the length.

Then,

 $x - 12$ = number of rods in the breadth;

$$\therefore x(x-12) = 160.$$

Solving,

$$x = 20 \text{ or } -8,$$

whence,

$$x - 12 = 8 \text{ or } -20.$$

Hence, the field is 20 rods long and 8 rods wide.

8. Let x = number of rods in each side.

Then, $(x + 4)^2 = x^2 + \left(\frac{x^2}{8} + 136\right)$.

Solving, $x = 40$ or 24 .

Hence, the field is either 40 rods square or 24 rods square.

9. Let x = number of rods in the length.

Then, $x - 8$ = number of rods in the width;

$$\therefore x(x - 8) = 240.$$

Solving, $x = 20$ or -12 ,

whence, $x - 8 = 12$ or -20 .

Hence, the lot is 20 rods long and 12 rods wide.

10. Let x = number of men in each rank.

Then, $9x + 3$ = number of men in each file;

$$\therefore x(9x + 3) = 600.$$

Solving, $x = 8$ or $-\frac{25}{3}$,

whence, $9x + 3 = 75$ or -72 .

Hence, the column was composed of men 8 abreast and 75 deep.

11. Let x = number of persons in the party.

Then, $\frac{60}{x}$ = number of dollars each paid;

$$\therefore \frac{60}{x + 5} = \frac{60}{x} - 1.$$

Solving, $x = 15$ or -20 .

Hence, there were 15 persons in the party.

12. Let x = number of days he worked.

Then, $\frac{30}{x}$ = number of dollars earned per day;

$$\therefore 30 = \left(\frac{30}{x} - 1\right)(x + 5).$$

Solving, $x = 10$ or -15 .

Hence, he worked 10 days.

13. Let x = smaller number.

Then, $x + 1$ = larger number;

$$\therefore x^2 + (x + 1)^2 = 61.$$

Solving, $x = 5$ or -6 ,

whence, $x + 1 = 6$ or -5 .

Hence, the numbers are 5 and 6, or -6 and -5 .

14. Let x = smaller number.

Then, $x + 1$ = larger number;

$$\therefore \frac{1}{x} + \frac{1}{x + 1} = \frac{9}{20}.$$

Solving, $x = 4$ or $-\frac{5}{9}$,

whence, $x + 1 = 5$ or $\frac{4}{9}$.

Hence, the numbers are 4 and 5.

15. Let x = number of inches in width of frame.
 Then, $(8 + 2x)(12 + 2x)$ = number of square inches in area of frame and picture;

$$\therefore (8 + 2x)(12 + 2x) = 2(8 \times 12).$$

Solving, $x = 2$ or -12 .

Hence, the frame was 2 inches wide.

16. Let x = number of barrels he bought.

Then, $\frac{100}{x}$ = number of dollars he paid for each,

and $x - 5$ = number of barrels he sold;

$$\therefore \left(\frac{100}{x} + 1 \right) (x - 5) = 100.$$

Solving, $x = 25$ or -20 ,

whence, $\frac{100}{x} = 4$ or -5 .

Hence, he bought 25 barrels at \$4 a barrel.

17. Let x = his per cent of gain and also the number of dollars the coat cost him.

Then, $11 = x + x \frac{x}{100}$.

Solving, $x = 10$ or -110 .

Hence, he gained 10 per cent.

18. Let x = number of miles per hour it traveled.

Then, $\frac{280}{x} = \frac{280}{x + 5} + 1$.

Solving, $x = 35$ or -40 .

Hence, the train traveled 35 miles an hour.

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19. Let x = number of rods in width of street.

Then, $(56 + 2x)(16 + 2x)$ = number of square rods in area of park and street;

$$\therefore (56 + 2x)(16 + 2x) = 56 \times 16 + 160 \times 4.$$

Solving, $x = 4$ or -40 .

Hence, the width of the street was 4 rods.

20. Let x = number of miles per hour he could row in still water.

Then, $x - 2$ = number of miles per hour upstream,

and $x + 2$ = number of miles per hour downstream;

whence, $\frac{8}{x - 2}$ = number of hours going upstream,

and $\frac{8}{x + 2}$ = number of hours returning;

$$\therefore \frac{8}{x - 2} + \frac{8}{x + 2} = 3$$

Solving, $x = 6$ or $-\frac{2}{3}$.

Hence, his rate of rowing in still water was 6 miles per hour.

21. Let x = number of rods in width of road.

Then, $(56 - 2x)(28 - 2x)$ = number of square rods in area of lot after road was made;

$$\therefore (56 - 2x)(28 - 2x) = 56 \times 28 - 160 \times 2.$$

Solving, $x = 2$ or 40 .

Since the lot was less than 40 rods wide, the value $x = 40$ must be rejected. Hence, the road was 2 rods wide.

22. Let x = number of yards in smaller lot.

Then, $20 - x$ = number of yards in larger lot;

$$\therefore x(20 - x) + (20 - x)x = 192.$$

Solving, $x = 8$ or 12 ,

whence, $20 - x = 12$ or 8 .

Hence, there were 8 yards in one lot and 12 in the other.

23. Let x = number of cents asked for 1 dozen eggs.

Then, $\frac{30}{x}$ = number of dozen eggs to be had for 30 cents,

whence, $\frac{360}{x}$ = number of eggs to be had for 30 cents.

Under the second supposition, $\frac{360}{x} - 2$, or $\frac{360 - 2x}{x}$, eggs can be had for 30 cents, and this raises the price 2 cents per dozen, or $\frac{1}{6}$ of a cent per egg. Hence, the price of 1 egg is

$$30 \div \frac{360 - 2x}{x}, \text{ or } \frac{15x}{180 - x} = \frac{x}{12} + \frac{1}{6}.$$

Solving, $x = 18$ or -20 .

Hence, the price of eggs is 18 cents a dozen.

24. Let x = number of miles an hour A traveled.

Then, $x - 1$ = number of miles an hour B traveled,

$$\frac{75}{x} = \text{number of hours it took A,}$$

and $\frac{75}{x - 1} = \text{number of hours it took B ;}$

$$\therefore \frac{75}{x} = \frac{75}{x - 1} - \frac{5}{2}.$$

Solving, $x = 6$ or -5 ,

whence, $x - 1 = 5$ or -6 .

Hence, A traveled 6 miles an hour, and B traveled 5 miles an hour.

25. Let x = number of gallons of liquid drawn each time.

Then, x = number of gallons of wine drawn the first time,

$81 - x$ = number of gallons of wine left,

and $\frac{81 - x}{81}$ of x , or $\frac{(81 - x)x}{81}$ = number of gallons of wine drawn the second time ;

$$\therefore 81 - x - \frac{(81 - x)x}{81} = 64.$$

Solving, $x = 153$ or 9 .

Since the cask held less than 153 gallons, the value $x = 153$ must be rejected. Hence, 9 gallons were drawn each time.

26. Let x = number of feet in circumference of fore wheel.

Then, $x + 5$ = number of feet in circumference of hind wheel.

$$\therefore \frac{5280}{x} = \frac{5280}{x+5} + 150.$$

Solving $x = 11$ or -16 .

whence, $x + 5 = 16$ or -11 .

Hence, the fore wheel is 11 feet in circumference and the hind wheel is 16 feet in circumference.

27. Let x = number of hours it will take the larger.

Then, $x + 2$ = number of hours it will take the smaller,

$$\frac{1}{x} = \text{part the larger can fill in 1 hour,}$$

and $\frac{1}{x+2}$ = part the smaller can fill in 1 hour;

$$\therefore \frac{1}{x} + \frac{1}{x+2} = \frac{1}{2\frac{2}{3}} = \frac{5}{12}.$$

Solving, $x = 4$ or $-\frac{8}{5}$.

whence, $x + 2 = 6$ or $\frac{2}{5}$.

Hence, the larger pipe can fill the cistern in 4 hours, and the smaller pipe in 6 hours.

28. Let x = number of men,

and let a represent the amount of work one man does in one day.

Then, since the ditch can be dug by x men in x days, or by $x + 6$ men in 8 days,

$$ax^2 = a \times 8(x + 6).$$

Solving, $x = 12$ or -4 .

Hence, there were 12 men.

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7. See next page.

8. See next page.

9. $x^4 - 18x^2 + 32 = 0.$

$$(x^2 - 16)(x^2 - 2) = 0.$$

$$(x-4)(x+4)(x-\sqrt{2})(x+\sqrt{2}) = 0.$$

$$\therefore x = 4 \text{ or } -4 \text{ or } +\sqrt{2} \text{ or } -\sqrt{2}.$$

$$x = \pm 4 \text{ or } \pm \sqrt{2}.$$

10. $3x^4 + 5x^2 - 8 = 0.$

$$(x^2 - 1)(3x^2 + 8) = 0.$$

$$(x-1)(x+1)(3x^2 + 8) = 0.$$

Since $(x-1)(x+1) = 0$, $x = \pm 1$.

Also, $3x^2 + 8 = 0.$

$$x^2 = -\frac{8}{3} = -2\frac{2}{3}.$$

$$\therefore x = \pm \frac{2}{3}\sqrt{-6}.$$

Hence, $x = \pm 1 \text{ or } \pm \frac{2}{3}\sqrt{-6}.$

11. $5x^4 + 6x^2 - 11 = 0.$

$$(x^2 - 1)(5x^2 + 11) = 0.$$

$$(x-1)(x+1)(5x^2 + 11) = 0.$$

Since $(x-1)(x+1) = 0$, $x = \pm 1$.

Also, $5x^2 + 11 = 0.$

$$x^2 = -\frac{11}{5} = -2\frac{1}{5}.$$

$$\therefore x = \pm \frac{1}{5}\sqrt{-55}.$$

Hence, $x = \pm 1 \text{ or } \pm \frac{1}{5}\sqrt{-55}.$

12. $2x^4 - 8x^2 - 90 = 0.$

$$x^4 - 4x^2 - 45 = 0.$$

$$(x^2 - 9)(x^2 + 5) = 0.$$

$$(x-3)(x+3)(x^2 + 5) = 0.$$

Since $(x-3)(x+3) = 0$, $x = \pm 3$.

Also, $x^2 + 5 = 0.$

$$x^2 = -5.$$

$$\therefore x = \pm \sqrt{-5}.$$

Hence, $x = \pm 3 \text{ or } \pm \sqrt{-5}.$

$$7. \quad x^4 - 13x^2 + 36 = 0.$$

$$(x^2 - 4)(x^2 - 9) = 0.$$

$$(x - 2)(x + 2)(x - 3)(x + 3) = 0.$$

$$\therefore x = 2 \text{ or } -2 \text{ or } 3 \text{ or } -3;$$

$$\text{that is, } x = \pm 2 \text{ or } \pm 3.$$

$$13. \quad x^4 - 5x^2 + 6 = 0.$$

$$\text{Let } x^2 = p, \text{ then, } x^2 = p^2,$$

$$\text{and } p^2 - 5p + 6 = 0.$$

$$(p - 2)(p - 3) = 0.$$

$$\therefore p = 2 \text{ or } 3;$$

$$\text{that is, } x^2 = 2 \text{ or } 3.$$

$$\therefore x = 16 \text{ or } 81.$$

$$8. \quad x^4 - 25x^2 + 144 = 0.$$

$$(x^2 - 9)(x^2 - 16) = 0.$$

$$(x - 3)(x + 3)(x - 4)(x + 4) = 0.$$

$$\therefore x = 3 \text{ or } -3 \text{ or } 4 \text{ or } -4;$$

$$\text{that is, } x = \pm 3 \text{ or } \pm 4.$$

$$14. \quad x^4 + 3x^2 - 28 = 0.$$

$$\text{Let } x^2 = p, \text{ then, } x^2 = p^2,$$

$$\text{and } p^2 + 3p - 28 = 0.$$

$$(p - 4)(p + 7) = 0.$$

$$\therefore p = 4 \text{ or } -7;$$

$$\text{that is, } x^2 = 4 \text{ or } -7.$$

$$\therefore x = 256 \text{ or } 2401.$$

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$$15. \quad x^4 - 3x^2 = -2.$$

$$\text{Let } x^2 = p, \text{ then, } x^2 = p^2,$$

$$\text{and } p^2 - 3p + 2 = 0.$$

$$(p - 1)(p - 2) = 0.$$

$$\therefore p = 1 \text{ or } 2;$$

$$\text{that is, } x^2 = 1 \text{ or } 2.$$

$$\therefore x = 1 \text{ or } 64.$$

$$16. \quad x^4 - x^2 = 6.$$

$$x^4 - x^2 - 6 = 0.$$

$$(x^2 - 3)(x^2 + 2) = 0.$$

$$\therefore x^2 = 3 \text{ or } -2.$$

$$x = 27 \text{ or } -8.$$

$$17. \quad x + 2\sqrt{x} = 3.$$

$$x + 2\sqrt{x} + 1 = 4.$$

$$\sqrt{x} + 1 = \pm 2.$$

$$\therefore \sqrt{x} = 1 \text{ or } -3.$$

$$x = 1 \text{ or } 9.$$

$$18. \quad x^3 - 2x^2 = 3.$$

$$x^3 - 2x^2 + 1 = 4.$$

$$x^3 - 1 = \pm 2.$$

$$\therefore x^3 = 3 \text{ or } -1.$$

$$x = 27 \text{ or } -1.$$

$$19.$$

$$\text{Let } x^{\frac{2}{3}} = p, \text{ then } x^3 = p^2, \text{ and}$$

$$\text{Factoring,}$$

$$x^3 + 8x^{\frac{2}{3}} - 9 = 0. \quad (1)$$

$$p^2 + 8p - 9 = 0. \quad (2)$$

$$(p - 1)(p + 9) = 0.$$

$$\therefore p = 1 \text{ or } -9; \quad (3)$$

$$x^{\frac{2}{3}} = 1 \text{ or } -9. \quad (4)$$

$$x^3 = 1 \text{ or } 81. \quad (5)$$

$$x = 1 \text{ or } 3\sqrt[3]{3}. \quad (6)$$

that is,

Squaring each member,

Taking the cube root of each member,

NOTE. — As in example 5, other roots may be obtained, if desired, by factoring, instead of taking like roots of each member, to find x .

Thus, from (5),

$$x^3 - 1 = 0, \quad (7)$$

and

$$x^3 - 81 = 0. \quad (8)$$

Factoring (7),

$$(x - 1)(x^2 + x + 1) = 0. \quad (9)$$

Factoring (8) in a similar manner, since $\sqrt[3]{81} = 3\sqrt[3]{3}$,

$$(x - 3\sqrt[3]{3})(x^2 + 3\sqrt[3]{3}x + 9\sqrt[3]{9}) = 0. \quad (10)$$

The first factors of (9) and (10) equated to zero give the values of x found in (6). The other two factors equated to zero and solved as quadratics give the remaining roots,

$$x = \frac{1}{2}(-1 \pm \sqrt{-3}) \text{ and } \frac{3}{2}(-\sqrt[3]{3} \pm \sqrt[3]{-243}).$$

$$20. \quad x^3 - x^2 - 2 = 0.$$

$$(x^2 + 1)(x^2 - 2) = 0.$$

$$\therefore x^2 = -1 \text{ or } 2.$$

$$x^2 = 1 \text{ or } 16.$$

$$\therefore x = 1 \text{ or } 2\sqrt{2}.$$

For other roots see note, Ex. 19.

$$21. \quad \sqrt[4]{x} + 3\sqrt{x} = 30.$$

Let $\sqrt[4]{x} = p$, then, $\sqrt{x} = p^2$,
and $3p^2 + p - 30 = 0.$

$$(p - 3)(3p + 10) = 0.$$

$$\therefore p = 3 \text{ or } -\frac{10}{3};$$

that is, $\sqrt[4]{x} = 3 \text{ or } -\frac{10}{3}.$

$$\therefore x = 81 \text{ or } \frac{10000}{81}.$$

$$22. \quad ax^{2n} + bx^n + c = 0.$$

$$4a^2x^{2n} + 4abx^n + b^2 = b^2 - 4ac.$$

$$2ax^n + b = \pm \sqrt{b^2 - 4ac}.$$

$$x^n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore x = \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)^{\frac{1}{n}}.$$

$$23. \quad x^3 - 4x - 5x^{\frac{1}{2}} = 0.$$

$$x^{\frac{1}{2}}(x^{\frac{5}{2}} + 1)(x^{\frac{1}{2}} - 5) = 0.$$

$$\therefore x^{\frac{1}{2}} = 0 \text{ or } -1 \text{ or } 5.$$

$$x = 0 \text{ or } 1 \text{ or } 25.$$

$$24. \quad x^{\frac{7}{2}} - x^2 - 12x^{\frac{1}{2}} = 0.$$

$$x^{\frac{1}{2}}(x^{\frac{5}{2}} - 4)(x^{\frac{3}{2}} + 3) = 0.$$

$$\therefore x^{\frac{1}{2}} = 0 \text{ or } x^{\frac{5}{2}} = 4 \text{ or } -3.$$

$$x = 0 \text{ or } x^3 = 16 \text{ or } 9.$$

$$\therefore x = 0 \text{ or } 2\sqrt[3]{2} \text{ or } \sqrt[3]{9}.$$

For other roots see note, Ex. 19.

$$25. \quad x - 3x^{\frac{3}{2}} + 2x^{\frac{1}{2}} = 0.$$

$$x^{\frac{1}{2}}(x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} - 2) = 0.$$

$$\therefore x^{\frac{1}{2}} = 0 \text{ or } x^{\frac{1}{2}} = 1 \text{ or } 2.$$

$$x = 0 \text{ or } 1 \text{ or } 16.$$

$$26. \quad 5x = x\sqrt{x} + 6\sqrt{x}.$$

$$x\sqrt{x} - 5x + 6\sqrt{x} = 0.$$

$$\sqrt{x}(\sqrt{x} - 2)(\sqrt{x} - 3) = 0.$$

$$\therefore \sqrt{x} = 0 \text{ or } 2 \text{ or } 3.$$

$$x = 0 \text{ or } 4 \text{ or } 9.$$

$$27. \quad 3x = x\sqrt[3]{x} + 2\sqrt[3]{x^2}.$$

$$x\sqrt[3]{x} - 3x + 2\sqrt[3]{x^2} = 0.$$

$$x^{\frac{4}{3}} - 3x + 2x^{\frac{2}{3}} = 0.$$

$$x^{\frac{2}{3}}(x^{\frac{2}{3}} - 1)(x^{\frac{1}{3}} - 2) = 0.$$

$$\therefore x^{\frac{2}{3}} = 0 \text{ or } x^{\frac{1}{3}} = 1 \text{ or } 2.$$

$$x = 0 \text{ or } 1 \text{ or } 8.$$

$$28. \quad x^{-\frac{1}{2}} - 3 - 4x^{\frac{1}{2}} = 0.$$

Multiplying by $x^{\frac{1}{2}}$,

$$1 - 3x^{\frac{1}{2}} - 4x = 0.$$

$$(1 + x^{\frac{1}{2}})(1 - 4x^{\frac{1}{2}}) = 0.$$

$$\therefore x^{\frac{1}{2}} = -1 \text{ or } \frac{1}{4}.$$

$$x = 1 \text{ or } \frac{1}{16}.$$

$$29. \quad x^{-\frac{1}{2}} - 6x^{\frac{1}{2}} = 1.$$

Multiplying by $x^{\frac{1}{2}}$,

$$1 - 6x = x^{\frac{1}{2}}.$$

$$1 - x^{\frac{1}{2}} - 6x = 0.$$

$$(1 - 3x^{\frac{1}{2}})(1 + 2x^{\frac{1}{2}}) = 0.$$

$$\therefore x^{\frac{1}{2}} = \frac{1}{3} \text{ or } \frac{1}{2}.$$

$$x = \frac{1}{9} \text{ or } \frac{1}{4}.$$

$$30. \quad x^{-3} + x^2 = 2x^{-\frac{1}{2}}.$$

Multiplying by x^3 ,

$$1 + x^5 = 2x^{\frac{5}{2}}.$$

$$1 - 2x^{\frac{5}{2}} + x^5 = 0.$$

$$(1 - x^{\frac{5}{2}})(1 - x^{\frac{5}{2}}) = 0$$

$$\therefore x^{\frac{5}{2}} = 1.$$

$$x^5 = 1.$$

$$\therefore x = 1.$$

1 occurs twice as a root of the equation. Other roots may be obtained by factoring $x^5 - 1$.

$$31. \quad x + 2x^{\frac{5}{2}} = 3x^{\frac{3}{2}}.$$

$$x - 3x^{\frac{3}{2}} + 2x^{\frac{5}{2}} = 0.$$

$$x(1 - x^{\frac{1}{2}})(1 - 2x^{\frac{1}{2}}) = 0.$$

$$\therefore x = 0 \text{ or } x^{\frac{1}{2}} = 1 \text{ or } \frac{1}{2}.$$

$$x = 0 \text{ or } 1 \text{ or } \frac{1}{4}.$$

$$32. \quad 2x + \sqrt{x} = 15x\sqrt{x}.$$

$$\sqrt{x} + 2x - 15x\sqrt{x} = 0.$$

$$\sqrt{x}(1 - 3\sqrt{x})(1 + 5\sqrt{x}) = 0.$$

$$\therefore \sqrt{x} = 0 \text{ or } \frac{1}{3} \text{ or } -\frac{1}{5}.$$

$$x = 0 \text{ or } \frac{1}{9} \text{ or } \frac{1}{25}.$$

$$33. \quad \sqrt{x} + 5 + 6x^{-\frac{1}{2}} = 0.$$

Multiplying by $x^{\frac{1}{2}}$,

$$x + 5x^{\frac{1}{2}} + 6 = 0.$$

$$(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} + 3) = 0.$$

$$\therefore x^{\frac{1}{2}} = -2 \text{ or } -3.$$

$$x = 4 \text{ or } 9.$$

$$34. \quad x^4 = 8x + 7x^2\sqrt{x}.$$

$$x^4 - 7x^{\frac{5}{2}} - 8x = 0.$$

$$x(x^{\frac{3}{2}} + 1)(x^{\frac{3}{2}} - 8) = 0.$$

$$\therefore x = 0 \text{ or } x^{\frac{3}{2}} = -1 \text{ or } 8.$$

$$x = 0 \text{ or } x^3 = 1 \text{ or } 64.$$

$$\therefore x = 0 \text{ or } 1 \text{ or } 4.$$

Other roots may be found as in Ex. 19.

$$35. \quad (x - 3)^2 + 2(x - 3) = 3.$$

Put p for $x - 3$ and p^2 for $(x - 3)^2$.

$$\text{Then, transposing } 3, \quad p^2 + 2p - 3 = 0.$$

$$\text{Factoring,} \quad (p - 1)(p + 3) = 0.$$

$$\therefore p = 1 \text{ or } -3,$$

$$x - 3 = 1 \text{ or } -3.$$

$$\therefore x = 4 \text{ or } 0.$$

that is,

$$36. \quad (x^2 + 1)^2 + 4(x^2 + 1) = 45.$$

Put p for $x^2 + 1$ and p^2 for $(x^2 + 1)^2$.

$$\text{Then, transposing } 45, \quad p^2 + 4p - 45 = 0.$$

$$\text{Factoring,} \quad (p - 5)(p + 9) = 0.$$

$$\therefore p = 5 \text{ or } -9,$$

$$x^2 + 1 = 5 \text{ or } -9.$$

$$x^2 = 4 \text{ or } -10.$$

$$\therefore x = \pm 2 \text{ or } \pm \sqrt{-10}$$

that is,

$$37. \quad (x^2 - 4)^2 - 3(x^2 - 4) = 10.$$

Put p for $x^2 - 4$ and p^2 for $(x^2 - 4)^2$.

$$\text{Then, transposing } 10, \quad p^2 - 3p - 10 = 0.$$

$$\text{Factoring,} \quad (p - 5)(p + 2) = 0.$$

$$\therefore p = 5 \text{ or } -2,$$

$$x^2 - 4 = 5 \text{ or } -2.$$

$$x^2 = 9 \text{ or } 2.$$

$$\therefore x = \pm 3 \text{ or } \pm \sqrt{2}.$$

that is,

$$38. \quad (x^2 - 2x)^2 - 2(x^2 - 2x) = 3.$$

Put p for $x^2 - 2x$ and p^2 for $(x^2 - 2x)^2$.

$$\text{Then,} \quad p^2 - 2p = 3.$$

$$\text{Completing the square,} \quad p^2 - 2p + 1 = 4.$$

$$p - 1 = \pm 2.$$

$$\therefore p = 3 \text{ or } -1,$$

$$x^2 - 2x = 3 \text{ or } -1.$$

$$x^2 - 2x + 1 = 4 \text{ or } 0.$$

$$x - 1 = \pm 2 \text{ or } \pm 0.$$

$$\therefore x = 3 \text{ or } -1 \text{ or } 1$$

that is,

Completing the square,

The roots of the given equation are 3, -1, 1, 1.

$$39. \quad (x^2 - x)^2 - (x^2 - x) - 132 = 0.$$

$$\text{Factoring,} \quad (x^2 - x - 12)(x^2 - x + 11) = 0.$$

$$(x - 4)(x + 3)(x^2 - x + 11) = 0.$$

$$\therefore x = 4 \text{ or } -3 \text{ or } x^2 - x + 11 = 0.$$

Solving the last equation,

$$x = \frac{1}{2}(1 \pm \sqrt{-43}).$$

Hence,

$$x = 4 \text{ or } -3 \text{ or } \frac{1}{2}(1 \pm \sqrt{-43}).$$

40. $x - 5 + 2\sqrt{x - 5} = 8.$

Put p for $\sqrt{x - 5}$ and p^2 for $x - 5$.

Then, $p^2 + 2p = 8.$

Completing the square, $p^2 + 2p + 1 = 9.$

$$p + 1 = \pm 3.$$

$$\therefore p = \sqrt{x - 5} = 2 \text{ or } -4.$$

$$x - 5 = 4 \text{ or } 16.$$

$$\therefore x = 9 \text{ or } 21.$$

41. $x^2 - 3x + 6 + 2\sqrt{x^2 - 3x + 6} = 24.$

Put p for $\sqrt{x^2 - 3x + 6}$ and p^2 for $x^2 - 3x + 6$.

Then, $p^2 + 2p = 24.$

Completing the square, $p^2 + 2p + 1 = 25.$

$$p + 1 = \pm 5.$$

$$\therefore p = \sqrt{x^2 - 3x + 6} = 4 \text{ or } -6.$$

$$x^2 - 3x + 6 = 16 \text{ or } 36.$$

Solving these two equations,

$$x = 5 \text{ or } -2 \text{ or } \frac{3}{2} \pm \frac{1}{2}\sqrt{129}.$$

42. $x^2 - 5x + 2\sqrt{x^2 - 5x - 2} = 10.$

Adding -2 , $x^2 - 5x - 2 + 2\sqrt{x^2 - 5x - 2} = 8.$

Put p for $\sqrt{x^2 - 5x - 2}$ and p^2 for $x^2 - 5x - 2$.

Then, $p^2 + 2p = 8.$

Completing the square, $p^2 + 2p + 1 = 9.$

$$p + 1 = \pm 3.$$

$$\therefore p = \sqrt{x^2 - 5x - 2} = 2 \text{ or } -4.$$

$$x^2 - 5x - 2 = 4 \text{ or } 16.$$

Solving these two equations,

$$x = 6 \text{ or } -1 \text{ or } \frac{5}{2} \pm \frac{1}{2}\sqrt{97}.$$

43. $x^2 - x - \sqrt{x^2 - x + 4} - 8 = 0.$

Separating -8 into $+4 - 12$,

$$x^2 - x + 4 - \sqrt{x^2 - x + 4} - 12 = 0.$$

Put p for $\sqrt{x^2 - x + 4}$ and p^2 for $x^2 - x + 4$.

Then, $p^2 - p - 12 = 0.$

Factoring, $(p - 4)(p + 3) = 0.$

$$\therefore p = \sqrt{x^2 - x + 4} = 4 \text{ or } -3.$$

$$x^2 - x + 4 = 16 \text{ or } 9.$$

Solving these two equations,

$$x = 4 \text{ or } -3 \text{ or } \frac{1}{2}(1 \pm \sqrt{21}).$$

44. $x^2 - 5x + 5\sqrt{x^2 - 5x + 1} = 49.$

Adding 1, $x^2 - 5x + 1 + 5\sqrt{x^2 - 5x + 1} = 50.$

Put p for $\sqrt{x^2 - 5x + 1}$ and p^2 for $x^2 - 5x + 1$.

Then, transposing 50, $p^2 + 5p - 50 = 0.$

Factoring, $(p - 5)(p + 10) = 0.$

$$\therefore p = \sqrt{x^2 - 5x + 1} = 5 \text{ or } -10.$$

$$x^2 - 5x + 1 = 25 \text{ or } 100.$$

Solving these two equations,

$$x = 8 \text{ or } -3 \text{ or } \frac{5}{2} \pm \frac{1}{2}\sqrt{421}.$$

45. $x + 10 = 2\sqrt{x + 10} + 5.$

Transposing, $x + 10 - 2\sqrt{x + 10} = 5.$

Put p for $\sqrt{x + 10}$ and p^2 for $x + 10.$

Then, $p^2 - 2p = 5.$

Completing the square, $p^2 - 2p + 1 = 6.$

$$p - 1 = \pm \sqrt{6}.$$

$$\therefore p = \sqrt{x + 10} = 1 \pm \sqrt{6}.$$

$$x + 10 = 7 \pm 2\sqrt{6}.$$

$$\therefore x = -3 \pm 2\sqrt{6}.$$

Squaring,

46. $x - 3 = 21 - 4\sqrt{x - 3}.$

Transposing, $x - 3 + 4\sqrt{x - 3} - 21 = 0.$

Put p for $\sqrt{x - 3}$ and p^2 for $x - 3.$

Then, $p^2 + 4p - 21 = 0.$

Factoring, $(p - 3)(p + 7) = 0.$

$$\therefore p = \sqrt{x - 3} = 3 \text{ or } -7.$$

$$x - 3 = 9 \text{ or } 49.$$

$$\therefore x = 12 \text{ or } 52.$$

47. $2x - 6\sqrt{2x - 1} = 8.$

Adding $-1,$ $2x - 1 - 6\sqrt{2x - 1} = 7.$

Put p for $\sqrt{2x - 1}$ and p^2 for $2x - 1.$

Then, transposing 7, $p^2 - 6p - 7 = 0.$

Factoring, $(p - 7)(p + 1) = 0.$

$$\therefore p = \sqrt{2x - 1} = 7 \text{ or } -1.$$

$$2x - 1 = 49 \text{ or } 1.$$

$$\therefore x = 25 \text{ or } 1.$$

48. $x = 11 - 3\sqrt{x + 7}.$

Adding 7 and transposing,

$$x + 7 + 3\sqrt{x + 7} - 18 = 0.$$

Put p for $\sqrt{x + 7}$ and p^2 for $x + 7.$

Then, $p^2 + 3p - 18 = 0.$

Factoring, $(p - 3)(p + 6) = 0.$

$$\therefore p = \sqrt{x + 7} = 3 \text{ or } -6.$$

$$x + 7 = 9 \text{ or } 36.$$

$$\therefore x = 2 \text{ or } 29.$$

49. $x + 2\sqrt{x + 3} = 21.$

Adding 3, $x + 3 + 2\sqrt{x + 3} = 24.$

Put p for $\sqrt{x + 3}$ and p^2 for $x + 3.$

Then, $p^2 + 2p = 24.$

Completing the square, $p^2 + 2p + 1 = 25.$

$$p + 1 = \pm 5.$$

$$\therefore p = \sqrt{x + 3} = 4 \text{ or } -6.$$

$$x + 3 = 16 \text{ or } 36.$$

$$\therefore x = 13 \text{ or } 33.$$

50. $2x - 3\sqrt{2x + 5} = -5.$

Adding 5, $2x + 5 - 3\sqrt{2x + 5} = 0.$

Put p for $\sqrt{2x + 5}$ and p^2 for $2x + 5.$

Then, $p^2 - 3p = 0.$

Factoring, $p(p - 3) = 0.$

$\therefore p = 0$ or $3,$

that is,

$\sqrt{2x + 5} = 0$ or $3.$

$2x + 5 = 0$ or $9.$

Solving these equations,

$x = -\frac{5}{2}$ or $2.$

51. $x^3 + x\sqrt{x} - 72 = 0.$

Put p for $x\sqrt{x}$, or $x^{\frac{3}{2}}$, and p^2 for $x^3.$

Then, $p^2 + p - 72 = 0.$

Factoring, $(p - 8)(p + 9) = 0.$

$\therefore p = 8$ or $-9,$

that is,

$x^{\frac{3}{2}} = 8$ or $-9.$

Squaring,

$x^3 = 64$ or $81.$

Taking the cube root,

$x = 4$ or $3\sqrt[3]{3}.$

Other roots may be obtained as in Ex. 19.

52. $x^{-\frac{2}{3}} - 5x^{-\frac{1}{3}} + 4 = 0.$

Multiplying by $x^{\frac{2}{3}},$ $1 - 5x^{\frac{1}{3}} + 4x^{\frac{2}{3}} = 0.$

Put p for $x^{\frac{1}{3}}$ and p^2 for $x^{\frac{2}{3}}.$

Then, $1 - 5p + 4p^2 = 0.$

Factoring, $(1 - p)(1 - 4p) = 0.$

$\therefore p = 1$ or $\frac{1}{4},$

that is,

$x^{\frac{1}{3}} = 1$ or $\frac{1}{4}.$

$\therefore x = 1$ or $\frac{1}{64}.$

53. See next page.

54. $\left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) = \frac{5}{4}.$ (1)

Put p for $x + \frac{1}{x}$ and p^2 for $\left(x + \frac{1}{x}\right)^2.$

Then,

$p^2 - 2p = \frac{5}{4}.$ (2)

Completing the square,

$p^2 - 2p + 1 = \frac{9}{4}.$

$p - 1 = \pm \frac{3}{2}.$

$\therefore p = \frac{5}{2}$ or $-\frac{1}{2},$ (3)

that is,

$x + \frac{1}{x} = \frac{5}{2}$ or $-\frac{1}{2}.$ (4)

Clearing equations (4) of fractions and transposing,

$2x^2 - 5x + 2 = 0$ (5)

or

$2x^2 + x + 2 = 0.$ (6)

Factoring (5),

$(x - 2)(2x - 1) = 0.$

$\therefore x = 2$ or $\frac{1}{2}.$

Solving (6) by the formula, § 295,

$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 2}}{2 \cdot 2}$

$= \frac{1}{4}(-1 \pm \sqrt{-15}).$

Hence,

$x = 2$ or $\frac{1}{2}$ or $\frac{1}{4}(-1 \pm \sqrt{-15}).$

$$53. \quad \left(\frac{12}{x} - 1\right)^2 + 8\left(\frac{12}{x} - 1\right) = 33.$$

Put p for $\frac{12}{x} - 1$ and p^2 for $\left(\frac{12}{x} - 1\right)^2$.

Then, transposing 33,

$$p^2 + 8p - 33 = 0.$$

Factoring,

$$(p - 3)(p + 11) = 0.$$

$$\therefore p = 3 \text{ or } -11.$$

that is,

$$\frac{12}{x} - 1 = 3 \text{ or } -11.$$

$$\frac{12}{x} = 4 \text{ or } -10.$$

$$\therefore x = 3 \text{ or } -\frac{6}{5}.$$

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$$55. \quad \left(\frac{1+x^2}{x}\right)^2 + 2\left(\frac{1+x^2}{x}\right) = 8. \quad (1)$$

Put p for $\frac{1+x^2}{x}$ and p^2 for $\left(\frac{1+x^2}{x}\right)^2$.

Then,

$$p^2 + 2p = 8. \quad (2)$$

Completing the square,

$$p^2 + 2p + 1 = 9.$$

$$p + 1 = \pm 3.$$

$$\therefore p = 2 \text{ or } -4, \quad (3)$$

that is,

$$\frac{1+x^2}{x} = 2 \text{ or } -4. \quad (4)$$

Clearing equations (4) of fractions and transposing,

$$x^2 - 2x + 1 = 0 \quad (5)$$

$$\text{or} \quad x^2 + 4x + 1 = 0. \quad (6)$$

$$\text{Factoring (5),} \quad (x-1)(x-1) = 0.$$

$$\therefore x = 1 \text{ or } 1.$$

Solving (6),

$$x = -2 \pm \sqrt{3}.$$

Hence, the roots of (1) are 1, 1, and $-2 \pm \sqrt{3}$.

56. See next page.

57. See next page.

$$58. \quad (x-a)^{\frac{3}{2}} - 3a^{\frac{1}{2}}(x-a)^{\frac{1}{2}} + 2a^{\frac{3}{2}} = 0.$$

Put p for $(x-a)^{\frac{1}{2}}$ and p^2 for $(x-a)^{\frac{1}{2}}$.

Then,

$$p^2 - 3a^{\frac{1}{2}}p + 2a^{\frac{3}{2}} = 0.$$

Factoring,

$$(p - a^{\frac{1}{2}})(p - 2a^{\frac{1}{2}}) = 0.$$

$$\therefore p = (x-a)^{\frac{1}{2}} = a^{\frac{1}{2}} \text{ or } 2a^{\frac{1}{2}}.$$

Cubing each member,

$$x-a = a \text{ or } 8a.$$

$$\therefore x = 2a \text{ or } 9a.$$

59. Let

Transposing,

$$x^3 = -1.$$

Factoring,

$$x^3 + 1 = 0.$$

$$(x+1)(x^2-x+1) = 0.$$

Equating each factor to zero and solving,

$$x = -1 \text{ or } \frac{1}{2}(1 \pm \sqrt{-3}).$$

Hence, the three cube roots of -1 are -1 , $\frac{1}{2}(1 + \sqrt{-3})$, and $\frac{1}{2}(1 - \sqrt{-3})$.

$$56. \quad \left(\frac{1-x}{x^2}\right)^2 - 4\left(\frac{1-x}{x^2}\right) = \frac{17}{16}. \quad (1)$$

Put p for $\frac{1-x}{x^2}$ and p^2 for $\left(\frac{1-x}{x^2}\right)^2$.

Then, adding 4 to each member, $p^2 - 4p + 4 = \frac{31}{16}$.

$$p - 2 = \pm \frac{3}{4}.$$

$$\therefore p = \frac{1-x}{x^2} = \frac{17}{4} \text{ or } -\frac{1}{4}.$$

$$\frac{1}{x^2} - \frac{1}{x} = \frac{17}{4} \text{ or } -\frac{1}{4}.$$

Completing the square,

$$\frac{1}{x^2} - \frac{1}{x} + \frac{1}{4} = \frac{18}{4} \text{ or } 0.$$

$$\frac{1}{x} - \frac{1}{2} = \pm \frac{3}{2}\sqrt{2} \text{ or } \pm 0.$$

$$\therefore \frac{1}{x} = \frac{1}{2}(1 \pm 3\sqrt{2}) \text{ or } \frac{1}{2}.$$

$$\therefore x = \frac{2}{1 \pm 3\sqrt{2}} \text{ or } 2.$$

$$\frac{2}{1 \pm 3\sqrt{2}} = \frac{2(1 \mp 3\sqrt{2})}{(1 \pm 3\sqrt{2})(1 \mp 3\sqrt{2})} = \frac{-2(-1 \pm 3\sqrt{2})}{1 - 18} = \frac{2}{17}(-1 \pm 3\sqrt{2}).$$

Hence, the roots of (1) are 2, 2, $\frac{2}{17}(-1 \pm 3\sqrt{2})$.

$$57. \quad \left(x - \frac{1}{x}\right)^2 + \frac{5}{6}\left(x - \frac{1}{x}\right) = 1. \quad (1)$$

Put p for $x - \frac{1}{x}$ and p^2 for $\left(x - \frac{1}{x}\right)^2$.

Then,

$$p^2 + \frac{5}{6}p = 1. \quad (2)$$

Clearing of fractions, etc.,

$$6p^2 + 5p - 6 = 0.$$

Factoring,

$$(2p + 3)(3p - 2) = 0.$$

$$\therefore p = x - \frac{1}{x} = -\frac{3}{2} \text{ or } \frac{2}{3}. \quad (3)$$

Clearing equations (3) of fractions, and transposing,

$$2x^2 + 3x - 2 = 0 \quad (4)$$

or

$$3x^2 - 2x - 3 = 0. \quad (5)$$

Factoring (4),

$$(2x - 1)(x + 2) = 0.$$

$$\therefore x = \frac{1}{2} \text{ or } -2.$$

Solving (5) by the formula, § 295,

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 3 \cdot (-3)}}{2 \cdot 3}$$

$$= \frac{1}{3}(1 \pm \sqrt{10}).$$

Hence,

$$x = \frac{1}{2} \text{ or } -2 \text{ or } \frac{1}{3}(1 \pm \sqrt{10}).$$

58, 59. See preceding page.

60. Let

$$x^3 = -8.$$

Transposing,

$$x^3 + 8 = 0.$$

Factoring,

$$(x + 2)(x^2 - 2x + 4) = 0.$$

Equating each factor to zero and solving,

$$x = -2 \text{ or } 1 \pm \sqrt{-3}.$$

Hence, the three cube roots of -8 are -2 , $1 + \sqrt{-3}$, and $1 - \sqrt{-3}$.

61. Let

$$x^4 = 1.$$

Transposing,

$$x^4 - 1 = 0.$$

Factoring,

$$(x - 1)(x + 1)(x^2 + 1) = 0.$$

Equating each factor to zero and solving,

$$x = 1 \text{ or } -1 \text{ or } \pm \sqrt{-1}.$$

Hence, the four fourth roots of 1 are 1, -1, $\sqrt{-1}$ and $-\sqrt{-1}$.

62.

$$x^3 - 28x^2 + 27 = 0.$$

Factoring,

$$(x^3 - 1)(x^3 - 27) = 0.$$

$$(x - 1)(x^2 + x + 1)(x - 3)(x^2 + 3x + 9) = 0.$$

Equating each factor to zero and solving,

$$x = 1, \frac{1}{2}(-1 \pm \sqrt{-3}), 3, \frac{3}{2}(-1 \pm \sqrt{-3}).$$

63.

$$x^3 - \frac{8}{x^3} = 7.$$

Clearing of fractions, etc.,

$$x^6 - 7x^3 - 8 = 0.$$

Factoring,

$$(x^3 + 1)(x^3 - 8) = 0.$$

$$(x + 1)(x^2 - x + 1)(x - 2)(x^2 + 2x + 4) = 0.$$

Equating each factor to zero and solving,

$$x = -1, \frac{1}{2}(1 \pm \sqrt{-3}), 2, -1 \pm \sqrt{-3}.$$

64.

$$x^4 - 16 = 0.$$

Factoring,

$$(x - 2)(x + 2)(x^2 + 4) = 0.$$

Equating each factor to zero and solving,

$$x = \pm 2, \pm 2\sqrt{-1}.$$

65.

$$x^4 + 2x^3 - x = 30.$$

Extracting the square root of the first member as far as possible, it is found that the first member lacks $\frac{1}{4}$ of being the square of $x^2 + x - \frac{1}{2}$.

Adding $\frac{1}{4}$ to each member,

$$x^4 + 2x^3 - x + \frac{1}{4} = 1\frac{3}{4}.$$

Extracting the square root,

$$x^2 + x - \frac{1}{2} = \pm \frac{1}{2}.$$

$$\therefore x^2 + x = 6 \text{ or } -5.$$

Solving these two equations,

$$x = 2 \text{ or } -3 \text{ or } \frac{1}{2}(-1 \pm \sqrt{-19}).$$

66.

$$x^4 - 4x^3 + 8x = -3.$$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 4 of being the square of $x^2 - 2x - 2$.

Adding 4 to each member,

$$x^4 - 4x^3 + 8x + 4 = 1.$$

Extracting the square root,

$$x^2 - 2x - 2 = \pm 1.$$

Adding 3 to each member,

$$x^2 - 2x + 1 = 4 \text{ or } 2.$$

Solving these two equations,

$$x = 3 \text{ or } -1 \text{ or } 1 \pm \sqrt{2}.$$

67.

$$x^4 - 2x^3 + x = 132.$$

Extracting the square root of the first member as far as possible, it is found that the first member lacks $\frac{1}{4}$ of being the square of $x^2 - x - \frac{1}{2}$.

Adding $\frac{1}{4}$ to each member, $x^4 - 2x^3 + x + \frac{1}{4} = 132\frac{1}{4}.$

Extracting the square root,

$$x^2 - x - \frac{1}{2} = \pm \frac{1}{2}.$$

$$\therefore x^2 - x = 12 \text{ or } -11.$$

Solving these two equations,

$$x = 4 \text{ or } -3 \text{ or } \frac{1}{2}(1 \pm \sqrt{-43}).$$

68. $x^4 - 6x^3 + 27x = 10.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks $\frac{81}{4}$ of being the square of $x^2 - 3x - \frac{9}{2}$.

Adding $\frac{81}{4}$ to each member,

$$x^4 - 6x^3 + 27x + \frac{81}{4} = \frac{121}{4}.$$

Extracting the square root, $x^2 - 3x - \frac{9}{2} = \pm \frac{11}{2}.$

$$x^2 - 3x = 10 \text{ or } -1.$$

Solving these two equations, $x = 5 \text{ or } -2 \text{ or } \frac{1}{2}(3 \pm \sqrt{5}).$

69. $x^4 + 2x^3 + 5x^2 + 4x - 60 = 0.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 64 of being the square of $x^2 + x + 2$.

Adding 64 to each member,

$$x^4 + 2x^3 + 5x^2 + 4x + 4 = 64.$$

Extracting the square root, $x^2 + x + 2 = \pm 8.$

$$x^2 + x = 6 \text{ or } -10.$$

Solving these two equations, $x = 2 \text{ or } -3 \text{ or } \frac{1}{2}(-1 \pm \sqrt{-39}).$

70. $x^4 + 6x^3 + 7x^2 - 6x - 8 = 0.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 9 of being the square of $x^2 + 3x - 1$.

Adding 9 to each member,

$$x^4 + 6x^3 + 7x^2 - 6x + 1 = 9.$$

Extracting the square root, $x^2 + 3x - 1 = \pm 3.$

$$x^2 + 3x = 4 \text{ or } -2.$$

Solving these two equations, $x = 1 \text{ or } -4 \text{ or } -1 \text{ or } -2.$

71. $x^4 - 6x^3 + 15x^2 - 18x + 8 = 0.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 1 of being the square of $x^2 - 3x + 3$.

Adding 1 to each member,

$$x^4 - 6x^3 + 15x^2 - 18x + 9 = 1.$$

Extracting the square root, $x^2 - 3x + 3 = \pm 1.$

$$x^2 - 3x = -2 \text{ or } -4.$$

Solving these two equations, $x = 1 \text{ or } 2 \text{ or } \frac{1}{2}(3 \pm \sqrt{-7}).$

72. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 1 of being the square of $x^2 - 5x + 3$.

Adding 1 to each member,

$$x^4 - 10x^3 + 35x^2 - 50x + 25 = 1.$$

Extracting the square root, $x^2 - 5x + 5 = \pm 1.$

$$x^2 - 5x = -4 \text{ or } -6.$$

Solving these two equations, $x = 1 \text{ or } 4 \text{ or } 2 \text{ or } 3.$

73. $16x^4 - 8x^3 - 31x^2 + 8x + 15 = 0.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 1 of being the square of $4x^2 - x - 4$.

Adding 1 to each member,

$$16x^4 - 8x^3 - 31x^2 + 8x + 16 = 1.$$

Extracting the square root, $4x^2 - x - 4 = \pm 1.$

$$4x^2 - x = 5 \text{ or } 3.$$

Solving these two equations, $x = -1 \text{ or } \frac{5}{4} \text{ or } 1 \text{ or } -\frac{3}{4}.$

74. $4x^4 - 4x^3 - 7x^2 + 4x + 3 = 0.$

Extracting the square root of the first member as far as possible, it is found that the first member lacks 1 of being the square of $2x^2 - x - 2$.

Adding 1 to each member,

$$4x^4 - 4x^3 - 7x^2 + 4x + 4 = 1.$$

Extracting the square root, $2x^2 - x - 2 = \pm 1.$

$$2x^2 - x = 3 \text{ or } 1.$$

Solving these two equations, $x = -1 \text{ or } \frac{3}{2} \text{ or } 1 \text{ or } -\frac{1}{2}.$

75.
$$\frac{x^2}{x+1} - \frac{x+1}{x^2} = \frac{7}{12}. \quad (1)$$

If $\frac{x^2}{x+1} = p,$
$$p - \frac{1}{p} = \frac{7}{12}. \quad (2)$$

Clearing of fractions, etc., $12p^2 - 7p - 12 = 0.$

$$\therefore p = \frac{7 \pm \sqrt{49 + 4 \cdot 12 \cdot 12}}{24} = \frac{4}{3} \text{ or } -\frac{3}{4};$$

that is,

$$\frac{x^2}{x+1} = \frac{4}{3} \text{ or } -\frac{3}{4}. \quad (3)$$

Clearing of fractions, etc., $3x^2 - 4x - 4 = 0 \quad (4)$

or $4x^2 + 3x + 3 = 0. \quad (5)$

Factoring (4), $(x-2)(3x+2) = 0.$

$$\therefore x = 2 \text{ or } -\frac{2}{3}.$$

Solving (5) by the formula, § 295,

$$x = \frac{-3 \pm \sqrt{9 - 4 \cdot 4 \cdot 3}}{2 \cdot 4}$$

$$= \frac{1}{8}(-3 \pm \sqrt{-39})$$

Hence,

$$x = 2 \text{ or } -\frac{2}{3} \text{ or } \frac{1}{8}(-3 \pm \sqrt{-39})$$

76.
$$\frac{x^2+x}{2} + \frac{2}{x^2+x} = 2.$$

If $\frac{x^2+x}{2} = p,$
$$p + \frac{1}{p} = 2.$$

Clearing of fractions, etc., $p^2 - 2p + 1 = 0.$

$$\therefore p = 1 \pm 0 = 1;$$

that is,

$$\frac{x^2+x}{2} = 1.$$

Clearing of fractions, etc., $x^2 + x - 2 = 0.$

Factoring, $(x-1)(x+2) = 0.$

$$\therefore x = 1 \text{ or } -2.$$

Each of these numbers occurs twice as a root.

77.
$$\frac{x^2+1}{4} + \frac{4}{x^2+1} = \frac{5}{2}.$$

If $\frac{x^2+1}{4} = p,$
$$p + \frac{1}{p} = \frac{5}{2}.$$

Clearing of fractions, etc., $2p^2 - 5p + 2 = 0.$

Factoring, $(p-2)(2p-1) = 0.$

$$\therefore p = \frac{x^2+1}{4} = 2 \text{ or } \frac{1}{2}.$$

Clearing of fractions, etc.,

$$x^2 = 7 \text{ or } 1.$$

$$\therefore x = \pm \sqrt{7} \text{ or } \pm 1.$$

$$78. \quad \frac{x+2}{x^2+4} + \frac{2(x^2+4)}{x+2} = \frac{51}{5}. \quad (1)$$

$$\text{If } \frac{x+2}{x^2+4} = p, \quad p + \frac{2}{p} = \frac{51}{5}. \quad (2)$$

Clearing of fractions, etc., $5p^2 - 51p + 10 = 0$.

Factoring, $(p-10)(5p-1) = 0$.

$$\therefore p = \frac{x+2}{x^2+4} = 10 \text{ or } \frac{1}{5}. \quad (3)$$

Clearing of fractions, etc., $10x^2 - x + 38 = 0$ (4)

or $x^2 - 5x - 6 = 0$. (5)

Solving (4) by the formula, § 295, $x = \frac{1 \pm \sqrt{1-4 \cdot 10 \cdot 38}}{2 \cdot 10}$

Factoring (5), $(x-6)(x+1) = 0$.
 $\therefore x = 6 \text{ or } -1$.

Hence, $x = 6 \text{ or } -1 \text{ or } \frac{1}{20}(1 \pm \sqrt{-1519})$.

$$79. \quad \frac{x^2+1}{x-1} - \frac{4(x-1)}{x^2+1} = \frac{21}{5}. \quad (1)$$

$$\text{If } \frac{x^2+1}{x-1} = p, \quad p - \frac{4}{p} = \frac{21}{5}. \quad (2)$$

Clearing of fractions, etc., $5p^2 - 21p - 20 = 0$.

Factoring, $(p-5)(5p+4) = 0$.

$$\therefore p = \frac{x^2+1}{x-1} = 5 \text{ or } -\frac{4}{5}. \quad (3)$$

Clearing of fractions, etc., $x^2 - 5x + 6 = 0$ (4)

or $5x^2 + 4x + 1 = 0$. (5)

Factoring (4), $(x-2)(x-3) = 0$.
 $\therefore x = 2 \text{ or } 3$.

Solving (5) by the formula, § 295, $x = \frac{-4 \pm \sqrt{16-4 \cdot 5 \cdot 1}}{2 \cdot 5}$

Hence, $x = 2 \text{ or } 3 \text{ or } \frac{1}{5}(-2 \pm \sqrt{-1})$.

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$$80. \quad \frac{x}{x^2-1} + \frac{x^2-1}{x} = -\frac{13}{6}. \quad (1)$$

$$\text{If } \frac{x^2-1}{x} = p, \quad \frac{1}{p} + p = -\frac{13}{6}. \quad (2)$$

Clearing of fractions, etc., $6p^2 + 13p + 6 = 0$.

Factoring, $(2p+3)(3p+2) = 0$.

$$\therefore p = \frac{x^2-1}{x} = -\frac{3}{2} \text{ or } -\frac{2}{3}. \quad (3)$$

Clearing of fractions, etc., $2x^2 + 3x - 2 = 0$ (4)

or $3x^2 + 2x - 3 = 0$. (5)

Factoring (4), $(2x-1)(x+2) = 0$.

$$\therefore x = \frac{1}{2} \text{ or } -2.$$

Solving (5) by the formula, § 295, $x = \frac{-2 \pm \sqrt{4-4 \cdot 3 \cdot (-3)}}{2 \cdot 3}$

Hence, $x = \frac{1}{2}(-1 \pm \sqrt{10})$.
 $x = \frac{1}{2} \text{ or } -2 \text{ or } \frac{1}{3}(-1 \pm \sqrt{10})$.

$$81. \quad x^2 + x + 1 - \frac{1}{x^2 + x + 1} = \frac{8}{3}. \quad (1)$$

$$\text{If } x^2 + x + 1 = p, \quad p - \frac{1}{p} = \frac{8}{3}. \quad (2)$$

$$\text{Clearing of fractions, etc.,} \quad 3p^2 - 8p - 3 = 0.$$

$$\text{Factoring,} \quad (p - 3)(3p + 1) = 0.$$

$$\therefore p = x^2 + x + 1 = 3 \text{ or } -\frac{1}{3}. \quad (3)$$

$$\text{Transposing, etc.,} \quad x^2 + x - 2 = 0. \quad (4)$$

$$\text{or} \quad 3x^2 + 3x + 4 = 0. \quad (5)$$

$$\text{Factoring (4),} \quad (x - 1)(x + 2) = 0.$$

$$\therefore x = 1 \text{ or } -2.$$

$$\text{Solving (5) by the formula, § 295,} \quad x = \frac{-3 \pm \sqrt{9 - 4 \cdot 3 \cdot 4}}{2 \cdot 3}$$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{-39}.$$

$$\text{Hence,} \quad x = 1 \text{ or } -2 \text{ or } -\frac{1}{2} \pm \frac{1}{2}\sqrt{-39}.$$

$$82. \quad x^2 - 3x + \frac{2}{x^2 - 3x + 2} = 1. \quad (1)$$

Adding 2 to each member and substituting p for $x^2 - 3x + 2$,

$$p + \frac{2}{p} = 3. \quad (2)$$

$$\text{Clearing of fractions, etc.,} \quad p^2 - 3p + 2 = 0.$$

$$\text{Factoring,} \quad (p - 1)(p - 2) = 0.$$

$$\therefore p = x^2 - 3x + 2 = 1 \text{ or } 2. \quad (3)$$

$$\text{Transposing, etc.,} \quad x^2 - 3x + 1 = 0. \quad (4)$$

$$\text{or} \quad x^2 - 3x = 0. \quad (5)$$

$$\text{Solving (4) by the formula, § 295,} \quad x = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{1}{2}(3 \pm \sqrt{5}).$$

$$\text{Factoring (5),} \quad x(x - 3) = 0.$$

$$\therefore x = 0 \text{ or } 3.$$

$$\text{Hence,} \quad x = 0 \text{ or } 3 \text{ or } \frac{1}{2}(3 \pm \sqrt{5}).$$

$$83. \quad x^2 - 2x + \frac{6}{x^2 - 2x - 6} = 11. \quad (1)$$

Adding -6 to each member and substituting p for $x^2 - 2x - 6$,

$$p + \frac{6}{p} = 5. \quad (2)$$

$$\text{Clearing of fractions, etc.,} \quad p^2 - 5p + 6 = 0.$$

$$\text{Factoring,} \quad (p - 2)(p - 3) = 0.$$

$$\therefore p = x^2 - 2x - 6 = 2 \text{ or } 3. \quad (3)$$

$$\text{Transposing, etc.,} \quad x^2 - 2x - 8 = 0. \quad (4)$$

$$\text{or} \quad x^2 - 2x - 9 = 0. \quad (5)$$

$$\text{Factoring (4),} \quad (x - 4)(x + 2) = 0.$$

$$\therefore x = 4 \text{ or } -2.$$

$$\text{Completing the square in (5),} \quad x^2 - 2x + 1 = 10.$$

$$x - 1 = \pm \sqrt{10}.$$

$$\therefore x = 1 \pm \sqrt{10}.$$

$$\text{Hence,} \quad x = 4 \text{ or } -2 \text{ or } 1 \pm \sqrt{10}.$$

$$84. \quad x^2 - x + \frac{2}{x^2 - x - 4} = 7. \quad (1)$$

Adding -4 to each member and substituting p for $x^2 - x - 4$,

$$p + \frac{2}{p} = 3. \quad (2)$$

Clearing of fractions, etc., $p^2 - 3p + 2 = 0$.

Factoring, $(p - 1)(p - 2) = 0$.

$$\therefore p = x^2 - x - 4 = 1 \text{ or } 2. \quad (3)$$

Transposing, etc., $x^2 - x - 5 = 0 \quad (4)$

or $x^2 - x - 6 = 0. \quad (5)$

Solving (4) by the formula, § 295,
$$x = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-5)}}{2}$$

$$= \frac{1}{2} (1 \pm \sqrt{21}).$$

Factoring (5), $(x - 3)(x + 2) = 0$.

$$\therefore x = 3 \text{ or } -2.$$

Hence, $x = 3 \text{ or } -2 \text{ or } \frac{1}{2} (1 \pm \sqrt{21}).$

$$85. \quad x^2 - 2x + \frac{4}{x^2 - 2x + 1} = 4.$$

Adding 1 to each member and substituting p for $x^2 - 2x + 1$,

$$p + \frac{4}{p} = 5.$$

Clearing of fractions, etc., $p^2 - 5p + 4 = 0$.

Factoring, $(p - 1)(p - 4) = 0$.

$$\therefore p = x^2 - 2x + 1 = 1 \text{ or } 4.$$

Extracting the square root, $x - 1 = \pm 1 \text{ or } \pm 2$.

$$\therefore x = 2 \text{ or } 0 \text{ or } 3 \text{ or } -1.$$

$$86. \quad \frac{1}{1 + x + x^2} + \frac{2}{\sqrt{1 + x + x^2}} - 3 = 0. \quad (1)$$

Let $\sqrt{1 + x + x^2} = p$

$$\text{Then,} \quad \frac{1}{p^2} + \frac{2}{p} - 3 = 0. \quad (2)$$

$$\text{Factoring,} \quad \left(\frac{1}{p} - 1\right) \left(\frac{1}{p} + 3\right) = 0.$$

$$\therefore \frac{1}{p} = 1 \text{ or } -3,$$

$$\text{whence,} \quad p = \sqrt{1 + x + x^2} = 1 \text{ or } -\frac{1}{3}. \quad (3)$$

$$\text{Squaring,} \quad 1 + x + x^2 = 1 \text{ or } \frac{1}{9}. \quad (4)$$

$$\text{From (4),} \quad x^2 + x = 0 \quad (5)$$

$$\text{or} \quad x^2 + x = -\frac{8}{9}. \quad (6)$$

$$\text{Factoring (5),} \quad x(x + 1) = 0,$$

$$\therefore x = 0 \text{ or } -1.$$

$$\text{Completing the square in (6),} \quad x^2 + x + \frac{1}{4} = -\frac{23}{36}.$$

$$\therefore x = -\frac{1}{2} \pm \frac{1}{6} \sqrt{-23}.$$

$$\text{Hence,} \quad x = 0 \text{ or } -1 \text{ or } -\frac{1}{2} \pm \frac{1}{6} \sqrt{-23}.$$

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$$2. \quad \begin{cases} x^2 + y^2 = 20, & (1) \\ x = 2y. & (2) \end{cases}$$

$$\text{Substituting (2) in (1),} \quad 4y^2 + y^2 = 20. \quad (3)$$

$$\text{Solving (3),} \quad y = \pm 2.$$

$$\text{Substituting } y = 2 \text{ in (2),} \quad x = 4.$$

$$\text{Substituting } y = -2 \text{ in (2),} \quad x = -4.$$

$$\text{Hence, when } x = 4, y = 2; \text{ when } x = -4, y = -2.$$

$$3. \quad \begin{cases} 10x + y = 3xy, & (1) \\ y - x = 2. & (2) \end{cases}$$

$$\text{From (2),} \quad y = x + 2. \quad (3)$$

$$\text{Substituting (3) in (1),} \quad 10x + x + 2 = 3x(x + 2). \quad (4)$$

$$\text{Solving (4),} \quad x = 2 \text{ or } -\frac{1}{3}. \quad (5)$$

$$\text{Substituting (5) in (3),} \quad y = 4 \text{ or } \frac{5}{3}.$$

$$4. \quad \begin{cases} x^2 + xy = 12, & (1) \\ x - y = 2. & (2) \end{cases}$$

$$\text{From (2),} \quad y = x - 2. \quad (3)$$

$$\text{Substituting (3) in (1),} \quad x^2 + x(x - 2) = 12. \quad (4)$$

$$\text{Solving (4),} \quad x = 3 \text{ or } -2. \quad (5)$$

$$\text{Substituting (5) in (3),} \quad y = 1 \text{ or } -4.$$

$$5. \quad \begin{cases} x = 6 - y, & (1) \\ x^3 + y^3 = 72. & (2) \end{cases}$$

$$\text{Substituting (1) in (2),} \quad 216 - 108y + 18y^2 - y^3 + y^3 = 72.$$

$$y^2 - 6y + 8 = 0. \quad (3)$$

$$\text{Solving (3),} \quad y = 4 \text{ or } 2. \quad (4)$$

$$\text{Substituting (4) in (1),} \quad x = 2 \text{ or } 4.$$

$$6. \quad \begin{cases} xy(x - 2y) = 10, & (1) \\ xy = 10. & (2) \end{cases}$$

$$\text{Substituting (2) in (1), etc.,} \quad x - 2y = 1. \quad (3)$$

$$\text{From (3),} \quad x = 2y + 1. \quad (4)$$

$$\text{Substituting (4) in (2),} \quad (2y + 1)y = 10. \quad (5)$$

$$\text{Solving (5),} \quad y = 2 \text{ or } -\frac{5}{2}. \quad (6)$$

$$\text{Substituting (6) in (4),} \quad x = 5 \text{ or } -\frac{1}{2}.$$

$$7. \quad \begin{cases} 3x(y + 1) = 12, & (1) \\ 3x = 2y. & (2) \end{cases}$$

$$\text{Substituting (2) in (1),} \quad 2y(y + 1) = 12. \quad (3)$$

$$\text{Solving (3),} \quad y = 2 \text{ or } -3. \quad (4)$$

$$\text{Substituting (4) in (2) and solving,} \quad x = \frac{4}{3} \text{ or } -2.$$

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11. See next page.

$$12. \quad \begin{cases} x + y = 8, & (1) \\ x^2 + y^2 = 34. & (2) \end{cases}$$

$$\text{Squaring (1),} \quad x^2 + 2xy + y^2 = 64. \quad (3)$$

$$\text{Subtracting (3) from (2),} \quad -2xy = -30. \quad (4)$$

$$\text{Adding (4) to (2),} \quad x^2 - 2xy + y^2 = 4. \quad (5)$$

$$\therefore x - y = \pm 2. \quad (6)$$

$$\text{From (1) and (6),} \quad x = 5 \text{ or } 3,$$

$$\text{and} \quad y = 3 \text{ or } 5.$$

11.
$$\begin{cases} x^2 + y^2 = 50, & (1) \\ xy = 7. & (2) \end{cases}$$

Multiplying (2) by 2, $2xy = 14. \quad (3)$

Adding (3) to (1), $x^2 + 2xy + y^2 = 64. \quad (4)$

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 36. \quad (5)$

From (4), $x + y = \pm 8. \quad (6)$

From (5), $x - y = \pm 6. \quad (7)$

Since (4) and (5) are derived separately, neither from the other,
 from (6) and (7), $\begin{cases} x + y = 8, \\ x - y = 6, \end{cases}$ or $\begin{cases} x + y = 8, \\ x - y = -6, \end{cases}$ or $\begin{cases} x + y = -8, \\ x - y = 6, \end{cases}$ or $\begin{cases} x + y = -8, \\ x - y = -6. \end{cases}$

From these equations,
$$\begin{aligned} x &= 7, 1, -1, -7; \\ y &= 1, 7, -7, -1. \end{aligned}$$

and

13. See next page.

14.
$$\begin{cases} x^2 + y^2 = 13, & (1) \\ x + y + xy = 11. & (2) \end{cases}$$

From (2), $x + y = 11 - xy. \quad (3)$

Squaring (3), $x^2 + 2xy + y^2 = 121 - 22xy + x^2y^2. \quad (4)$

Subtracting (1) from (4), $2xy = 108 - 22xy + x^2y^2.$

$x^2y^2 - 24xy + 108 = 0. \quad (5)$

$(xy - 6)(xy - 18) = 0.$

$\therefore xy = 6 \text{ or } 18. \quad (6)$

Subtracting (6) from (2), $x + y = 5 \text{ or } -7. \quad (7)$

Multiplying (6) by 2, $2xy = 12 \text{ or } 36. \quad (8)$

Subtracting $2xy = 12$ from (1), $x^2 - 2xy + y^2 = 1,$

whence, $x - y = \pm 1. \quad (9)$

Subtracting $2xy = 36$ from (1), $x^2 - 2xy + y^2 = -23,$

whence, $x - y = \pm \sqrt{-23}. \quad (10)$

When $xy = 6$, $\begin{cases} x + y = 5, \\ x - y = \pm 1. \end{cases}$ When $xy = 18$, $\begin{cases} x + y = -7, \\ x - y = \pm \sqrt{-23}. \end{cases}$

From these equations, $x = 3, 2, \frac{1}{2}(-7 + \sqrt{-23}), \frac{1}{2}(-7 - \sqrt{-23});$

and $y = 2, 3, \frac{1}{2}(-7 - \sqrt{-23}), \frac{1}{2}(-7 + \sqrt{-23}).$

15.
$$\begin{cases} x^4 + y^4 = 17, & (1) \\ x + y = 3. & (2) \end{cases}$$

Raising (2) to the fourth power,

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81. \quad (3)$

Subtracting (1) from (3), $4x^3y + 6x^2y^2 + 4xy^3 = 64. \quad (4)$

Dividing (4) by 2, $2x^3y + 3x^2y^2 + 2xy^3 = 32. \quad (5)$

$2xy \times \text{square of (2)}, \quad 2x^3y + 4x^2y^2 + 2xy^3 = 18xy. \quad (6)$

Subtracting (5) from (6), $x^2y^2 = 18xy - 32. \quad (7)$

Solving (7) for xy , $xy = 2 \text{ or } 16. \quad (8)$

Squaring (2), $x^2 + 2xy + y^2 = 9. \quad (9)$

Subtracting $4xy = 8$ from (9), $x^2 - 2xy + y^2 = 1, \quad (10)$

whence, $x - y = \pm 1.$

Subtracting $4xy = 64$ from (9), $x^2 - 2xy + y^2 = -55, \quad (11)$

whence, $x - y = \pm \sqrt{-55}.$

When $xy = 2$, $\begin{cases} x + y = 3, \\ x - y = \pm 1. \end{cases}$ When $xy = 16$, $\begin{cases} x + y = 3, \\ x - y = \pm \sqrt{-55}. \end{cases}$

From these equations, $x = 2, 1, \frac{1}{2}(3 + \sqrt{-55}), \frac{1}{2}(3 - \sqrt{-55});$

and $y = 1, 2, \frac{1}{2}(3 - \sqrt{-55}), \frac{1}{2}(3 + \sqrt{-55}).$

$$13. \quad \begin{cases} x + y = 9, \\ x^3 + y^3 = 243. \end{cases} \quad (1)$$

$$\text{Cubing (1),} \quad x^3 + 3x^2y + 3xy^2 + y^3 = 729. \quad (2)$$

$$\text{Subtracting (2) from (3),} \quad 3x^2y + 3xy^2 = 486. \quad (3)$$

$$xy(x + y) = 162. \quad (4)$$

$$\text{Substituting (1) in (4),} \quad 9xy = 162. \quad (5)$$

$$\therefore xy = 18. \quad (6)$$

$$\text{Squaring (1),} \quad x^2 + 2xy + y^2 = 81. \quad (7)$$

$$\text{Subtracting (5) } \times 4 \text{ from (6),} \quad x^2 - 2xy + y^2 = 9. \quad (8)$$

$$\therefore x - y = \pm 3. \quad (9)$$

$$\text{From (1) and (7),} \quad x = 6 \text{ or } 3, \quad (10)$$

$$\text{and} \quad y = 3 \text{ or } 6. \quad (11)$$

$$16. \quad \begin{cases} x^4 + x^2y^2 + y^4 = 21, \\ x^2 + xy + y^2 = 7. \end{cases} \quad (1)$$

$$\text{From (2),} \quad x^2 + y^2 = 7 - xy. \quad (2)$$

$$\text{Squaring (3),} \quad x^4 + 2x^2y^2 + y^4 = 49 - 14xy + x^2y^2. \quad (3)$$

$$\text{Subtracting (1) from (4),} \quad x^2y^2 = 28 - 14xy + x^2y^2. \quad (4)$$

$$\text{Solving (5) for } xy, \quad xy = 2. \quad (5)$$

$$\text{Adding (6) to (2),} \quad x^2 + 2xy + y^2 = 9, \quad (6)$$

$$\text{whence,} \quad x + y = \pm 3. \quad (7)$$

$$\text{Subtracting (6) } \times 3 \text{ from (2),} \quad x^2 - 2xy + y^2 = 1, \quad (8)$$

$$\text{whence,} \quad x - y = \pm 1. \quad (9)$$

Since (7) and (8) have been derived separately, we have

$$\begin{cases} x + y = 3, & \text{or} & \begin{cases} x + y = 3, \\ x - y = -1, \end{cases} & \text{or} & \begin{cases} x + y = -3, \\ x - y = 1, \end{cases} & \text{or} & \begin{cases} x + y = -3, \\ x - y = -1. \end{cases} \end{cases}$$

$$\text{From these equations,} \quad x = 2, 1, -1, -2; \quad (10)$$

$$\text{and} \quad y = 1, 2, -2, -1. \quad (11)$$

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$$18. \quad \begin{cases} xy + 3y^2 = 20, \\ x^2 - 3xy = -8. \end{cases} \quad (1)$$

$$\text{Assume} \quad x = vy. \quad (2)$$

$$\text{Substituting (3) in (1),} \quad vy^2 + 3y^2 = 20. \quad (3)$$

$$\text{Substituting (3) in (2),} \quad v^2y^2 - 3vy^2 = -8. \quad (4)$$

$$\text{From (4) and (5),} \quad y^2 = \frac{20}{v + 3} = \frac{-8}{v^2 - 3v}. \quad (5)$$

$$\text{Clearing of fractions, etc.,} \quad 5v^2 - 13v + 6 = 0.$$

$$\text{Factoring,} \quad (v - 2)(5v - 3) = 0.$$

$$\therefore v = 2 \text{ or } \frac{3}{5}.$$

$$\text{Substituting 2 for } v \text{ in (6),} \quad \begin{cases} y = 2 \text{ or } -2, \\ x = 4 \text{ or } -4, \end{cases} \quad \text{when } v = 2.$$

$$\text{Substituting } \frac{3}{5} \text{ for } v \text{ in (6),} \quad \begin{cases} y = \frac{5}{3}\sqrt{2} \text{ or } -\frac{5}{3}\sqrt{2}, \\ x = \sqrt{2} \text{ or } -\sqrt{2}, \end{cases} \quad \text{when } v = \frac{3}{5}.$$

$$\text{whence, by (3),} \quad \begin{cases} x = 4, -4, \sqrt{2}, -\sqrt{2}; \\ y = 2, -2, \frac{5}{3}\sqrt{2}, -\frac{5}{3}\sqrt{2}. \end{cases}$$

Hence,

19.

$$\begin{cases} x^2 + xy = 12, & (1) \\ xy + 2y^2 = 5. & (2) \end{cases}$$

Assume

$$x = vy. \quad (3)$$

Substituting (3) in (1),

$$v^2y^2 + vy^2 = 12. \quad (4)$$

Substituting (3) in (2),

$$vy^2 + 2y^2 = 5. \quad (5)$$

From (4) and (5),

$$y^2 = \frac{12}{v^2 + v} = \frac{5}{v + 2}. \quad (6)$$

Clearing of fractions, etc., $5v^2 - 7v - 24 = 0$.

Factoring,

$$(v - 3)(5v + 8) = 0.$$

$$\therefore v = 3 \text{ or } -\frac{8}{5}.$$

Substituting 3 for v in (6),

$$y = 1 \text{ or } -1, \quad \text{when } v = 3.$$

whence, by (3),

$$x = 3 \text{ or } -3, \quad \text{when } v = 3.$$

Substituting $-\frac{8}{5}$ for v in (6),

$$y = \frac{5}{2}\sqrt{2} \text{ or } -\frac{5}{2}\sqrt{2}, \quad \text{when } v = -\frac{8}{5}.$$

whence, by (3),

$$x = -4\sqrt{2} \text{ or } 4\sqrt{2}, \quad \text{when } v = -\frac{8}{5}.$$

Hence,

$$\begin{cases} x = 3, -3, 4\sqrt{2}, -4\sqrt{2}; \\ y = 1, -1, -\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2}. \end{cases}$$

20.

$$\begin{cases} x^2 + 2y^2 = 44, & (1) \\ xy - y^2 = 8. & (2) \end{cases}$$

Assume

$$x = vy. \quad (3)$$

Substituting (3) in (1),

$$v^2y^2 + 2y^2 = 44. \quad (4)$$

Substituting (3) in (2),

$$vy^2 - y^2 = 8. \quad (5)$$

From (4) and (5),

$$y^2 = \frac{44}{v^2 + 2} = \frac{8}{v - 1}. \quad (6)$$

Clearing of fractions, etc., $2v^2 - 11v + 15 = 0$.

$$(v - 3)(2v - 5) = 0.$$

$$\therefore v = 3 \text{ or } \frac{5}{2}.$$

Substituting 3 for v in (6),

$$y = 2 \text{ or } -2, \quad \text{when } v = 3.$$

whence, by (3),

$$x = 6 \text{ or } -6, \quad \text{when } v = 3.$$

Substituting $\frac{5}{2}$ for v in (6),

$$y = \frac{4}{3}\sqrt{3} \text{ or } -\frac{4}{3}\sqrt{3}, \quad \text{when } v = \frac{5}{2}.$$

whence, by (3),

$$x = \frac{10}{3}\sqrt{3} \text{ or } -\frac{10}{3}\sqrt{3}, \quad \text{when } v = \frac{5}{2}.$$

Hence,

$$\begin{cases} x = 6, -6, \frac{10}{3}\sqrt{3}, -\frac{10}{3}\sqrt{3}; \\ y = 2, -2, \frac{4}{3}\sqrt{3}, -\frac{4}{3}\sqrt{3}. \end{cases}$$

21.

$$\begin{cases} x^2 - xy - y^2 = 20, & (1) \\ x^2 - 3xy + 2y^2 = 8. & (2) \end{cases}$$

Assume

$$x = vy. \quad (3)$$

Substituting (3) in (1),

$$v^2y^2 - vy^2 - y^2 = 20. \quad (4)$$

Substituting (3) in (2),

$$v^2y^2 - 3vy^2 + 2y^2 = 8. \quad (5)$$

From (4) and (5),

$$y^2 = \frac{20}{v^2 - v - 1} = \frac{8}{v^2 - 3v + 2}. \quad (6)$$

Clearing of fractions, etc., $3v^2 - 13v + 12 = 0$.

Factoring,

$$(v - 3)(3v - 4) = 0.$$

$$\therefore v = 3 \text{ or } \frac{4}{3}.$$

Substituting 3 for v in (6),

$$y = 2 \text{ or } -2, \quad \text{when } v = 3.$$

whence, by (3),

$$x = 6 \text{ or } -6, \quad \text{when } v = 3.$$

Substituting $\frac{4}{3}$ for v in (6),

$$y = 6\sqrt{-1} \text{ or } -6\sqrt{-1}, \quad \text{when } v = \frac{4}{3}.$$

whence, by (3),

$$x = 8\sqrt{-1} \text{ or } -8\sqrt{-1}, \quad \text{when } v = \frac{4}{3}.$$

Hence,

$$\begin{cases} x = 6, -6, 8\sqrt{-1}, -8\sqrt{-1}; \\ y = 2, -2, 6\sqrt{-1}, -6\sqrt{-1}. \end{cases}$$

22.

$$\begin{cases} x^2 - xy + y^2 = 21, \\ x^2 + 2y^2 = 27. \end{cases} \quad (1)$$

Assume

$$x = vy. \quad (2)$$

Substituting (3) in (1),

$$v^2y^2 - vy^2 + y^2 = 21. \quad (3)$$

Substituting (3) in (2),

$$v^2y^2 + 2y^2 = 27. \quad (4)$$

From (4) and (5),

$$y^2 = \frac{21}{v^2 - v + 1} = \frac{27}{v^2 + 2}. \quad (5)$$

Clearing of fractions, etc.,

$$2v^2 - 9v - 5 = 0.$$

Factoring,

$$(v - 5)(2v + 1) = 0.$$

$$\therefore v = 5 \text{ or } -\frac{1}{2}.$$

Substituting 5 for v in (6),

$$y = 1 \text{ or } -1, \quad \left. \begin{matrix} x = 5 \text{ or } -5, \end{matrix} \right\} \text{ when } v = 5.$$

whence, by (3),

Substituting $-\frac{1}{2}$ for v in (6),

$$y = 2\sqrt{3} \text{ or } -2\sqrt{3}, \quad \left. \begin{matrix} x = -\sqrt{3} \text{ or } \sqrt{3}, \end{matrix} \right\} \text{ when } v = -\frac{1}{2}.$$

whence, by (3),

Hence,

$$\begin{cases} x = 5, -5, \sqrt{3}, -\sqrt{3}; \\ y = 1, -1, -2\sqrt{3}, 2\sqrt{3}. \end{cases}$$

23.

$$\begin{cases} 2x^2 - 3xy + 2y^2 = 100, \\ x^2 - y^2 = 75. \end{cases} \quad (1)$$

Assume

$$x = vy. \quad (2)$$

Substituting (3) in (1),

$$2v^2y^2 - 3vy^2 + 2y^2 = 100. \quad (3)$$

Substituting (3) in (2),

$$v^2y^2 - y^2 = 75. \quad (4)$$

From (4) and (5),

$$y^2 = \frac{100}{v^2 - 3} = \frac{75}{v^2 - 1}. \quad (5)$$

Reducing (6),

$$2v^2 - 9v + 10 = 0.$$

Factoring,

$$(v - 2)(2v - 5) = 0.$$

$$\therefore v = 2 \text{ or } \frac{5}{2}.$$

Substituting 2 for v in (6),

$$y = 5 \text{ or } -5, \quad \left. \begin{matrix} x = 10 \text{ or } -10, \end{matrix} \right\} \text{ when } v = 2.$$

whence, by (3),

Substituting $\frac{5}{2}$ for v in (6),

$$y = \frac{10}{\sqrt{7}} \text{ or } -\frac{10}{\sqrt{7}}, \quad \left. \begin{matrix} x = \frac{25}{\sqrt{7}} \text{ or } -\frac{25}{\sqrt{7}}, \end{matrix} \right\} \text{ when } v = \frac{5}{2}.$$

whence, by (3),

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29.

$$\begin{cases} x^2 + y^2 = 53, \\ x - y = 5. \end{cases} \quad (1)$$

Squaring (2),

$$x^2 - 2xy + y^2 = 25. \quad (2)$$

Subtracting (3) from (1),

$$2xy = 28. \quad (3)$$

Adding (4) to (1),

$$x^2 + 2xy + y^2 = 81, \quad (4)$$

whence,

$$x + y = \pm 9. \quad (5)$$

From (5) and (2),

$$x = 7 \text{ or } -2,$$

and

$$y = 2 \text{ or } -7.$$

30.

$$\begin{cases} x^2 + y^2 = 28, \\ x + y = 4. \end{cases} \quad (1)$$

Dividing (1) by (2),

$$x^2 - xy + y^2 = 7. \quad (2)$$

Squaring (2),

$$x^2 + 2xy + y^2 = 16. \quad (3)$$

Subtracting (3) from (4),

$$3xy = 9, \quad (4)$$

whence,

$$xy = 3. \quad (5)$$

Subtracting (5) from (3),

$$x^2 - 2xy + y^2 = 4, \quad (6)$$

whence,

$$x - y = \pm 2. \quad (6)$$

From (2) and (6),

$$x = 3 \text{ or } 1,$$

and

$$y = 1 \text{ or } 3.$$

$$\begin{aligned} 31. \quad & \begin{cases} 1 + x = y, \\ x^2 + y^2 = 61. \end{cases} \end{aligned} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Substituting $1 + x$ for y in (2), $x^2 + 1 + 2x + x^2 = 61$.

$$\begin{aligned} \text{Factoring,} \quad & \begin{aligned} x^2 + x - 30 &= 0. \\ (x - 5)(x + 6) &= 0. \end{aligned} \end{aligned}$$

$$\begin{aligned} \therefore x &= 5 \text{ or } -6. \\ \text{Substituting (3) in (1),} \quad & y = 6 \text{ or } -5. \end{aligned} \quad (3)$$

$$32. \quad \begin{cases} 1 + x = y, \\ 1 + x^3 = \frac{y^3}{4}. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$\text{Dividing (2) by (1),} \quad 1 - x + x^2 = \frac{y^2}{4}. \quad (3)$$

$$\text{Substituting } 1 + x \text{ for } y \text{ in (3),} \quad 1 - x + x^2 = \frac{(1 + x)^2}{4}. \quad (4)$$

$$\begin{aligned} \text{Reducing (4),} \quad & x^2 - 2x + 1 = 0. \\ \therefore x &= 1. \end{aligned} \quad \begin{matrix} (5) \\ (6) \end{matrix}$$

$$\text{Substituting (6) in (1),} \quad y = 2.$$

Since $x - 1 = 0$ occurs *twice* as a factor of (5), the roots of the given equations are $x = 1, 1$; $y = 2, 2$.

$$33. \quad \begin{cases} x^2 + y^2 = 40, \\ xy = 12. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$\begin{aligned} \text{Adding (2) } \times 2 \text{ to (1),} \quad & x^2 + 2xy + y^2 = 64, \\ \text{whence,} \quad & x + y = \pm 8. \end{aligned} \quad (3)$$

$$\begin{aligned} \text{Subtracting (2) } \times 2 \text{ from (1),} \quad & x^2 - 2xy + y^2 = 16, \\ \text{whence,} \quad & x - y = \pm 4. \end{aligned} \quad (4)$$

Solving the four sets of equations given in (3) and (4),

$$\begin{aligned} x &= 6, 2, -2, -6, \\ y &= 2, 6, -6, -2. \end{aligned}$$

and

34. See next page.

$$35. \quad \begin{cases} x^2 + 3xy - y^2 = 43, \\ x + 2y = 10. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$\text{From (2),} \quad x = 10 - 2y. \quad (3)$$

$$\begin{aligned} \text{Substituting (3) in (1),} \quad & 100 - 40y + 4y^2 + 30y - 6y^2 - y^2 = 43. \\ & 3y^2 + 10y - 57 = 0. \\ & (y - 3)(3y + 19) = 0. \\ \therefore y &= 3 \text{ or } -\frac{19}{3}. \\ \text{Substituting (4) in (3),} \quad & x = 4 \text{ or } \frac{68}{3}. \end{aligned} \quad (4)$$

$$36. \quad \begin{cases} x^2 + xy + y^2 = 19, \\ x^3 - y^3 = 19. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$\text{Dividing (2) by (1),} \quad x - y = 1. \quad (3)$$

$$\text{Squaring (3),} \quad x^2 - 2xy + y^2 = 1. \quad (4)$$

$$\begin{aligned} \text{Subtracting (4) from (1),} \quad & 3xy = 18, \\ \text{whence,} \quad & xy = 6. \end{aligned} \quad (5)$$

$$\begin{aligned} \text{Adding (5) to (1),} \quad & x^2 + 2xy + y^2 = 25, \\ \text{whence,} \quad & x + y = \pm 5. \end{aligned} \quad (6)$$

$$\begin{aligned} \text{From (6) and (3),} \quad & x = 3 \text{ or } -2, \\ \text{and} \quad & y = 2 \text{ or } -3. \end{aligned}$$

$$34. \quad \begin{cases} x(x+y) = x, & (1) \\ y(x-y) = -1. & (2) \end{cases}$$

It is evident that $x = 0$ satisfies (1).

Substituting 0 for x in (2),

$$-y^2 = -1.$$

$$\therefore y = \pm 1. \quad (3)$$

Dividing (1) by x ,

$$x + y = 1. \quad (4)$$

Multiplying (4) by y ,

$$y(x+y) = y. \quad (5)$$

Subtracting (2) from (5),

$$2y^2 = y + 1.$$

$$2y^2 - y - 1 = 0.$$

$$(y-1)(2y+1) = 0.$$

$$\therefore y = 1 \text{ or } -\frac{1}{2}. \quad (6)$$

Substituting (6) in (4),

$$x = 0 \text{ or } \frac{3}{2}.$$

Hence,

$$\begin{cases} x = 0, 0, 0, \frac{3}{2}; \\ y = 1, -1, 1, -\frac{1}{2}. \end{cases}$$

$$37. \quad \begin{cases} x^2 + 3xy = y^2 + 23, & (1) \\ x + 3y = 9. & (2) \end{cases}$$

From (1),

$$x(x+3y) = y^2 + 23. \quad (3)$$

From (2),

$$x = 9 - 3y. \quad (4)$$

Substituting (4) and (2) in (3),

$$(9-3y)9 = y^2 + 23.$$

$$y^2 + 27y - 58 = 0.$$

Factoring,

$$(y-2)(y+29) = 0.$$

$$\therefore y = 2 \text{ or } -29. \quad (5)$$

Substituting (5) in (4),

$$x = 3 \text{ or } 96.$$

$$38. \quad \begin{cases} 2x^2 + xy - 5y^2 = 20, & (1) \\ 2x - 3y = 1. & (2) \end{cases}$$

From (2),

$$2x = 3y + 1. \quad (3)$$

Multiplying (1) by 2,

$$4x^2 + 2xy - 10y^2 = 40. \quad (4)$$

Squaring (2),

$$4x^2 - 12xy + 9y^2 = 1. \quad (5)$$

Subtracting (5) from (4),

$$14xy - 19y^2 = 39. \quad (6)$$

Substituting (3) in (6),

$$7(3y+1)y - 19y^2 = 39. \quad (7)$$

$$2y^2 + 7y - 39 = 0. \quad (7)$$

Factoring,

$$(y-3)(2y+13) = 0.$$

$$\therefore y = 3 \text{ or } -\frac{13}{2}. \quad (8)$$

Substituting (8) in (3),

$$x = 5 \text{ or } -\frac{37}{2}.$$

$$39. \quad \begin{cases} \frac{x^2}{y^2} + \frac{4x}{y} = 21, & (1) \\ x - y = 2. & (2) \end{cases}$$

$$\text{Completing the square in (1), } \frac{x^2}{y^2} + \frac{4x}{y} + 4 = 25. \quad (3)$$

$$\text{Extracting the square root, } \frac{x}{y} + 2 = \pm 5. \quad (4)$$

Solving equations (4) for x ,

$$x = 3y \text{ or } -7y. \quad (5)$$

Substituting $3y$ for x in (2),

$$y = 1,$$

$$x = 3.$$

$$y = -\frac{1}{4},$$

$$x = \frac{7}{4}.$$

whence,

Substituting $-7y$ for x in (2),

whence,

$$\begin{cases} x = 3, \frac{7}{4}; \\ y = 1, -\frac{1}{4}. \end{cases}$$

Hence,

$$\begin{aligned} 40. \quad & \begin{cases} x^2 - xy = 48, & (1) \\ xy - y^2 = 12. & (2) \end{cases} \end{aligned}$$

$$\text{From (1),} \quad x(x - y) = 48. \quad (3)$$

$$\text{From (2),} \quad y(x - y) = 12. \quad (4)$$

$$\text{Dividing (3) by (4),} \quad \frac{x}{y} = 4. \quad (5)$$

$$\therefore x = 4y. \quad (5)$$

$$\text{Substituting (5) in (2),} \quad 4y^2 - y^2 = 12. \quad (6)$$

$$\therefore y = 2 \text{ or } -2. \quad (6)$$

$$\text{Substituting (6) in (5),} \quad x = 8 \text{ or } -8.$$

NOTE. — Since the equations are homogeneous quadratics as far as their unknown terms are concerned, we may assume that $x = vy$. Assuming $x = vy$ as a third equation, it is found that $v = 1$ or 4 . In the above solution only the value $v = 4$ is considered. If $v = 1$, then $x = y$, and (3) and (4) become $x \cdot 0 = 48$ and $y \cdot 0 = 12$, that is, $x = \frac{48}{0}$ and $y = \frac{12}{0}$. The interpretation of such expressions will be given later. Such roots are called *infinite roots*. The symbol for an infinite number is ∞ , read '*infinity*.'

$$41. \quad \begin{cases} x^2 + xy = -6, & (1) \\ xy + y^2 = 15. & (2) \end{cases}$$

$$\text{From (1),} \quad x(x + y) = -6. \quad (3)$$

$$\text{From (2),} \quad y(x + y) = 15. \quad (4)$$

$$\text{Dividing (4) by (3),} \quad \frac{y}{x} = -\frac{5}{2}, \quad (5)$$

$$\text{whence,} \quad \frac{y}{x} = -\frac{5}{2}x. \quad (5)$$

$$\text{Substituting (5) in (1),} \quad x^2 - \frac{5}{2}x^2 = -6, \quad (6)$$

$$\text{whence,} \quad x = 2 \text{ or } -2. \quad (6)$$

$$\text{Substituting (6) in (5),} \quad y = -5 \text{ or } 5.$$

If $x = vy$, $v = -1$ or $-\frac{2}{5}$. The value $v = -1$ gives rise to infinite roots. See note, Ex. 40.

42. See next page.

$$43. \quad \begin{cases} x^2 - xy = 8, & (1) \\ xy + y^2 = 12. & (2) \end{cases}$$

$$\text{Assume} \quad x = vy. \quad (3)$$

$$\text{Substituting (3) in (1),} \quad v^2y^2 - vy^2 = 8. \quad (4)$$

$$\text{Substituting (3) in (2),} \quad vy^2 + y^2 = 12. \quad (5)$$

$$\text{From (4) and (5),} \quad y^2 = \frac{8}{v^2 - v} = \frac{12}{v + 1}. \quad (6)$$

$$\text{Clearing of fractions, etc.,} \quad 3v^2 - 5v - 2 = 0.$$

$$\text{Factoring,} \quad (v - 2)(3v + 1) = 0.$$

$$\therefore v = 2 \text{ or } -\frac{1}{3}.$$

$$\text{Substituting 2 for } v \text{ in (6),} \quad y = 2 \text{ or } -2, \quad \left. \begin{matrix} y = 2 \text{ or } -2, \\ x = 4 \text{ or } -4, \end{matrix} \right\} \text{ when } v = 2.$$

$$\text{whence, by (3),} \quad x = 4 \text{ or } -4, \quad \left. \begin{matrix} y = 3\sqrt{2} \text{ or } -3\sqrt{2}, \\ x = -\sqrt{2} \text{ or } \sqrt{2}, \end{matrix} \right\} \text{ when } v = -\frac{1}{3}.$$

$$\text{Substituting } -\frac{1}{3} \text{ for } v \text{ in (6),} \quad y = 3\sqrt{2} \text{ or } -3\sqrt{2}, \quad \left. \begin{matrix} y = 3\sqrt{2} \text{ or } -3\sqrt{2}, \\ x = -\sqrt{2} \text{ or } \sqrt{2}, \end{matrix} \right\} \text{ when } v = -\frac{1}{3}.$$

$$\text{whence,} \quad x = -\sqrt{2} \text{ or } \sqrt{2}, \quad \left. \begin{matrix} y = 3\sqrt{2} \text{ or } -3\sqrt{2}, \\ x = -\sqrt{2} \text{ or } \sqrt{2}, \end{matrix} \right\} \text{ when } v = -\frac{1}{3}.$$

$$\text{Hence,} \quad \begin{cases} x = 4, -4, \sqrt{2}, -\sqrt{2}; \\ y = 2, -2, -3\sqrt{2}, 3\sqrt{2}. \end{cases}$$

42.

$$\begin{cases} 4xy = 96 - x^2y^2, & (1) \\ x + y = 6. & (2) \end{cases}$$

Solving (1) for xy ,

$$xy = 8 \text{ or } -12. \quad (3)$$

Squaring (2),

$$x^2 + 2xy + y^2 = 36. \quad (4)$$

Subtracting $4xy = 32$ from (4),

$$x^2 - 2xy + y^2 = 4,$$

whence,

$$x - y = \pm 2. \quad (5)$$

Subtracting $4xy = -48$ from (4), $x^2 - 2xy + y^2 = 84$,

whence,

$$x - y = \pm 2\sqrt{21}. \quad (6)$$

From (2) and (5), $x = 4$ or 2 and $y = 2$ or 4 .From (2) and (6), $x = 3 + \sqrt{21}$ or $3 - \sqrt{21}$ and $y = 3 - \sqrt{21}$ or $3 + \sqrt{21}$.

Hence,

$$\begin{cases} x = 4, 2; 3 + \sqrt{21}, 3 - \sqrt{21}; \\ y = 2, 4, 3 - \sqrt{21}, 3 + \sqrt{21}. \end{cases}$$

44.

$$\begin{cases} x^2 - xy = 6, & (1) \\ x^2 + y^2 = 61. & (2) \end{cases}$$

Assume

$$x = vy. \quad (3)$$

Substituting (3) in (1),

$$v^2y^2 - vy^2 = 6. \quad (4)$$

Substituting (3) in (2),

$$v^2y^2 + y^2 = 61. \quad (5)$$

From (4) and (5),

$$y^2 = \frac{6}{v^2 - v} = \frac{61}{v^2 + 1}. \quad (6)$$

Clearing of fractions, etc., $55v^2 - 61v - 6 = 0$.

Factoring,

$$(5v - 6)(11v + 1) = 0.$$

$$\therefore v = \frac{6}{5} \text{ or } -\frac{1}{11}.$$

Substituting $\frac{6}{5}$ for v in (6),

$$\begin{cases} y = 5 \text{ or } -5, \\ x = 6 \text{ or } -6, \end{cases} \text{ when } v = \frac{6}{5}.$$

whence,

Substituting $-\frac{1}{11}$ for v in (6),

$$\begin{cases} y = \frac{1}{2}\sqrt{2} \text{ or } -\frac{1}{2}\sqrt{2}, \\ x = -\frac{1}{2}\sqrt{2} \text{ or } \frac{1}{2}\sqrt{2}, \end{cases} \text{ when } v = -\frac{1}{11}.$$

whence,

$$\begin{cases} x = 6, -6, \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}; \\ y = 5, -5, -\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}. \end{cases}$$

Hence,

45.

$$\begin{cases} x^2 + xy = 77, & (1) \\ xy - y^2 = 12. & (2) \end{cases}$$

Assume

$$x = vy. \quad (3)$$

Substituting (3) in (1),

$$v^2y^2 + vy^2 = 77. \quad (4)$$

Substituting (3) in (2),

$$vy^2 - y^2 = 12. \quad (5)$$

From (4) and (5),

$$y^2 = \frac{77}{v^2 + 1} = \frac{12}{v - 1}. \quad (6)$$

Clearing of fractions, etc., $12v^2 - 65v + 77 = 0$.

Factoring,

$$(4v - 7)(3v - 11) = 0.$$

$$\therefore v = \frac{7}{4} \text{ or } \frac{11}{3}.$$

Substituting $\frac{7}{4}$ for v in (6),

$$\begin{cases} y = 4 \text{ or } -4, \\ x = 7 \text{ or } -7, \end{cases} \text{ when } v = \frac{7}{4}.$$

whence,

Substituting $\frac{11}{3}$ for v in (6),

$$\begin{cases} y = \frac{3}{2}\sqrt{2} \text{ or } -\frac{3}{2}\sqrt{2}, \\ x = \frac{11}{2}\sqrt{2} \text{ or } -\frac{11}{2}\sqrt{2}, \end{cases} \text{ when } v = \frac{11}{3}.$$

whence,

$$\begin{cases} x = 7, -7, \frac{11}{2}\sqrt{2}, -\frac{11}{2}\sqrt{2}; \\ y = 4, -4, \frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2}. \end{cases}$$

Hence,

$$46. \quad \begin{cases} 2x - y = 2, \\ 2x^2 + y^2 = \frac{3}{2}. \end{cases} \quad (1)$$

$$\text{From (1),} \quad y = 2x - 2. \quad (2)$$

$$\text{Substituting (2) in (2), } 2x^2 + 4x^2 - 8x + 4 = \frac{3}{2}. \quad (3)$$

$$\text{Completing the square,} \quad x^2 - \frac{4}{3}x = -\frac{1\frac{5}{6}}{\frac{4}{3}} = -\frac{1}{\frac{3}{2}} = -\frac{2}{3}. \\ x - \frac{2}{3} = \pm \frac{1}{3}. \\ \therefore x = \frac{5}{6} \text{ or } \frac{1}{6}. \quad (4)$$

$$\text{Substituting (4) in (3),} \quad y = -\frac{1}{3} \text{ or } -1.$$

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$$47. \quad \begin{cases} 2x^2 + 3xy + y^2 = 20, \\ 5x^2 + 4y^2 = 41. \end{cases} \quad (1)$$

$$\text{Multiplying (1) by 41, } 82x^2 + 123xy + 41y^2 = 41 \cdot 20. \quad (2)$$

$$\text{Multiplying (2) by 20, } 100x^2 + 80y^2 = 20 \cdot 41. \quad (3)$$

$$\text{Subtracting (3) from (4), } 18x^2 - 123xy + 39y^2 = 0. \quad (4)$$

$$\text{Factoring,} \quad 6x^2 - 41xy + 13y^2 = 0. \quad (5)$$

$$\text{Factoring,} \quad (3x - y)(2x - 13y) = 0. \\ \therefore x = \frac{1}{3}y \text{ or } \frac{13}{2}y. \quad (6)$$

$$\text{Substituting } \frac{1}{3}y \text{ for } x \text{ in (2),} \quad y = \frac{2}{3} \text{ or } -\frac{2}{3}, \\ \text{whence,} \quad x = 1 \text{ or } -1.$$

$$\text{Substituting } \frac{13}{2}y \text{ for } x \text{ in (2),} \quad y = \frac{2}{11}\sqrt{21} \text{ or } -\frac{2}{11}\sqrt{21}, \\ \text{whence,} \quad x = \frac{13}{11}\sqrt{21} \text{ or } -\frac{13}{11}\sqrt{21}.$$

$$\text{Hence,} \quad \begin{cases} x = 1, -1, \frac{13}{11}\sqrt{21}, -\frac{13}{11}\sqrt{21}; \\ y = \frac{2}{11}\sqrt{21}, -\frac{2}{11}\sqrt{21}. \end{cases}$$

$$48. \quad \begin{cases} 2xy - y^2 = 12, \\ 3xy + 5x^2 = 104. \end{cases} \quad (1)$$

$$\text{Multiplying (1) by 26,} \quad 52xy - 26y^2 = 312. \quad (2)$$

$$\text{Multiplying (2) by 3,} \quad 9xy + 15x^2 = 312. \quad (3)$$

$$\text{Subtracting (3) from (4), } 15x^2 - 43xy + 26y^2 = 0. \quad (4)$$

$$\text{Factoring,} \quad (x - 2y)(15x - 13y) = 0. \quad (5)$$

$$\therefore x = 2y \text{ or } \frac{13}{15}y. \quad (6)$$

$$\text{Substituting } 2y \text{ for } x \text{ in (1),} \quad y = 2 \text{ or } -2, \\ \text{whence,} \quad x = 4 \text{ or } -4.$$

$$\text{Substituting } \frac{13}{15}y \text{ for } x \text{ in (1),} \quad y = \frac{6}{11}\sqrt{55} \text{ or } -\frac{6}{11}\sqrt{55}, \\ \text{whence,} \quad x = \frac{26}{11}\sqrt{55} \text{ or } -\frac{26}{11}\sqrt{55}.$$

$$\text{Hence,} \quad \begin{cases} x = 4, -4, \frac{26}{11}\sqrt{55}, -\frac{26}{11}\sqrt{55}; \\ y = 2, -2, \frac{6}{11}\sqrt{55}, -\frac{6}{11}\sqrt{55}. \end{cases}$$

$$49. \quad \begin{cases} x^2 + xy + y^2 = 151, \\ x^2 + y^2 = 106. \end{cases} \quad (1)$$

$$\text{Subtracting (2) from (1),} \quad xy = 45. \quad (2)$$

$$\text{Adding (3) to (1),} \quad x^2 + 2xy + y^2 = 196, \quad (3)$$

$$\text{whence,} \quad x + y = \pm 14. \quad (4)$$

$$\text{Subtracting (3) } \times 3 \text{ from (1), } x^2 - 2xy + y^2 = 16, \\ \text{whence,} \quad x - y = \pm 4. \quad (5)$$

$$\text{Since (4) and (5) are derived separately, neither from the other,}$$

$$\begin{cases} x + y = 14, \\ x - y = 4, \end{cases} \text{ or } \begin{cases} x + y = 14, \\ x - y = -4, \end{cases} \text{ or } \begin{cases} x + y = -14, \\ x - y = 4, \end{cases} \text{ or } \begin{cases} x + y = -14, \\ x - y = -4. \end{cases}$$

$$\text{Solving these equations.} \quad x = 9, 5, -5, -9; \\ \text{and} \quad y = 5, 9, -9, -5.$$

50.

$$\begin{cases} x^2 + xy + y^2 = 84, & (1) \\ x - \sqrt{xy} + y = 6. & (2) \end{cases}$$

$$\text{Dividing (1) by (2), } x + \sqrt{xy} + y = 14. \quad (3)$$

$$\text{Adding (2) and (3) and dividing by 2, } x + y = 10. \quad (4)$$

$$\text{Subtracting (4) from (3), } \sqrt{xy} = 4, \quad (5)$$

$$\text{whence, } xy = 16. \quad (5)$$

$$\text{Subtracting (5) } \times 3 \text{ from (1), } x^2 - 2xy + y^2 = 36, \quad (6)$$

$$\text{whence, } x - y = \pm 6. \quad (6)$$

$$\text{From (4) and (6), } x = 8 \text{ or } 2, \quad (6)$$

$$\text{and } y = 2 \text{ or } 8. \quad (6)$$

51.

$$\begin{cases} 4x^2 - 2xy + y^2 = 13, & (1) \\ 8x^3 + y^3 = 65. & (2) \end{cases}$$

$$\text{Dividing (2) by (1), } 2x + y = 5. \quad (3)$$

$$\text{Squaring (3), } 4x^2 + 4xy + y^2 = 25. \quad (4)$$

$$\text{Subtracting (1) from (4), } 6xy = 12, \quad (5)$$

$$\text{whence, } 8xy = 16. \quad (5)$$

$$\text{Subtracting (5) from (4), } 4x^2 - 4xy + y^2 = 9, \quad (6)$$

$$\text{whence, } 2x - y = \pm 3. \quad (6)$$

$$\text{From (4) and (6), } x = 2 \text{ or } \frac{1}{2}, \quad (6)$$

$$\text{and } y = 1 \text{ or } 4. \quad (6)$$

52.

$$\begin{cases} 6x^2 + 6y^2 = 13xy, & (1) \\ x^2 - y^2 = 20. & (2) \end{cases}$$

$$\text{From (1), } 6x^2 - 13xy + 6y^2 = 0. \quad (3)$$

$$\text{Factoring, } (2x - 3y)(3x - 2y) = 0. \quad (3)$$

$$\therefore y = \frac{2}{3}x \text{ or } \frac{3}{2}x. \quad (3)$$

$$\text{Substituting } \frac{2}{3}x \text{ for } y \text{ in (2), } x = 6 \text{ or } -6, \quad (3)$$

$$\text{whence, } y = 4 \text{ or } -4. \quad (3)$$

$$\text{Substituting } \frac{3}{2}x \text{ for } y \text{ in (2), } x = 4\sqrt{-1} \text{ or } -4\sqrt{-1}, \quad (3)$$

$$\text{whence, } x = 6\sqrt{-1} \text{ or } -6\sqrt{-1}. \quad (3)$$

53.

$$\begin{cases} x^2 + y^2 - 3(x + y) = 8, & (1) \\ x + y + xy = 11. & (2) \end{cases}$$

$$\text{Adding (2) } \times 2 \text{ to (1), } x^2 + 2xy + y^2 - (x + y) = 30. \quad (3)$$

$$\text{Completing the square, } (x + y)^2 - (x + y) + \frac{1}{4} = 30\frac{1}{4} = 1\frac{3}{4}. \quad (3)$$

$$\text{Extracting the square root, } x + y - \frac{1}{2} = \pm \frac{3}{2}, \quad (4)$$

$$\text{whence, } x + y = 6 \text{ or } -5. \quad (4)$$

$$\text{Subtracting (4) from (2), when } x + y = 6, \quad xy = 5, \quad (5)$$

$$\text{and when } x + y = -5, \quad xy = 16. \quad (6)$$

$$\text{To form } x^2 - 2xy + y^2, \text{ it is necessary to subtract } 2xy - 3(x + y) \text{ from (1).}$$

$$\text{From (5), when } x + y = 6, \quad 2xy - 3(x + y) = -8. \quad (7)$$

$$\text{From (6), when } x + y = -5, \quad 2xy - 3(x + y) = 47. \quad (8)$$

$$\text{Subtracting (7) from (1), } x^2 - 2xy + y^2 = 16. \quad (9)$$

$$\text{Therefore, when } x + y = 6, \quad x - y = \pm 4. \quad (9)$$

$$\text{Subtracting (8) from (1), } x^2 - 2xy + y^2 = -39. \quad (10)$$

$$\text{Therefore, when } x + y = -5, \quad x - y = \pm \sqrt{-39}. \quad (10)$$

$$\text{From (9), } x = 5 \text{ or } 1, \quad (10)$$

$$\text{and } y = 1 \text{ or } 5. \quad (10)$$

$$\text{From (10), } x = \frac{1}{2}(-5 + \sqrt{-39}) \text{ or } \frac{1}{2}(-5 - \sqrt{-39}), \quad (10)$$

$$\text{and } y = \frac{1}{2}(-5 - \sqrt{-39}) \text{ or } \frac{1}{2}(-5 + \sqrt{-39}). \quad (10)$$

$$\text{Hence, } \begin{cases} x = 5, 1, \frac{1}{2}(-5 + \sqrt{-39}), \frac{1}{2}(-5 - \sqrt{-39}); \\ y = 1, 5, \frac{1}{2}(-5 - \sqrt{-39}), \frac{1}{2}(-5 + \sqrt{-39}). \end{cases}$$

54.

$$\begin{cases} 3xy + 2x + y = 25, & (1) \\ \frac{9x}{y} = \frac{4y}{x}. & (2) \end{cases}$$

From (2),

$$9x^2 = 4y^2, \quad (3)$$

whence,

$$3x = 2y. \quad (4)$$

or

$$3x = -2y. \quad (4)$$

Substituting (3) in (1),

$$2y^2 + \frac{2}{3}(2y) + y = 25.$$

Clearing of fractions, etc.,

$$6y^2 + 7y - 75 = 0.$$

Factoring,

$$(y - 3)(6y + 25) = 0.$$

Therefore, when $3x = 2y$,

$$y = 3 \text{ or } -\frac{25}{6},$$

whence,

$$x = 2 \text{ or } -\frac{25}{9}.$$

Substituting (4) in (1),

$$-2y^2 - \frac{2}{3}(2y) + y = 25.$$

Clearing of fractions, etc.,

$$6y^2 + y + 75 = 0.$$

Solving,

$$y = \frac{-1 \pm \sqrt{1 - 4 \cdot 6 \cdot 75}}{12} = -\frac{1}{12}(1 \mp \sqrt{-1799}),$$

whence, since $3x = -2y$, $x = \frac{1}{18}(1 \mp \sqrt{-1799})$.

Hence,

$$\begin{cases} x = 2, -\frac{25}{9}, \frac{1}{18}(1 + \sqrt{-1799}), \frac{1}{18}(1 - \sqrt{-1799}); \\ y = 3, -\frac{25}{6}, -\frac{1}{12}(1 + \sqrt{-1799}), -\frac{1}{12}(1 - \sqrt{-1799}). \end{cases}$$

55.

$$\begin{cases} x^2 + xy = 40, & (1) \\ 27 + 2y^2 = 3xy. & (2) \end{cases}$$

Assume

$$x = vy. \quad (3)$$

Substituting (3) in (1),

$$v^2y^2 + vy^2 = 40. \quad (4)$$

Substituting (3) in (2),

$$27 + 2y^2 = 3vy^2. \quad (5)$$

From (4) and (5),

$$y^2 = \frac{40}{v^2 + v} = \frac{27}{3v - 2}. \quad (6)$$

Clearing of fractions, etc.,

$$27v^2 - 93v + 80 = 0.$$

Factoring,

$$(3v - 5)(9v - 16) = 0.$$

$$\therefore v = \frac{5}{3} \text{ or } \frac{16}{9}.$$

Substituting $\frac{5}{3}$ for v in (6),

$$y = 3 \text{ or } -3, \quad \left. \begin{matrix} y = 3 \text{ or } -3, \\ x = 5 \text{ or } -5, \end{matrix} \right\} \text{ when } v = \frac{5}{3}.$$

whence,

Substituting $\frac{16}{9}$ for v in (6),

$$y = \frac{9}{10}\sqrt{10} \text{ or } -\frac{9}{10}\sqrt{10}, \quad \left. \begin{matrix} y = \frac{9}{10}\sqrt{10} \text{ or } -\frac{9}{10}\sqrt{10}, \\ x = \frac{8}{5}\sqrt{10} \text{ or } -\frac{8}{5}\sqrt{10}, \end{matrix} \right\} \text{ when } v = \frac{16}{9}.$$

whence,

$$x = \frac{8}{5}\sqrt{10} \text{ or } -\frac{8}{5}\sqrt{10},$$

Hence,

$$\begin{cases} x = 5, -5, \frac{8}{5}\sqrt{10}, -\frac{8}{5}\sqrt{10}; \\ y = 3, -3, \frac{9}{10}\sqrt{10}, -\frac{9}{10}\sqrt{10}. \end{cases}$$

56.

$$\begin{cases} xy^2 + xy = 24, & (1) \\ xy^3 + x = 56. & (2) \end{cases}$$

From (1),

$$xy(y + 1) = 24. \quad (3)$$

From (2),

$$x(y^3 + 1) = 56. \quad (4)$$

Dividing (4) by (3),

$$\frac{y^2 - y + 1}{y} = \frac{7}{3}.$$

Clearing of fractions, etc.,

$$3y^2 - 10y + 3 = 0. \quad (5)$$

Factoring,

$$(y - 3)(3y - 1) = 0.$$

$$\therefore y = 3 \text{ or } \frac{1}{3}. \quad (6)$$

From (4),

$$x = \frac{56}{y^3 + 1}. \quad (7)$$

Substituting (6) in (7),

$$x = 2 \text{ or } 54.$$

57.

$$\begin{cases} x^2 + y^2 = 82, \\ x + y = 4. \end{cases} \quad (1)$$

Let

$$\begin{cases} x + y = 4, \\ x = u + v, \end{cases} \quad (2)$$

and

$$x = u + v, \quad (3)$$

$$y = u - v. \quad (4)$$

Substituting (3) and (4) in (1) and (2), and dividing by 2,

$$u^4 + 6u^2v^2 + v^4 = 41, \quad (5)$$

and

$$u = 2. \quad (6)$$

Substituting (6) in (5),

$$v^4 + 24v^2 - 25 = 0. \quad (7)$$

Factoring,

$$(v-1)(v+1)(v^2+25) = 0.$$

$$\therefore v = 1 \text{ or } -1 \text{ or } 5\sqrt{-1} \text{ or } -5\sqrt{-1}. \quad (8)$$

By (3), (6) + (8) gives,

$$x = 3 \text{ or } 1 \text{ or } 2 + 5\sqrt{-1} \text{ or } 2 - 5\sqrt{-1}.$$

By (4), (6) - (8) gives,

$$y = 1 \text{ or } 3 \text{ or } 2 - 5\sqrt{-1} \text{ or } 2 + 5\sqrt{-1}.$$

58. See next page.

59.

$$\begin{cases} x^2 + y^2 - 78 = x + y, \\ xy + x + y = 39. \end{cases} \quad (1)$$

$$xy + x + y = 39. \quad (2)$$

Adding (2) $\times 2$ to (1), etc., $x^2 + 2xy + y^2 + x + y = 156.$ (3)

Completing the square,

$$(x+y)^2 + (x+y) + \frac{1}{4} = \frac{156\frac{1}{4}}{4}.$$

Extracting the square root,

$$x + y + \frac{1}{2} = \pm \frac{13}{2}.$$

$$\therefore x + y = 12 \quad (4)$$

or

$$x + y = -13. \quad (5)$$

When $x + y = 12$, from (2),

$$xy = 27. \quad (6)$$

When $x + y = -13$, from (2),

$$xy = 52. \quad (7)$$

The square of $(x - y)$ may be obtained by adding $x + y - 2xy =$ its value to (1).

From (6) and (4),

$$x + y - 2xy = -42. \quad (8)$$

From (7) and (5),

$$x + y - 2xy = -117. \quad (9)$$

Adding (8) to (1),

$$x^2 - 2xy + y^2 = 36; \quad (10)$$

therefore, when $x + y = 12$,

$$x - y = \pm 6.$$

Adding (9) to (1),

$$x^2 - 2xy + y^2 = -39; \quad (11)$$

therefore, when $x + y = -13$,

$$x - y = \pm \sqrt{-39}.$$

From (4) and (10),

$$x = 9 \text{ or } 3 \text{ and } y = 3 \text{ or } 9.$$

From (5) and (11), $x = \frac{1}{2}(-13 + \sqrt{-39})$ or $\frac{1}{2}(-13 - \sqrt{-39})$,

and

$$y = \frac{1}{2}(-13 - \sqrt{-39}) \text{ or } \frac{1}{2}(-13 + \sqrt{-39}).$$

60.

$$\begin{cases} x^2 + 2xy + 3y^2 = 43, \\ 2x^2 + 3xy + 4y^2 = 62. \end{cases} \quad (1)$$

$$x^2 + xy + y^2 = 19. \quad (2)$$

Subtracting (1) from (2),

$$xy + 2y^2 = 24. \quad (3)$$

Subtracting (2) from (1) $\times 2$,

$$xy + 2y^2 = 24. \quad (4)$$

Multiplying (3) by 24,

$$24x^2 + 24xy + 24y^2 = 24 \cdot 19. \quad (5)$$

Multiplying (4) by 19,

$$19xy + 38y^2 = 19 \cdot 24. \quad (6)$$

Subtracting (6) from (5),

$$24x^2 + 5xy - 14y^2 = 0. \quad (7)$$

Factoring,

$$(3x - 2y)(8x + 7y) = 0.$$

$$\therefore x = \frac{2}{3}y \text{ or } -\frac{7}{8}y. \quad (8)$$

Substituting $\frac{2}{3}y$ for x in (4),

$$y = 3 \text{ or } -3,$$

whence,

$$x = 2 \text{ or } -2.$$

Substituting $-\frac{7}{8}y$ for x in (4),

$$y = \frac{8}{3}\sqrt{3} \text{ or } -\frac{8}{3}\sqrt{3},$$

whence,

$$x = -\frac{7}{3}\sqrt{3} \text{ or } \frac{7}{3}\sqrt{3}.$$

Hence,

$$\begin{cases} x = 2, -2, \frac{7}{3}\sqrt{3}, -\frac{7}{3}\sqrt{3}; \\ y = 3, -3, -\frac{8}{3}\sqrt{3}, \frac{8}{3}\sqrt{3}. \end{cases}$$

58. $\begin{cases} x^4 - y^4 = 369, \\ x^2 - y^2 = 9. \end{cases}$ (1)
 Dividing (1) by (2), $x^2 + y^2 = 41.$ (2)
 From (2) and (3), $x^2 = 25,$ (3)
 whence, $x = \pm 5;$ (4)
 also $y^2 = 16,$
 whence, $y = \pm 4.$ (5)
 Since (4) and (5) are derived separately, $x = 5, 5, -5, -5,$
 when $y = 4, -4, 4, -4;$
 that is, $x = \pm 5, y = \pm 4.$

61. $\begin{cases} x^2 - xy + 2y^2 = 46, \\ x^2 + xy + 3y^2 = 111. \end{cases}$ (1)
 Multiplying (1) by 111, $111x^2 - 111xy + 222y^2 = 111 \cdot 46.$ (2)
 Multiplying (2) by 46, $46x^2 + 46xy + 138y^2 = 46 \cdot 111.$ (3)
 Subtracting (4) from (3), $65x^2 - 157xy + 84y^2 = 0.$ (4)
 Factoring, $(5x - 4y)(13x - 21y) = 0.$
 $\therefore x = \frac{4}{13}y$ or $\frac{21}{13}y.$ (5)
 Subtracting (1) from (2), $2xy + y^2 = 65.$ (6)
 Substituting $\frac{4}{13}y$ for x in (6), $y = 5$ or $-5,$
 whence, $x = 4$ or $-4.$
 Substituting $\frac{21}{13}y$ for x in (6), $y = \frac{13}{11}\sqrt{11}$ or $-\frac{13}{11}\sqrt{11},$
 whence, $x = \frac{21}{11}\sqrt{11}$ or $-\frac{21}{11}\sqrt{11}.$

62. $\begin{cases} x^2 - 7xy + 12y^2 = 0, \\ xy + 3y = 2x + 21. \end{cases}$ (1)
 Factoring (1), $(x - 4y)(x - 3y) = 0.$ (2)
 $\therefore x = 4y$ or $3y.$ (3)
 Substituting $4y$ for x in (2), $4y^2 - 5y - 21 = 0.$
 Factoring, $(y - 3)(4y + 7) = 0.$
 $\therefore y = 3$ or $-\frac{7}{4},$
 whence, $x = 12$ or $-\frac{7}{4}.$
 Substituting $3y$ for x in (2), $y^2 - y = 7.$
 Completing the square, $y^2 - y + \frac{1}{4} = \frac{29}{4}.$
 Solving, $y = \frac{1}{2}(1 + \sqrt{29})$ or $\frac{1}{2}(1 - \sqrt{29}),$
 whence, $x = \frac{3}{2}(1 + \sqrt{29})$ or $\frac{3}{2}(1 - \sqrt{29}).$

63. See next page.

64. $\begin{cases} x + y = 25, \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$ (1)
 Squaring (2), $x + 2\sqrt{xy} + y = 49.$ (2)
 Multiplying (1) by (2), $2x + 2y = 50.$ (3)
 Subtracting (3) from (4), $x - 2\sqrt{xy} + y = 1,$ (4)
 whence, $\sqrt{x} - \sqrt{y} = \pm 1.$ (5)
 From (2) and (5), $\sqrt{x} = 4$ or $3,$
 and $\sqrt{y} = 3$ or $4.$
 Hence, $x = 16$ or $9,$
 and $y = 9$ or $16.$

$$63. \quad \begin{cases} x^3 - y^3 = 37, \\ xy(y - x) = -12. \end{cases} \quad (1)$$

$$\text{Changing signs in (2),} \quad xy(x - y) = 12. \quad (2)$$

$$\text{Dividing (1) by (3),} \quad \frac{x^2 + xy + y^2}{xy} = \frac{37}{12}. \quad (3)$$

$$\text{Clearing of fractions, etc.,} \quad 12x^2 - 25xy + 12y^2 = 0.$$

$$\text{Factoring,} \quad (3x - 4y)(4x - 3y) = 0. \quad (4)$$

$$\text{Substituting } \frac{3}{4}x \text{ for } y \text{ in (3),} \quad \therefore y = \frac{3}{4}x \text{ or } \frac{4}{3}x. \quad (4)$$

$$\text{Transposing and factoring, } (x - 4)(x^2 + 4x + 16) = 0. \quad (5)$$

Equating each factor to zero and solving,

$$x = 4 \text{ or } 2(-1 + \sqrt{-3}) \text{ or } 2(-1 - \sqrt{-3}),$$

$$\text{whence, since } y = \frac{3}{4}x, \quad y = 3 \text{ or } \frac{3}{2}(-1 + \sqrt{-3}) \text{ or } \frac{3}{2}(-1 - \sqrt{-3}).$$

$$\text{Substituting } \frac{4}{3}x \text{ for } y \text{ in (2),} \quad x^3 = -27.$$

$$\text{Transposing and factoring, } (x + 3)(x^2 - 3x + 9) = 0.$$

Equating each factor to zero and solving,

$$x = -3 \text{ or } \frac{3}{2}(1 + \sqrt{-3}) \text{ or } \frac{3}{2}(1 - \sqrt{-3}),$$

$$\text{whence, since } y = \frac{4}{3}x, \quad y = -4 \text{ or } 2(1 + \sqrt{-3}) \text{ or } 2(1 - \sqrt{-3}).$$

$$\text{Hence, } \begin{cases} x = 4, -3, 2(-1 + \sqrt{-3}), 2(-1 - \sqrt{-3}), \frac{3}{2}(1 + \sqrt{-3}), \frac{3}{2}(1 - \sqrt{-3}); \\ y = 3, -4, \frac{3}{2}(-1 + \sqrt{-3}), \frac{3}{2}(-1 - \sqrt{-3}), 2(1 + \sqrt{-3}), 2(1 - \sqrt{-3}). \end{cases}$$

$$65. \quad \begin{cases} x^3 + y^3 = 225y, \\ x^2 - y^2 = 75. \end{cases} \quad (1)$$

$$\text{Dividing (1) by (2), } \frac{x^3 + y^3}{x^2 - y^2} = \frac{x^2 - xy + y^2}{x - y} = 3y. \quad (2)$$

$$\text{Clearing of fractions, etc.,} \quad x^2 - 4xy + 4y^2 = 0.$$

$$\text{Factoring,} \quad (x - 2y)(x - 2y) = 0. \quad (4)$$

$$\therefore x = 2y. \quad (5)$$

$$\text{Substituting (5) in (2),} \quad y = 5 \text{ or } -5, \quad (6)$$

$$\text{whence,} \quad x = 10 \text{ or } -10. \quad (7)$$

Since the factors of (4) are equal, each of the values in (6) and (7) is twice used as a root.

66. See next page.

$$67. \quad \begin{cases} x^2 + y^2 = 3xy + 5, \\ x^4 + y^4 = 2. \end{cases} \quad (1)$$

$$\text{Let} \quad x = u + v, \quad (2)$$

$$\text{and} \quad y = u - v. \quad (3)$$

$$\text{Substituting (3) and (4) in (1),} \quad u^2 = 5(v^2 - 1), \quad (4)$$

$$\text{and in (2),} \quad u^4 + 6u^2v^2 + v^4 = 1. \quad (5)$$

$$\text{Substituting (5) in (6),} \quad 56v^4 - 80v^2 + 24 = 0. \quad (6)$$

$$7v^4 - 10v^2 + 3 = 0.$$

$$(v - 1)(v + 1)(7v^2 - 3) = 0.$$

$$\therefore v = 1, -1, \frac{1}{7}\sqrt{21}, -\frac{1}{7}\sqrt{21}. \quad (7)$$

$$\text{Substituting (7) in (5),} \quad u = 0, 0, \pm \frac{2}{7}\sqrt{-35}, \pm \frac{2}{7}\sqrt{-35}. \quad (8)$$

$$\therefore \begin{cases} x = u + v = 1, -1, \frac{1}{7}(\pm 2\sqrt{-35} + \sqrt{21}), \frac{1}{7}(\pm 2\sqrt{-35} - \sqrt{21}); \\ y = u - v = -1, 1, \frac{1}{7}(\pm 2\sqrt{-35} - \sqrt{21}), \frac{1}{7}(\pm 2\sqrt{-35} + \sqrt{21}). \end{cases}$$

$$66. \quad \begin{cases} x^2 + xy + 2y^2 = 11, & (1) \\ 2x^2 + 5y^2 = 22. & (2) \end{cases}$$

$$\text{Multiplying (1) by 2,} \quad 2x^2 + 2xy + 4y^2 = 22. \quad (3)$$

$$\text{Subtracting (2) from (3),} \quad 2xy - y^2 = 0. \quad (4)$$

$$\text{Factoring,} \quad y(2x - y) = 0. \quad (5)$$

$$\therefore y = 0 \text{ or } 2x.$$

$$\text{Substituting 0 for } y \text{ in (2),} \quad x = \sqrt{11} \text{ or } -\sqrt{11}.$$

$$\text{Substituting } 2x \text{ for } y \text{ in (2),} \quad x = 1 \text{ or } -1,$$

$$\text{whence,} \quad y = 2 \text{ or } -2.$$

$$\text{Hence,} \quad \begin{cases} x = \sqrt{11}, -\sqrt{11}, 1, -1; \\ y = 0, 0, 2, -2. \end{cases}$$

$$68. \quad \begin{cases} x^2 + y^2 = \frac{13}{x-y}, & (1) \\ x^2y - xy^2 = 6. & (2) \end{cases}$$

$$\text{From (1),} \quad (x^2 + y^2)(x - y) = 13. \quad (3)$$

$$\text{From (2),} \quad xy(x - y) = 6. \quad (4)$$

$$\text{Multiplying (3) by 6,} \quad (6x^2 + 6y^2)(x - y) = 78. \quad (5)$$

$$\text{Multiplying (4) by 13,} \quad 13xy(x - y) = 78. \quad (6)$$

$$\text{Subtracting (6) from (5),} \quad (6x^2 - 13xy + 6y^2)(x - y) = 0.$$

$$\text{Factoring,} \quad (2x - 3y)(3x - 2y)(x - y) = 0.$$

$$\therefore x = \frac{2}{3}y \text{ or } \frac{3}{2}y \text{ or } y.$$

$$\text{Substituting } \frac{2}{3}y \text{ for } x \text{ in (4),} \quad y^3 = 8.$$

$$\text{Transposing and factoring,} \quad (y - 2)(y^2 + 2y + 4) = 0.$$

$$\therefore y = 2 \text{ or } -1 + \sqrt{-3} \text{ or } -1 - \sqrt{-3},$$

$$\text{whence,} \quad x = 3 \text{ or } \frac{2}{3}(-1 + \sqrt{-3}) \text{ or } \frac{2}{3}(-1 - \sqrt{-3}).$$

$$\text{Substituting } \frac{3}{2}y \text{ for } x \text{ in (4),} \quad y^3 = -27.$$

$$\text{Transposing and factoring,} \quad (y + 3)(y^2 - 3y + 9) = 0.$$

$$\therefore y = -3 \text{ or } \frac{1}{2}(1 + \sqrt{-3}) \text{ or } \frac{1}{2}(1 - \sqrt{-3}),$$

$$\text{whence,} \quad x = -2 \text{ or } 1 + \sqrt{-3} \text{ or } 1 - \sqrt{-3}.$$

$$\text{Substituting } y \text{ for } x \text{ in (4),} \quad 0 \cdot y^2 = 6.$$

$$\therefore x = y = \pm \sqrt[3]{6} = \pm \infty. \quad (\text{See note, Ex. 40.})$$

$$\text{Hence,} \quad \begin{cases} x = 3, -2, \frac{2}{3}(-1 + \sqrt{-3}), \frac{2}{3}(-1 - \sqrt{-3}), 1 + \sqrt{-3}, \\ \quad 1 - \sqrt{-3}, +\infty, -\infty; \\ y = 2, -3, -1 + \sqrt{-3}, -1 - \sqrt{-3}, \frac{3}{2}(1 + \sqrt{-3}), \\ \quad \frac{3}{2}(1 - \sqrt{-3}), +\infty, -\infty. \end{cases}$$

$$69. \quad \begin{cases} (x + y)(x^2 + y^2) = 65, & (1) \\ (x - y)(x^2 - y^2) = 5. & (2) \end{cases}$$

$$\text{Dividing (1) by (2),} \quad \frac{x^2 + y^2}{(x - y)^2} = 13. \quad (3)$$

$$\text{Clearing of fractions, etc.,} \quad 6x^2 - 13xy + 6y^2 = 0.$$

$$\text{Factoring,} \quad (2x - 3y)(3x - 2y) = 0.$$

$$\therefore x = \frac{2}{3}y \text{ or } \frac{3}{2}y. \quad (4)$$

Substituting $\frac{2}{3}y$ for x in (2), $y^3 = 8$.

Transposing and factoring, $(y - 2)(y^2 + 2y + 4) = 0$.

$$\therefore y = 2 \text{ or } -1 + \sqrt{-3} \text{ or } -1 - \sqrt{-3},$$

whence, $x = 3 \text{ or } \frac{2}{3}(-1 + \sqrt{-3}) \text{ or } \frac{2}{3}(-1 - \sqrt{-3})$.

Substituting $\frac{2}{3}y$ for x in (2), $y^3 = 27$.

Transposing and factoring, $(y - 3)(y^2 + 3y + 9) = 0$.

$$\therefore y = 3 \text{ or } \frac{2}{3}(-1 + \sqrt{-3}) \text{ or } \frac{2}{3}(-1 - \sqrt{-3}),$$

whence, $x = 2 \text{ or } -1 + \sqrt{-3} \text{ or } -1 - \sqrt{-3}$.

$$\text{Hence, } \begin{cases} x = 3, 2, \frac{2}{3}(-1 + \sqrt{-3}), \frac{2}{3}(-1 - \sqrt{-3}), -1 + \sqrt{-3}, \\ \quad \quad \quad -1 - \sqrt{-3}; \\ y = 2, 3, -1 + \sqrt{-3}, -1 - \sqrt{-3}, \frac{2}{3}(-1 + \sqrt{-3}), \\ \quad \quad \quad \frac{2}{3}(-1 - \sqrt{-3}). \end{cases}$$

70.

$$\begin{cases} x^2 + y = x - y^2 + 42, & (1) \\ xy = 20. & (2) \end{cases}$$

From (1), transposing, $x^2 + y^2 - (x - y) = 42$. (3)

Subtracting (2) $\times 2$ from (3), and completing the square,

$$x^2 - 2xy + y^2 - (x - y) + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}. \quad (4)$$

Extracting the square root, $x - y - \frac{1}{2} = \pm \frac{3}{2}$,

whence, $x - y = 2 \text{ or } -1$. (5)

Squaring (5), $x^2 - 2xy + y^2 = 4 \text{ or } 1$. (6)

Adding (2) $\times 4$ to (6), $x^2 + 2xy + y^2 = 84 \text{ or } 81$,

whence, $x + y = \pm 2\sqrt{21} \text{ or } \pm 9$. (7)

$$\text{Hence, } \begin{cases} x + y = 9, & \text{or } \begin{cases} x + y = -9, & \text{or } \begin{cases} x + y = 2\sqrt{21}, \\ x - y = 2, \end{cases} \\ x - y = -1, & \text{or } \begin{cases} x + y = -2\sqrt{21}, \\ x - y = 2. \end{cases} \end{cases} \end{cases}$$

Solving these equations, $x = 4, -5, 1 + \sqrt{21}, 1 - \sqrt{21}$,

and $y = 5, -4, -1 + \sqrt{21}, -1 - \sqrt{21}$.

71.

$$\begin{cases} x + y + 2\sqrt{x + y} = 24, & (1) \\ x - y + 3\sqrt{x - y} = 10. & (2) \end{cases}$$

Completing squares, $x + y + 2\sqrt{x + y} + 1 = 25$, (3)

and $x - y + 3\sqrt{x - y} + \frac{9}{4} = \frac{49}{4}$. (4)

Extracting square roots, $\sqrt{x + y} + 1 = \pm 5$,

whence, $\sqrt{x + y} = 4 \text{ or } -6$; (5)

and $\sqrt{x - y} + \frac{3}{2} = \pm \frac{7}{2}$,

whence, $\sqrt{x - y} = 2 \text{ or } -5$. (6)

Squaring (5), $x + y = 16 \text{ or } 36$. (7)

Squaring (6), $x - y = 4 \text{ or } 25$. (8)

Since (7) and (8) have been derived separately, we have the equations

$$\begin{cases} x + y = 16, & \text{or } \begin{cases} x + y = 16, & \text{or } \begin{cases} x + y = 36, & \text{or } \begin{cases} x + y = 36, \\ x - y = 25. \end{cases} \end{cases} \\ x - y = 4, & \text{or } \begin{cases} x + y = 16, & \text{or } \begin{cases} x + y = 36, \\ x - y = 4. \end{cases} \end{cases} \end{cases}$$

Solving these equations, $x = 10, \frac{41}{2}, 20, \frac{61}{2}$,

and $y = 6, -\frac{9}{2}, 16, \frac{11}{2}$.

$$72. \quad \begin{cases} x^2 + y^2 + 6\sqrt{x^2 + y^2} = 55, \\ x^2 - y^2 = 7. \end{cases} \quad (1)$$

(2)

Completing the square in (1),

$$x^2 + y^2 + 6\sqrt{x^2 + y^2} + 9 = 64. \quad (3)$$

Extracting the square root, $\sqrt{x^2 + y^2} + 3 = \pm 8$,

whence,

$$\sqrt{x^2 + y^2} = 5 \text{ or } -11. \quad (4)$$

Squaring (4),

$$x^2 + y^2 = 25 \text{ or } 121. \quad (5)$$

From (5) and (2),

$$x^2 = 16 \text{ or } 64,$$

and

$$y^2 = 9 \text{ or } 57,$$

that is,

$$\begin{cases} x^2 = 16, & \text{or} \\ y^2 = 9, & \end{cases} \quad \text{or} \quad \begin{cases} x^2 = 64, \\ y^2 = 57. \end{cases}$$

From these equations,

$$x = \pm 4, \pm 8;$$

and

$$y = \pm 3, \pm \sqrt{57}.$$

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$$73. \quad \begin{cases} \frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^3 - x^2y}{y^3 - x^2y} = \frac{3}{8}, \\ x^2 = y^2 + 16. \end{cases} \quad (1)$$

(2)

Since $\frac{x^3 - x^2y}{y^3 - x^2y} = -\frac{x^3 - x^2y}{x^2y - y^3} = -\frac{x^2(x-y)}{y(x^2 - y^2)} = -\frac{x^2}{y(x+y)}$, (1) becomes,

$$\frac{x+y}{y} - \frac{2x}{x+y} - \frac{x^2}{y(x+y)} = \frac{3}{8}. \quad (3)$$

Uniting terms in (3),

$$\frac{y^2}{y(x+y)} = \frac{y}{x+y} = \frac{3}{8}. \quad (4)$$

Clearing of fractions, etc.,

$$y = \frac{3}{8}x. \quad (5)$$

Substituting (5) in (2),

$$x^2 = \frac{9}{64}x^2 + 16.$$

$$x^2 = 25.$$

$$\therefore x = 5 \text{ or } -5,$$

$$y = 3 \text{ or } -3.$$

whence, by (5),

$$74. \quad \begin{cases} x^2 - xy = a^2 + b^2, \\ xy - y^2 = 2ab. \end{cases} \quad (1)$$

(2)

Subtracting (2) from (1),

$$x^2 - 2xy + y^2 = a^2 - 2ab + b^2,$$

whence,

$$x - y = a - b \text{ or } -(a - b). \quad (3)$$

Adding (2) to (1),

$$x^2 - y^2 = a^2 + 2ab + b^2. \quad (4)$$

Dividing (4) by (3),

$$x + y = \frac{(a+b)^2}{a-b} \text{ or } -\frac{(a+b)^2}{a-b}. \quad (5)$$

From (3),

$$x - y = \frac{(a-b)^2}{a-b} \text{ or } -\frac{(a-b)^2}{a-b}. \quad (6)$$

From (5) and (6),

$$x = \frac{a^2 + b^2}{a-b} \text{ or } -\frac{a^2 + b^2}{a-b},$$

and

$$y = \frac{2ab}{a-b} \text{ or } -\frac{2ab}{a-b}.$$

75.

$$\begin{cases} x - 2y = 2(a+b), \\ xy + 2y^2 = 2b(b-a). \end{cases} \quad (1)$$

Multiplying (1) by y ,

$$xy - 2y^2 = 2y(a+b). \quad (2)$$

Subtracting (3) from (2),

$$4y^2 = 2b(b-a) - 2y(a+b). \quad (3)$$

Transposing, etc., $2y^2 + (a+b)y + b(a-b) = 0$. (4)

Factoring (4), $(y+b)(2y+a-b) = 0$.

$$\therefore y = -b \text{ or } \frac{b-a}{2}. \quad (5)$$

From (1), $x = 2(y+a+b)$. (6)

Substituting (5) in (6), $x = 2a \text{ or } a+3b$.

Hence,
$$\begin{cases} x = 2a, a+3b; \\ y = -b, \frac{b-a}{2}. \end{cases}$$

76.

$$\begin{cases} x^3 + y^3 = 2a(a^2 + 3b^2), \\ x^2y + xy^2 = 2a(a^2 - b^2). \end{cases} \quad (1)$$

Multiplying (2) by 3,

$$3x^2y + 3xy^2 = 2a(3a^2 - 3b^2). \quad (3)$$

Adding (1) and (3), $x^3 + 3x^2y + 3xy^2 + y^3 = 2a \cdot 4a^2 = 8a^3$. (4)

Extracting the cube root,

$$x + y = 2a. \quad (5)$$

Dividing (2) by (5),

$$xy = a^2 - b^2. \quad (6)$$

Squaring (5),

$$x^2 + 2xy + y^2 = 4a^2. \quad (7)$$

Subtracting (6) $\times 4$ from (7), $x^2 - 2xy + y^2 = 4b^2$,

$$x - y = \pm 2b. \quad (8)$$

whence,

From (5) and (8), $x = a + b \text{ or } a - b$,

and $y = a - b \text{ or } a + b$.

The roots just given and others may be obtained as follows; but the student will not be able to perform the operations involved until he has studied the subject of imaginary and complex numbers.

From (4), $(x+y)^3 - (2a)^3 = 0$. (9)

Putting $x+y = z$ in (9), $z^3 - (2a)^3 = 0$.

Factoring, $(z-2a)(z^2+2az+4a^2) = 0$.

Equating each factor to zero and solving,

$$z = x+y = 2a \text{ or } a(-1+\sqrt{-3}) \text{ or } a(-1-\sqrt{-3}). \quad (10)$$

Dividing (2) by (10), the corresponding values of xy are

$$xy = a^2 - b^2 \text{ or } \frac{1}{2}(a^2 - b^2)(-1-\sqrt{-3}) \text{ or } \frac{1}{2}(a^2 - b^2)(-1+\sqrt{-3}). \quad (11)$$

For $x+y = 2a$ and $xy = a^2 - b^2$ the solution is given above.

Squaring (10), $(x+y)^2 = a^2(-1+\sqrt{-3})^2 \text{ or } a^2(-1-\sqrt{-3})^2$
 $= 2a^2(-1-\sqrt{-3}) \text{ or } 2a^2(-1+\sqrt{-3}). \quad (12)$

(11) $\times 4$, $4xy = (2a^2 - 2b^2)(-1-\sqrt{-3}) \text{ or } (2a^2 - 2b^2)(-1+\sqrt{-3}). \quad (13)$

(12) - (13), $(x-y)^2 = 2b^2(-1-\sqrt{-3}) \text{ or } 2b^2(-1+\sqrt{-3})$
 $= b^2(-1+\sqrt{-3})^2 \text{ or } b^2(-1-\sqrt{-3})^2. \quad (14)$

whence,

$$x - y = \pm b(-1+\sqrt{-3}) \text{ or } \pm b(-1-\sqrt{-3}). \quad (15)$$

From (10) and (15), taken with values previously obtained,

$$\begin{cases} x = a+b, a-b, \frac{1}{2}(a+b)(-1+\sqrt{-3}), \frac{1}{2}(a-b)(-1+\sqrt{-3}), \\ \quad \frac{1}{2}(a+b)(-1-\sqrt{-3}), \frac{1}{2}(a-b)(-1-\sqrt{-3}); \\ y = a-b, a+b, \frac{1}{2}(a-b)(-1+\sqrt{-3}), \frac{1}{2}(a+b)(-1+\sqrt{-3}), \\ \quad \frac{1}{2}(a-b)(-1-\sqrt{-3}), \frac{1}{2}(a+b)(-1-\sqrt{-3}). \end{cases}$$

Each value of y differs from the corresponding value of x only in the sign of b .

$$77. \quad \begin{cases} \frac{a-x}{x} + \frac{b+y}{y} = \frac{ay+bx}{a^2-b^2}, \\ x^2+y^2 = 2(a^2+b^2). \end{cases} \quad (1)$$

$$\text{Reducing (1),} \quad \begin{aligned} \frac{a}{x} - 1 + \frac{b}{y} + 1 &= \frac{ay+bx}{a^2-b^2}, \\ \frac{ay+bx}{xy} &= \frac{ay+bx}{a^2-b^2}. \end{aligned} \quad (2)$$

$$\text{Multiplying (3) by 2,} \quad \therefore xy = a^2 - b^2. \quad (3)$$

$$\text{Adding (4) to (2),} \quad 2xy = 2(a^2 - b^2). \quad (4)$$

$$\text{whence,} \quad x^2 + 2xy + y^2 = 4a^2, \quad (5)$$

$$\text{Subtracting (4) from (2),} \quad x + y = \pm 2a. \quad (5)$$

$$\text{whence,} \quad x^2 - 2xy + y^2 = 4b^2, \quad (6)$$

Since (5) and (6) have been derived separately, neither from the other,

$$\begin{cases} x+y=2a, \text{ or } x+y=2a, \text{ or } x+y=-2a, \text{ or } x+y=-2a, \\ x-y=2b, \text{ or } x-y=-2b, \text{ or } x-y=2b, \text{ or } x-y=-2b. \end{cases}$$

Solving these equations, $x = a + b, a - b, -a + b, -a - b,$

and $y = a - b, a + b, -a - b, -a + b,$

the values of x and y in each case differing in the sign of b .

1. Let

and

Then,

and

Solving,

and

Hence, the numbers are 4 and 8.

$x =$ one number,

$y =$ the other.

$$x + y = 12,$$

$$xy = 32.$$

$$x = 8 \text{ or } 4,$$

$$y = 4 \text{ or } 8.$$

2. Let

and

Then,

and

Solving,

and

Hence, the numbers are 6 and 11.

$x =$ one number,

$y =$ the other.

$$x + y = 17,$$

$$x^2 + y^2 = 157.$$

$$x = 11 \text{ or } 6,$$

$$y = 6 \text{ or } 11.$$

3. Let

and

Then,

and

Solving,

and

Hence, the numbers are 6 and 5, or -5 and -6 .

$x =$ the greater number,

$y =$ the less number.

$$x - y = 1,$$

$$x^2 - y^2 = 91.$$

$$x = 6 \text{ or } -5,$$

$$y = 5 \text{ or } -6.$$

4. Let

and

Then,

and

Solving,

and

Hence, the numbers are 81 and 1.

$x =$ one number,

$y =$ the other.

$$x + y = 82,$$

$$\sqrt{x} + \sqrt{y} = 10.$$

$$x = 81 \text{ or } 1,$$

$$y = 1 \text{ or } 81.$$

5. Let x = number of rods in length of garden,
and y = number of rods in width of garden.

Then, since the length and width together are one half of the distance around the garden, and their product is the area in square rods,

$$x + y = 26,$$

$$\text{and } xy = 160.$$

$$\text{Solving, } x = 16 \text{ or } 10,$$

$$\text{and } y = 10 \text{ or } 16.$$

Hence, the garden is 16 rods long and 10 rods wide.

6. Let x = the numerator,
and y = the denominator.

$$\text{Then, } x^2 + y^2 = 89, \quad (1)$$

and since $\frac{x}{y}$ represents the fraction and $\frac{y}{x}$ its reciprocal,

$$\frac{x}{y} - \frac{y}{x} = \frac{39}{40}. \quad (2)$$

$$\text{Clearing (2) of fractions, etc., } 40x^2 - 39xy - 40y^2 = 0.$$

$$\text{Factoring, } (5x - 8y)(8x + 5y) = 0.$$

$$\therefore y = \frac{5}{8}x \text{ or } -\frac{8}{5}x. \quad (3)$$

$$\text{Substituting (3) in (1), } x = 8 \text{ or } -8 \text{ or } 5 \text{ or } -5,$$

$$\text{whence, } y = 5 \text{ or } -5 \text{ or } -8 \text{ or } 8.$$

Hence, the fraction is $\frac{8}{5}$ or $-\frac{5}{8}$.

7. Let x = one number,
and y = the other.

$$\text{Then, } xy = 2(x + y) + 8, \quad (1)$$

$$\text{and } xy = x^2 + y^2 - 48. \quad (2)$$

$$\text{Multiplying (1) by 3, } 3xy = 6(x + y) + 24. \quad (3)$$

$$\text{Subtracting (2) from (3), } 2xy = -(x^2 + y^2) + 6(x + y) + 72.$$

$$\text{Transposing, } x^2 + 2xy + y^2 - 6(x + y) - 72 = 0.$$

$$\text{Factoring, } (x + y - 12)(x + y + 6) = 0.$$

$$\therefore x + y = 12 \text{ or } -6. \quad (4)$$

$$\text{Substituting (4) in (1), } xy = 32 \text{ or } -4. \quad (5)$$

$$\text{Solving (4) and (5), } x = 8, 4, -3 + \sqrt{13}, -3 - \sqrt{13},$$

$$y = 4, 8, -3 - \sqrt{13}, -3 + \sqrt{13}.$$

Hence, the numbers are 8 and 4 or $-3 + \sqrt{13}$ and $-3 - \sqrt{13}$.

8. Let x = tens' digit,
and y = units' digit.

$$\text{Then, } 10x + y - 63 = 10y + x, \quad (1)$$

$$\text{and } (10x + y)(x + y) = 729. \quad (2)$$

$$\text{From (1), } y = x - 7. \quad (3)$$

$$\text{Substituting (3) in (2), } (11x - 7)(2x - 7) = 729. \quad (4)$$

$$\text{Solving (4), } x = 8 \text{ or } -\frac{15}{2}.$$

$$\text{Rejecting the second value, from (3), } y = 1.$$

Hence, the number is 81.

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9. Let x = number of yards,
 and y = number of cents per yard.
 Then, $xy = 600$, (1)
 and $(x + 5)(y - 4) = 600$. (2)
 Subtracting (2) from (1), $4x - 5y + 20 = 0$, (3)
 whence, $5y = 4x + 20$, (4)
 From (1), $x \cdot 5y = 3000$. (5)
 Substituting (4) in (5), $4x^2 + 20x = 3000$,
 whence, $x = 25$ or -30 . (6)
 Substituting (6) in (1), $y = 24$ or -20 .
 Hence, the man bought 25 yards at 24 cents per yard.

10. Let x = the greater number,
 and y = the less number.
 Then, $x(x - y) = 160$, (1)
 and $y(x - y) = 96$. (2)
 Dividing (2) by (1), etc., $y = \frac{96}{160}x = \frac{3}{5}x$. (3)
 Substituting (3) in (1), $x = 20$ or -20 ,
 whence, by (3), $y = 12$ or -12 .
 Hence, the numbers are 20 and 12. The second values are inadmissible, because -20 is not greater than -12 .

11. Let x = number of rods in length of garden,
 and y = number of rods in width.
 Then, $xy = 54$, (1)
 and since an addition of $1\frac{1}{2}$ rods on all sides makes each dimension 3 rods greater and doubles the area, $(x + 3)(y + 3) = 108$. (2)
 Subtracting (1) from (2), $3x + 3y + 9 = 54$,
 whence, $x + y = 15$. (3)
 Solving (3) and (1), $x = 9$ or 6 ,
 and $y = 6$ or 9 .
 Hence, the garden was 9 rods long and 6 rods wide.

12. Let x = number of rods in length of field,
 and y = number of rods in width of field.
 Then, $x + y = 100$,
 and $xy = 15 \cdot 160 = 2400$.
 Solving, $x = 60$ or 40 ,
 and $y = 40$ or 60 .
 Hence, the field is 60 rods by 40 rods.

13. Let x = number of rods in length of field,
 and y = number of rods in width of field.
 Then, $xy = 20 \cdot 160 = 3200$, (1)
 and $(x - 20)(y - 8) = 12 \cdot 160 = 1920$. (2)
 Subtracting (2) from (1), $8x + 20y - 160 = 1280$.
 Solving (1) and (3), $2x + 5y = 360$. (3)
 and $x = 100$ or 80 ,
 and $y = 32$ or 40 .
 Hence, the field is 100 rods by 32 rods, or 80 rods by 40 rods.

14. Let x = number of dollars asked for each cow,
and y = number of dollars asked for each sheep.

Then,
$$\frac{100}{y} - \frac{100}{x} = 16, \quad (1)$$

and
$$3x - 12y = 15. \quad (2)$$

Solving,
$$x = 25 \text{ or } -\frac{5}{4},$$

and
$$y = 5 \text{ or } -\frac{3}{8}.$$

Hence, \$25 was the price of a cow, and \$5 the price of a sheep.

15. See next page.

16. Let x = number of dollars in larger loan,
and $y\%$ = rate of interest on larger loan.

Then,
$$\frac{xy}{100} = \text{number of dollars interest yielded by each investment}$$

$$1000 - x = \text{number of dollars in smaller loan,}$$

and
$$\frac{xy}{100} \div (1000 - x), \text{ or } \frac{xy}{1000 - x} \% = \text{rate of interest on smaller loan;}$$

$$\therefore \frac{x \cdot xy}{1000 - x} = 3600, \quad (1)$$

and
$$(1000 - x)y = 1600. \quad (2)$$

Multiplying (1) by (2),
$$x^2 y^2 = 36 \cdot 16 \cdot 100 \cdot 100,$$

whence,
$$xy = \pm 6 \cdot 4 \cdot 100 = \pm 2400. \quad (3)$$

Since the product of principal and rate cannot be negative, the negative value in (3) is rejected.

Adding (3) and (2),
$$1000y = 4000.$$

$$\therefore y = 4. \quad (4)$$

Substituting (4) in (3),
$$x = 600. \quad (5)$$

From (4) and (5),
$$1000 - x = 400,$$

and
$$\frac{xy}{1000 - x} = \frac{2400}{400} = 6.$$

Hence, the sums invested were \$600 at 4% and \$400 at 6%.

17. Let x = the numerator,
and y = the denominator.

Then,
$$\frac{x+2}{y-2} = \frac{y}{x}, \quad (1)$$

and
$$\frac{x-3}{y+3} = \frac{3x}{7y}. \quad (2)$$

Clearing (1) of fractions,
$$x^2 + 2x = y^2 - 2y. \quad (3)$$

Completing squares,
$$x^2 + 2x + 1 = y^2 - 2y + 1.$$

Extracting the square root,
$$x + 1 = y - 1 \text{ or } -y + 1. \quad (4)$$

whence,
$$x = y - 2 \text{ or } -y. \quad (5)$$

Clearing (2) of fractions, etc.,
$$4xy - 21y - 9x = 0.$$

Substituting $y - 2$ for x in (5),
$$y = 9 \text{ or } \frac{1}{2},$$

whence,
$$x = 7 \text{ or } -\frac{3}{2}.$$

Substituting $-y$ for x in (5),
$$y = 0 \text{ or } -3,$$

whence,
$$x = 0 \text{ or } 3.$$

Hence, the fraction is $\frac{7}{9}$.

The other values of x and y , which do not give arithmetical fractions, are disregarded.

15. Let x = number of dollars each man contributed,
 and y = number of dollars each woman contributed.
 Then, $x = y + 2$,
 and $\frac{15}{x} + \frac{15}{y} = 8$.
 Solving, $x = 5$ or $\frac{3}{4}$,
 and $y = 3$ or $-\frac{5}{4}$.
 Hence, each man contributed \$5, and each woman \$3.

18. Let x = one number,
 and y = the other.
 Then, $xy - (x + y) = 59$,
 and $x^2 + y^2 = 170$.
 Solving, $x = 11$ or 7 or $-8 + \sqrt{21}$ or $-8 - \sqrt{21}$,
 and $y = 7$ or 11 or $-8 - \sqrt{21}$ or $-8 + \sqrt{21}$.
 Hence, the numbers are 11 and 7, or $-8 + \sqrt{21}$ and $-8 - \sqrt{21}$.

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19. Let x = number of gold coins,
 and y = number of silver coins.
 Then, $x + y = 15$, (1)
 and xy dollars + yx cents = 5050 cents,
 that is, $100xy + xy = 5050$. (2)
 Solving (1) and (2), $x = 10$ or 5 ,
 and $y = 5$ or 10 .

Hence, there were 10 gold coins and 5 silver coins or 5 gold coins and 10 silver coins.

20. See next page.

21. Let x = number of yards in circumference of fore wheel,
 and y = number of yards in circumference of hind wheel.
 Then, $\frac{240}{x} - \frac{240}{y} = 12$, (1)
 and $\frac{240}{x+1} - \frac{240}{y+1} = 8$. (2)
 Dividing (1) by 12 and clearing of fractions,
 $-20x + 20y = xy$. (3)
 Dividing (2) by 8, clearing of fractions, transposing, etc.,
 $-31x + 29y = xy + 1$. (4)
 Subtracting (3) from (4), $-11x + 9y = 1$,
 whence, $y = \frac{11x+1}{9}$. (5)
 Solving (5) and (3), or (5) and (4), $x = 4$ or $-\frac{5}{11}$,
 and $y = 5$ or $-\frac{4}{9}$.
 Hence, the fore wheel is 4 yards and the hind wheel 5 yards in circumference.

20. Let x = number of days it will take the first man,
and y = number of days it will take the second.

Then
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6\frac{2}{3}} = \frac{3}{20},$$

and
$$x - y = 3.$$

Solving,
$$x = 15 \text{ or } \frac{4}{3},$$

and
$$y = 12 \text{ or } -\frac{5}{3}.$$

Hence, the first man can do the work alone in 15 days, and the second can do it alone in 12 days.

22. Let x = number of dollars in the principal,
and y = number of per cent in the rate.

Then,
$$x + .01 xy = 11130, \quad (1)$$

and
$$(x + 100)(1 + .01 y - .01) = 11130. \quad (2)$$

Simplifying (2),
$$.99x + y + 99 + .01 xy = 11130. \quad (3)$$

Subtracting (3) from (1),
$$.01x - y - 99 = 0, \quad (4)$$

whence,
$$x = 100y + 9900. \quad (4)$$

Substituting (4) in (1) and simplifying,
$$y^2 + 199y - 1230 = 0. \quad (5)$$

Factoring (5),
$$(y - 6)(y + 205) = 0.$$

$$\therefore y = 6 \text{ or } -205.$$

Rejecting the second value and substituting the first in (4),

$$x = 10500.$$

Hence, the principal was \$10500 and the rate was 6 %.

23. Let $10x + y$ = first number,
and $10y + x$ = second number.

Then,
$$(10x + y)(x + 10y) = 4032, \quad (1)$$

and
$$\frac{10x + y}{x + 10y} = \frac{7}{4}. \quad (2)$$

Multiplying (1) by (2),
$$(10x + y)^2 = 7056. \quad (3)$$

Extracting the square root,
$$10x + y = \pm 84,$$

whence, rejecting the negative value,
$$y = 84 - 10x. \quad (4)$$

Substituting (4) in (1),
$$84(840 - 99x) = 4032. \quad (5)$$

Solving (5),
$$x = 8. \quad (6)$$

Substituting (6) in (4),
$$y = 4.$$

Hence, the numbers are 84 and 48.

24. Let x = number of miles per hour he walks,
and y = number of miles per hour he rows.

Then, since he walks 4 miles and rows 8 miles,

$$\frac{4}{x} + \frac{8}{y} = 3, \quad (1)$$

and
$$\frac{4}{x} + \frac{8}{y - 2} = 5. \quad (2)$$

Subtracting (1) from (2),
$$\frac{8}{y - 2} - \frac{8}{y} = 2. \quad (3)$$

Simplifying,
$$y^2 - 2y - 8 = 0. \quad (4)$$

Solving (4),
$$y = 4 \text{ or } -2. \quad (5)$$

Substituting (5) in (1),
$$x = 4 \text{ or } \frac{4}{3}.$$

Hence, his rates of walking and rowing are each 4 miles an hour.

25. Let x = number of miles each traveled,
 and y = number of miles per hour A traveled.
 Then, $\frac{x}{y}$ = number of hours it took A,
 and $\frac{x}{y-2}$ = number of hours it took C;

$$\therefore \frac{20}{y-2} + \frac{x-20}{y} = \frac{x}{y-2} - \frac{2}{3}, \quad (1)$$

$$\text{and} \quad \frac{20}{y-2} + \frac{x-20}{y} = \frac{x}{y} + \frac{1}{3}. \quad (2)$$

$$\text{Simplifying (1),} \quad y^2 - 2y = 3x - 60. \quad (3)$$

$$\text{Subtracting (1) from (2), and simplifying,} \quad y^2 - 2y = 2x. \quad (4)$$

$$\text{Subtracting (4) from (3), etc.,} \quad x = 60. \quad (5)$$

$$\text{Substituting (5) in (4),} \quad y = 12 \text{ or } -10.$$

Rejecting the negative value, it is seen that the distance each traveled was 60 miles; A rode 12 miles an hour, C 10 miles an hour, and B 10 miles an hour for 20 miles and 12 miles an hour for 40 miles.

26. Let x = one number,
 and y = the other.

$$\text{Then,} \quad xy = x^2 - y^2, \quad (1)$$

$$\text{and} \quad x^3 - y^3 = x^2 + y^2. \quad (2)$$

$$\text{From (1),} \quad 2(x+y)(x-y) = 2xy. \quad (3)$$

$$\text{From (2),} \quad (x^2 + xy + y^2)(x-y) = x^2 + y^2. \quad (4)$$

$$\text{Subtracting (3) from (4),} \quad [x^2 + xy + y^2 - 2(x+y)](x-y) = x^2 - 2xy + y^2. \quad (5)$$

$$\text{Dividing by } x-y, \quad x^2 + xy + y^2 - 2x - 2y = x - y.$$

$$\text{Transposing, etc.,} \quad x^2 + xy + y^2 - 3x - y = 0. \quad (6)$$

$$\text{From (1),} \quad -x^2 + xy + y^2 = 0. \quad (7)$$

$$\text{Subtracting (7) from (6),} \quad 2x^2 - 3x - y = 0,$$

$$\text{whence,} \quad y = 2x^2 - 3x. \quad (8)$$

$$\text{Substituting (8) in (1),} \quad 2x^3 - 3x^2 = x^2 - 4x^4 + 12x^3 - 9x^2. \quad (9)$$

$$\text{Dividing by } x^2, \text{ etc.,} \quad 4x^2 - 10x = -5. \quad (10)$$

$$\text{Solving (10),} \quad x = \frac{1}{4}(5 + \sqrt{5}) \text{ or } \frac{1}{4}(5 - \sqrt{5}).$$

$$y = \frac{1}{2}\sqrt{5} \text{ or } -\frac{1}{2}\sqrt{5}.$$

Since $x = 0$ will satisfy (9) and since (5) arises from (8), we have the additional roots, $x = 0$ and $y = 0$.

Hence, the numbers are $\frac{1}{4}(5 + \sqrt{5})$ and $\frac{1}{2}\sqrt{5}$ or $\frac{1}{4}(5 - \sqrt{5})$ and $-\frac{1}{2}\sqrt{5}$ or 0 and 0.

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$$3. \quad ax^2 + bx + c = x^2 - 5x - 75 = 0.$$

Since $b^2 - 4ac = 25 + 300 = 325$, the roots are real and unequal (Prin. 1), and irrational (Prin. 2). Since c is negative, the roots have opposite signs and, b being negative, the positive root is the greater numerically (Prin. 3).

$$4. \quad ax^2 + bx + c = x^2 + 5x + 6 = 0.$$

Since $b^2 - 4ac = 25 - 24 = 1 = 1^2$, the roots are real and unequal (Prin. 1), and rational (Prin. 2). Since c is positive and b is positive, both roots are negative (Prin. 3).

5. $ax^2 + bx + c = x^2 + 7x - 30 = 0.$

Since $b^2 - 4ac = 49 + 120 = 169 = 13^2$, the roots are real and unequal (Prin. 1), and rational (Prin. 2). Since c is negative, the roots have opposite signs and, b being positive, the negative root is the greater numerically (Prin. 3).

6. $ax^2 + bx + c = x^2 - 3x + 5 = 0.$

Since $b^2 - 4ac = 9 - 20 = -11$, both roots are imaginary (Prin. 1).

7. $ax^2 + bx + c = x^2 + 3x - 5 = 0.$

Since $b^2 - 4ac = 9 + 20 = 29$, the roots are real and unequal (Prin. 1), and irrational (Prin. 2). Since c is negative, the roots have opposite signs and, b being positive, the negative root is the greater numerically (Prin. 3).

8. $ax^2 + bx + c = x^2 + x + 2 = 0.$

Since $b^2 - 4ac = 1 - 8 = -7$, both roots are imaginary (Prin. 1).

9. $ax^2 + bx + c = x^2 + x - 2 = 0.$

Since $b^2 - 4ac = 1 + 8 = 9 = 3^2$, the roots are real and unequal (Prin. 1), and rational (Prin. 2). Since c is negative, the roots have opposite signs and, b being positive, the negative root is the greater numerically (Prin. 3).

10. $ax^2 + bx + c = 4x^2 - 4x + 1 = 0.$

Since $b^2 - 4ac = 16 - 16 = 0$, the roots are real and equal (Prin. 1), and rational (Prin. 2). Since c is positive and b is negative, both roots are positive (Prin. 3).

11. $ax^2 + bx + c = 4x^2 + 6x - 4 = 0.$

Since $b^2 - 4ac = 36 + 64 = 100 = 10^2$, the roots are real and unequal (Prin. 1), and rational (Prin. 2). Since c is negative, the roots have opposite signs and, b being positive, the negative root is the greater numerically (Prin. 3).

12. $ax^2 + bx + c = 2x^2 - 9x + 4 = 0.$

Since $b^2 - 4ac = 81 - 32 = 49 = 7^2$, the roots are real and unequal (Prin. 1), and rational (Prin. 2). Since c is positive and b is negative, both roots are positive (Prin. 3).

13. $ax^2 + bx + c = 4x^2 + 16x + 7 = 0.$

Since $b^2 - 4ac = 256 - 112 = 144 = 12^2$, the roots are real and unequal (Prin. 1), and rational (Prin. 2). Since c is positive and b is positive, both roots are negative (Prin. 3).

14. $ax^2 + bx + c = 9x^2 + 12x + 4 = 0.$

Since $b^2 - 4ac = 144 - 144 = 0$, the roots are real and equal (Prin. 1), and rational (Prin. 2). Since c is positive and b is positive, both roots are negative (Prin. 3).

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2. The equation is $(x - 6)(x - 4) = 0$, or $x^2 - 10x + 24 = 0$.

3. The equation is $(x - 5)(x + 3) = 0$, or $x^2 - 2x - 15 = 0$.

4. The equation is $(x - 3)(x + \frac{1}{3}) = 0$, or $x^2 - \frac{8}{3}x - 1 = 0$, or, multiplying by 3, $3x^2 - 8x - 3 = 0$.

5. The equation is $(x - \frac{2}{3})(x - \frac{5}{3}) = 0$, or $x^2 - \frac{7}{3}x + \frac{10}{9} = 0$, or, multiplying by 9, $9x^2 - 21x + 10 = 0$.

6. The equation is $(x + 2)(x + \frac{1}{2}) = 0$, or $x^2 + \frac{5}{2}x + 1 = 0$, or, multiplying by 2, $2x^2 + 5x + 2 = 0$.

7. The equation is $(x + \frac{1}{2})(x + \frac{3}{2}) = 0$, or $x^2 + 2x + \frac{3}{4} = 0$, or, multiplying by 4, $4x^2 + 8x + 3 = 0$.

8. The equation is $(x - a)(x + 3a) = 0$, or $x^2 + 2ax - 3a^2 = 0$.

9. The equation is $(x - a - 2)(x - a + 2) = 0$, or $x^2 - 2ax + a^2 - 4 = 0$.

10. The equation is $(x - b - 1)(x - b + 1) = 0$, or $x^2 - 2bx + b^2 - 1 = 0$.

11. The equation is $(x - a - b)(x - a + b) = 0$, or $x^2 - 2ax + a^2 - b^2 = 0$.

12. The equation is $(x - \sqrt{a} + \sqrt{b})(x - \sqrt{b}) = 0$, or $x^2 - x\sqrt{a} + \sqrt{ab} - b = 0$.

13. The equation is $(x - \frac{1}{2}a - \frac{1}{2}\sqrt{b})(x - \frac{1}{2}a + \frac{1}{2}\sqrt{b}) = 0$, or, multiplying each factor by 2,

$$(2x - a - \sqrt{b})(2x - a + \sqrt{b}) = 0, \text{ or } 4x^2 - 4ax + a^2 - b = 0.$$

14. The sum of the roots is 6 and their product is $9 - 2$, or 7. Hence, the equation is $x^2 - 6x + 7 = 0$.

15. The sum of the roots is -4 and their product is $4 - 5$, or -1 . Hence, the equation is $x^2 + 4x - 1 = 0$.

16. The sum of the roots is 4 and their product is $4 - \frac{3}{2}$, or 1. Hence, the equation is $x^2 - 4x + 1 = 0$.

17. The sum of the roots is -3 and their product is $\frac{9}{4} - \frac{6}{4}$, or $\frac{3}{4}$. Hence, the equation is $x^2 + 3x + \frac{3}{4} = 0$, or $4x^2 + 12x + 3 = 0$.

18. The sum of the roots is -1 and their product is $\frac{1}{4} - \frac{2}{4}$, or $-\frac{1}{4}$. Hence, the equation is $x^2 + x - \frac{1}{4} = 0$, or $4x^2 + 4x - 1 = 0$.

19. The sum of the roots is $4a$ and their product is $4a^2 - a^2 \cdot 4 \cdot 5$, or $4a^2 - 20a^2$, or $-16a^2$. Hence, the equation is $x^2 - 4ax - 16a^2 = 0$.

GENERAL REVIEW

| | |
|--|---|
| <p>1. $\begin{aligned} & x\sqrt{y} + y\sqrt{x} + \sqrt{xy} \\ & - x\sqrt{y} - y\sqrt{x} + x^2y^{\frac{1}{2}} \\ & \quad x\sqrt{y} - y\sqrt{x} - \sqrt{xy} \\ & \underline{- 2x\sqrt{y} + y\sqrt{x} - 3\sqrt{xy}} \\ \text{Sum} = & -x\sqrt{y} \quad - 2\sqrt{xy} \end{aligned}$</p> | <p>2. $\begin{aligned} & 2a + 3b - 3y \\ & - a - 3b + 2y \\ \text{Sum} = & a - y \quad (1) \\ & \quad a - b - y \\ & \quad \underline{a + b + y} \\ \text{Diff.} = & -2b - 2y \quad (2) \\ (1) - (2), & \quad a + 2b + y. \end{aligned}$</p> |
|--|---|

3. The number is $(b - a) - a$, or $b - 2a$.

5. The number is $(a - b) - (b - a + c) = a - b - b + a - c = 2a - 2b - c$.

$$\begin{aligned} 6. \quad & a - \{b - a - [a - b - (2a + b) + (2a - b) - a] - b\} \\ & = a - \{-a - [-b - (2a + b) + (2a - b)]\} \\ & = a - \{-a - [-b - 2a - b + 2a - b]\} \\ & = a - \{-a - [-3b]\} \\ & = a - \{-a + 3b\} \\ & = a + a - 3b = 2a - 3b. \end{aligned}$$

19. If 1 is substituted for x , $x^3 - 3x + 2 = 1 - 3 + 2 = 0$.

Therefore, $x - 1$ is a factor of $x^3 - 3x + 2$.

By dividing by $x - 1$, $x^3 - 3x + 2 = (x - 1)(x^2 + x - 2)$
 $= (x - 1)(x - 1)(x + 2)$.

20. In $x^n + 3ax^{n-1} - 4a^n$ substitute a for x .

Then, $x^n + 3ax^{n-1} - 4a^n = a^n + 3a^n - 4a^n = 0$.

Therefore, by the factor law, $x^n + 3ax^{n-1} - 4a^n$ is divisible by $x - a$.

$$\begin{aligned} 21. \quad a^{12} - 1 &= (a^6 + 1)(a^6 - 1) \\ &= (a^2 + 1)(a^4 - a^2 + 1)(a^3 + 1)(a^3 - 1) \\ &= (a^2 + 1)(a^4 - a^2 + 1)(a + 1)(a^2 - a + 1)(a - 1)(a^2 + a + 1). \end{aligned}$$

$$\begin{aligned} 22. \quad &4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2 \\ &= (2ad + 2bc + a^2 - b^2 - c^2 + d^2)(2ad + 2bc - a^2 + b^2 + c^2 - d^2) \\ &= [a^2 + 2ad + d^2 - (b^2 - 2bc + c^2)][b^2 + 2bc + c^2 - (a^2 - 2ad + d^2)] \\ &= [(a + d)^2 - (b - c)^2][(b + c)^2 - (a - d)^2] \\ &= (a + d + b - c)(a + d - b + c)(b + c + a - d)(b + c - a + d) \\ &= (a + b - c + d)(a - b + c + d)(a + b + c - d)(-a + b + c + d). \end{aligned}$$

$$25. \quad \begin{array}{r} 2x^4 + x^3 \qquad \qquad - \quad x - 12 \\ 2x^4 + x^3 - 4x^2 + 7x - 15 \end{array}$$

$$\begin{array}{r|l} 3 - 8x + 4x^2 & \begin{array}{r} -12 - x \\ -12 + 32x - 16x^2 \\ -33x + 16x^2 + x^3 \\ -33x + 88x^2 - 44x^3 \\ -72x^2 + 45x^3 + 2x^4 \\ -72x^2 + 192x^3 - 96x^4 \\ -49x^3 \end{array} + \begin{array}{r} x^3 + 2x^4 \\ x^3 \\ 44x^3 \\ -44x^3 \\ 45x^3 \\ 192x^3 \\ -147x^3 \end{array} \\ \hline 1 - 2x & \begin{array}{r} 3 - 8x + 4x^2 \\ 3 - 2x \end{array} \end{array} \quad \begin{array}{l} -4 \\ -11x \\ -24x^2 \end{array}$$

$\therefore 3 - 2x$ or $2x - 3 = \text{H.C.D. of } 2d \text{ and } 3d \text{ polynomials.}$

By trial it is found that $2x - 3$ is an exact divisor of the first polynomial,
 $4x^4 - 11x^2 + 11x - 12$.

Hence, H.C.D. = $2x - 3$.

$$28. \quad \frac{x^2 - 5x + 6}{3x^2 - 4x - 4} = \frac{(x - 2)(x - 3)}{(x - 2)(3x + 2)} = \frac{x - 3}{3x + 2}$$

$$29. \quad \frac{x^3 - 5x^2 + 4}{x^3 - 2x^2 + 1} = \frac{(x - 1)(x^2 - 4x - 4)}{(x - 1)(x^2 - x - 1)} = \frac{x^2 - 4x - 4}{x^2 - x - 1}$$

$$30. \quad \frac{a^2 - b^2 - c^2 - 2bc}{a^2 - b^2 + c^2 + 2ac} = \frac{(a + b + c)(a - b - c)}{(a + c + b)(a + c - b)} = \frac{a - b - c}{a - b + c}$$

31. See next page.

$$\begin{aligned} 32. \quad \frac{x + y}{2x - 2y} + \frac{x - y}{2x + 2y} + \frac{4x^2y^2}{y^4 - x^4} &= \frac{(x + y)^2 + (x - y)^2}{2(x^2 - y^2)} - \frac{4x^2y^2}{x^4 - y^4} \\ &= \frac{x^2 + y^2}{x^2 - y^2} - \frac{4x^2y^2}{x^4 - y^4} \\ &= \frac{x^4 + 2x^2y^2 + y^4 - 4x^2y^2}{x^4 - y^4} \\ &= \frac{(x^2 - y^2)(x^2 - y^2)}{(x^2 + y^2)(x^2 - y^2)} = \frac{x^2 - y^2}{x^2 + y^2} \end{aligned}$$

$$\begin{aligned}
 31. \quad \frac{x}{x+1} - \frac{x}{1-x} + \frac{x^2}{x^2-1} &= \frac{x}{x+1} + \frac{x}{x-1} + \frac{x^2}{x^2-1} \\
 &= \frac{x(x-1) + x(x+1) + x^2}{x^2-1} = \frac{3x^2}{x^2-1}.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \frac{1}{(a-b)(b-c)} - \frac{1}{(c-b)(c-a)} + \frac{1}{(c-a)(a-b)} \\
 &= \frac{1}{(a-b)(b-c)} + \frac{1}{(b-c)(c-a)} + \frac{1}{(c-a)(a-b)} \\
 &= \frac{(c-a) + (a-b) + (b-c)}{(a-b)(b-c)(c-a)} = \frac{0}{(a-b)(b-c)(c-a)} = 0.
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \left(a + \frac{1}{a}\right) \left(a^2 + \frac{1}{a^2}\right) \left(a - \frac{1}{a}\right) &= \left(a + \frac{1}{a}\right) \left(a - \frac{1}{a}\right) \left(a^2 + \frac{1}{a^2}\right) \\
 &= \left(a^2 - \frac{1}{a^2}\right) \left(a^2 + \frac{1}{a^2}\right) = a^4 - \frac{1}{a^4}.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \frac{1}{x - \frac{1}{x + \frac{1}{x}}} - \frac{1}{x + \frac{1}{x - \frac{1}{x}}} &= \frac{1}{x - \frac{x}{x^2+1}} - \frac{1}{x + \frac{x}{x^2-1}} \\
 &= \frac{x^2+1}{(x^3+x) - x} - \frac{x^2-1}{(x^3-x) + x} \\
 &= \frac{(x^2+1) - (x^2-1)}{x^3} = \frac{2}{x^3}.
 \end{aligned}$$

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$$\begin{aligned}
 36. \quad \left[\frac{x}{1 + \frac{1}{x}} + 1 - \frac{1}{x+1} \right] \div \left[\frac{x}{1 - \frac{1}{x}} - x - \frac{1}{x-1} \right] \\
 &= \left(\frac{x^2}{x+1} + 1 - \frac{1}{x+1} \right) \div \left(\frac{x^2}{x-1} - x - \frac{1}{x-1} \right) \\
 &= \left(\frac{x^2-1}{x+1} + 1 \right) \div \left(\frac{x^2-1}{x-1} - x \right) = (x-1+1) \div (x+1-x) = x.
 \end{aligned}$$

37. Prove that

$$\frac{a}{b} \times \frac{p}{q} = \frac{ap}{bq}.$$

Def. of fraction,

$$\frac{a}{b} \times \frac{p}{q} = (a \div b) \times (p \div q)$$

Associative Law,

$$= a \div b \times p \div q$$

Commutative Law,

$$= a \times p \div b \div q$$

Associative Law,

$$= ap \div bq$$

Def. of fraction,

$$= \frac{ap}{bq}.$$

38. Prove that

$$\frac{a}{b} \div \frac{m}{n} = \frac{an}{bm}.$$

Def. of fraction,

$$\frac{a}{b} \div \frac{m}{n} = (a \div b) \div (m \div n)$$

Associative Law, § 104, 2,

$$= (a \div b) \div m \times n$$

Associative Law,

$$= a \div b \div m \times n$$

Commutative Law,

$$= a \times n \div b \div m$$

Associative Law,

$$= (an) \div (bm)$$

Def. of fraction,

$$= \frac{an}{bm}.$$

$$\begin{aligned} 39. \quad \left(\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} \right) \div \left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x}} \right) &= \frac{x-y}{\sqrt{xy}} \div \frac{\sqrt{x}-\sqrt{y}}{\sqrt{xy}} \\ &= \frac{x-y}{\sqrt{x}-\sqrt{y}} = \sqrt{x} + \sqrt{y}. \end{aligned}$$

$$41. \quad (2a + 3b)^4$$

$$\begin{aligned} &= (2a)^4 + 4(2a)^3(3b) + 6(2a)^2(3b)^2 + 4(2a)(3b)^3 + (3b)^4 \\ &= 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4. \end{aligned}$$

$$42. \quad (\sqrt{x} + \sqrt[3]{y})^6$$

$$\begin{aligned} &= (\sqrt{x})^6 + 6(\sqrt{x})^5(\sqrt[3]{y}) + 15(\sqrt{x})^4(\sqrt[3]{y})^2 + 20(\sqrt{x})^3(\sqrt[3]{y})^3 \\ &\quad + 15(\sqrt{x})^2(\sqrt[3]{y})^4 + 6(\sqrt{x})(\sqrt[3]{y})^5 + (\sqrt[3]{y})^6 \\ &= x^3 + 6x^2\sqrt{x}\sqrt[3]{y} + 15x^2\sqrt[3]{y}^2 + 20xy\sqrt{x} + 15xy\sqrt[3]{y} + 6y\sqrt{x}\sqrt[3]{y}^2 + y^2. \end{aligned}$$

$$44. \quad \frac{a^2 + 2a\sqrt{ab} + 3ab + 2b\sqrt{ab} + b^2}{a^2} \mid \frac{a + \sqrt{ab} + b}{a}$$

$$\begin{array}{r|l} 2a & 2a\sqrt{ab} \\ 2a + \sqrt{ab} & 2a\sqrt{ab} + ab \\ 2a + 2\sqrt{ab} & 2ab \\ 2a + 2\sqrt{ab} + b & 2ab + 2b\sqrt{ab} + b^2 \end{array}$$

$$45. \quad \frac{8a^3 - 36a^2b + 54ab^2 - 27b^3}{8a^3} \mid \frac{2a - 3b}{8a^3}$$

$$\begin{array}{r|l} 12a^2 & -36a^2b \\ 12a^2 - 18ab + 9b^2 & -36a^2b + 54ab^2 - 27b^3 \end{array}$$

$$\begin{array}{r|l} 2a^{\frac{1}{2}} & b \\ 2a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b & b + \frac{1}{2}a^{-1}b^2 \\ 2a^{\frac{1}{2}} + a^{-\frac{1}{2}}b & -\frac{1}{4}a^{-1}b^2 \\ 2a^{\frac{1}{2}} + a^{-\frac{1}{2}}b - \frac{1}{4}a^{-\frac{3}{2}}b^2 & -\frac{1}{4}a^{-1}b^2 - \frac{1}{8}a^{-2}b^3 + \frac{1}{8}a^{-\frac{3}{2}}b^4 \\ 2a^{\frac{1}{2}} & \frac{1}{8}a^{-2}b^3 - \frac{1}{8}a^{-\frac{3}{2}}b^4 \end{array}$$

47. $4 \cdot 82 \cdot 68 \cdot 09 \mid 2197$

$$\begin{array}{r}
 4 \overline{) 82} \\
 41 \overline{) 41} \\
 429 \overline{) 38 \ 61} \\
 4387 \overline{) 3 \ 07 \ 09}
 \end{array}$$

$2 \cdot 197 \mid 18$

$$\begin{array}{r}
 1 \overline{) 18} \\
 300 \overline{) 1 \ 197} \\
 90 \overline{) 9} \\
 399 \overline{) 1 \ 197}
 \end{array}$$

Therefore, the sixth root of 4826809 is 18.

48. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{3}} \times \sqrt{6} = \frac{1}{3}\sqrt{6}.$

49. $\sqrt[4]{25 a^4} = a \sqrt[4]{25} = a \sqrt[4]{5^2} = a (5)^{\frac{1}{2}} = a (5)^{\frac{1}{2}} = a \sqrt{5}.$

50. $\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2} \sqrt{2} = \frac{1}{2} \text{ of } 1.4142 + = .7071 +.$

51. $(2 + \sqrt{8})(1 - \sqrt{2}) = (2 + 2\sqrt{2})(1 - \sqrt{2}) = 2(1 + \sqrt{2})(1 - \sqrt{2})$
 $= 2(1 - 2) = -2.$

52. $\frac{\sqrt{3} + 3\sqrt{2}}{\sqrt{6} + 2} = \frac{(\sqrt{3} + 3\sqrt{2})(\sqrt{6} - 2)}{(\sqrt{6} + 2)(\sqrt{6} - 2)} = \frac{3\sqrt{2} - 2\sqrt{3} + 6\sqrt{3} - 6\sqrt{2}}{6 - 4}$
 $= 2\sqrt{3} - \frac{3}{2}\sqrt{2}.$

53. Prove that $x^0 = 1.$

Since $x^m \div x^m = x^{m-m}$, or x^0 , and since $x^m \div x^m = 1$, Ax. 1, $x^0 = 1.$

54. Prove that $ax^{-5} = \frac{a}{x^5}.$

$$ax^{-5} = \frac{ax^{-5} \cdot x^5}{x^5} = \frac{ax^{-5+5}}{x^5} = \frac{ax^0}{x^5} = \frac{a}{x^5}.$$

55. Prove that $x^{\frac{2}{3}} = \sqrt[3]{x^2}$; also that $x^{\frac{2}{3}} = (\sqrt[3]{x})^2.$

$$x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} = x^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3}} = x^2; \therefore x^{\frac{2}{3}} = \sqrt[3]{x^2}.$$

$$x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} = x^{\frac{2}{3} + \frac{2}{3}} = x^{\frac{4}{3}}; \therefore x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2 \text{ by the first part of the proof}$$

56. $125^{\frac{2}{3}} = (\sqrt[3]{125})^2 = 5^2 = 25.$

$$\left(\frac{x^5}{32}\right)^{-\frac{2}{3}} = \left[\left(\frac{x}{2}\right)^5\right]^{-\frac{2}{3}} = \left(\frac{x}{2}\right)^{-8} = \left[\left(\frac{x}{2}\right)^3\right]^{-1} = \left(\frac{x^3}{8}\right)^{-1} = \frac{8}{x^3}.$$

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57.

$$\frac{1}{a-b} + \frac{a-b}{x} = \frac{1}{a+b} + \frac{a+b}{x}.$$

Transposing,

$$\frac{1}{a-b} - \frac{1}{a+b} = \frac{a+b}{x} - \frac{a-b}{x}.$$

Uniting terms,

$$\frac{2b}{a^2 - b^2} = \frac{2b}{x}.$$

$$\therefore x = a^2 - b^2.$$

58.
$$\frac{x - \frac{1}{a}}{c} + \frac{x - \frac{1}{b}}{a} + \frac{x - \frac{1}{c}}{b} = 0.$$

Clearing of fractions,

$$abx - b + bcx - c + cax - a = 0.$$

$$\therefore x = \frac{a + b + c}{ab + bc + ca}.$$

59.
$$mx^2 - nx = mn.$$

Completing the square, $4m^2x^2 - 4mnx + n^2 = 4m^2n + n^2.$

Extracting the square root, $2mx - n = \pm \sqrt{4m^2n + n^2}.$

$$\therefore x = \frac{n}{2m} \pm \frac{1}{2m} \sqrt{4m^2n + n^2}.$$

60.
$$x^4 + \frac{1}{2} = \frac{3x^2}{2}.$$

Clearing of fractions and transposing, $2x^4 - 3x^2 + 1 = 0.$

Factoring, $(x^2 - 1)(2x^2 - 1) = 0.$

Equating each factor to zero and solving, $x = \pm 1$ or $\pm \frac{1}{2}\sqrt{2}.$

61.
$$x^6 + 8 = 9x^3.$$

Transposing, $x^6 - 9x^3 + 8 = 0.$

Factoring, $(x - 1)(x^2 + x + 1)(x - 2)(x^2 + 2x + 4) = 0.$

Equating each factor to zero and solving, $x = 1$ or $\frac{1}{2}(-1 \pm \sqrt{-3})$ or 2 or $-1 \pm \sqrt{-3}.$

62.
$$(1 + x)^5 + (1 - x)^5 = 242.$$

Expanding, uniting terms, etc., $x^4 + 2x^2 - 24 = 0.$

Factoring, $(x^2 - 4)(x^2 + 6) = 0.$

Equating each factor to zero and solving, $x = \pm 2$ or $\pm \sqrt{-6}.$

63.
$$\sqrt{x - 9} = \sqrt{x} - 1.$$

Squaring, $x - 9 = x - 2\sqrt{x} + 1$

Canceling, etc., $5 = \sqrt{x}.$

Squaring, $x = 25.$

64.
$$x^2 + \sqrt{x^2 + 16} = 14.$$

Adding 16 to each member and transposing,

$$(x^2 + 16) + \sqrt{x^2 + 16} - 30 = 0.$$

Factoring, $(\sqrt{x^2 + 16} - 5)(\sqrt{x^2 + 16} + 6) = 0.$

$$\therefore \sqrt{x^2 + 16} = 5 \text{ or } -6.$$

Squaring, $x^2 + 16 = 25 \text{ or } 36.$

$$x^2 = 9 \text{ or } 20.$$

$$\therefore x = \pm 3 \text{ or } \pm 2\sqrt{5}.$$

$$65. \quad \left(\frac{4}{x} + x\right)^2 - \left(\frac{4}{x} + x\right) = 20.$$

Transposing and factoring,

$$\left(\frac{4}{x} + x - 5\right) \left(\frac{4}{x} + x + 4\right) = 0.$$

$$\therefore \frac{4}{x} + x = 5 \text{ or } -4.$$

Clearing of fractions and transposing we have the equations

$$x^2 - 5x + 4 = 0,$$

$$\text{whence,} \quad x = 1 \text{ or } 4,$$

$$\text{and} \quad x^2 + 4x + 4 = 0,$$

$$\text{whence,} \quad x = -2 \text{ or } -2.$$

Hence, the roots are 1, 4, -2, -2.

$$66. \quad x + x^2 + (1 + x + x^2)^2 = 55.$$

Adding 1 to each member and transposing,

$$(1 + x + x^2)^2 + (1 + x + x^2) - 56 = 0.$$

$$\text{Factoring,} \quad (1 + x + x^2 - 7)(1 + x + x^2 + 8) = 0.$$

Equating each factor to zero and solving,

$$x = 2 \text{ or } -3 \text{ or } \frac{1}{2}(-1 \pm \sqrt{-35}).$$

$$67. \quad \frac{1+x}{1+x+\sqrt{1+x^2}} = a - \frac{1+x}{1-x+\sqrt{1+x^2}}.$$

$$\text{Rationalizing,} \quad \frac{(1+x-\sqrt{1+x^2})(1+x)}{2x} = a - \frac{(1-x-\sqrt{1+x^2})(1+x)}{-2x},$$

$$\text{or} \quad \frac{(1+x-\sqrt{1+x^2})(1+x)}{2x} = a + \frac{(1-x-\sqrt{1+x^2})(1+x)}{2x}.$$

$$\text{Canceling, etc.,} \quad \frac{2x(1+x)}{2x} = a,$$

$$\text{whence,} \quad 1+x = a.$$

$$\therefore x = a - 1.$$

$$68. \quad \begin{cases} x+y=8, & (1) \\ y+z=4, & (2) \\ z+x=6. & (3) \end{cases}$$

Adding and dividing by 2,

$$x+y+z=9. \quad (4)$$

Subtracting (2), (3), and (1) in turn from (4),

$$x=5, y=3, z=1.$$

$$69. \quad \begin{cases} \frac{1}{x} + \frac{1}{y} = 10, & (1) \\ \frac{3}{x} + \frac{2}{y} = 10. & (2) \end{cases}$$

Subtracting (1) \times 2 from (2),

$$\frac{1}{x} = -10.$$

$$\therefore x = -\frac{1}{10}.$$

Subtracting (2) from (1) \times 3,

$$\frac{1}{y} = 20.$$

$$\therefore y = \frac{1}{20}.$$

$$\begin{array}{rcl}
 70. & \left\{ \begin{array}{l} 2x + 3y + z = 9, \\ x + 2y + 3z = 13, \\ 3x + y + 2z = 11. \end{array} \right. & \begin{array}{l} (1) \\ (2) \\ (3) \end{array} \\
 \text{Adding (1), (2), and (3),} & 6x + 6y + 6z = 33. & (4) \\
 \text{Dividing (4) by 3,} & 2x + 2y + 2z = 11. & (5) \\
 \text{Subtracting (5) from (3),} & x - y = 0, & (6)
 \end{array}$$

$$\begin{array}{rcl}
 \text{whence,} & x = y. & (6) \\
 \text{Subtracting (5) from (1),} & y - z = -2, & (7) \\
 \text{whence,} & z = y + 2, \text{ or } x + 2. & (7) \\
 \text{Substituting (6) and (7) in (2), } & x + 2x + 3x + 6 = 13. & \\
 & \therefore x = \frac{7}{6}, & \\
 \text{whence, by (6),} & y = \frac{7}{6}, & \\
 \text{and, by (7),} & z = \frac{19}{6}. &
 \end{array}$$

$$\begin{array}{rcl}
 71. & \left\{ \begin{array}{l} ax + y + z = 2(a + 1), \\ x + ay + z = 3a + 1, \\ x + y + az = a^2 + 3. \end{array} \right. & \begin{array}{l} (1) \\ (2) \\ (3) \end{array} \\
 \text{Adding (1), (2), and (3), } & (a + 2)(x + y + z) = a^2 + 5a + 6. & (4) \\
 \text{Dividing (4) by } a + 2, & x + y + z = a + 3. & (5) \\
 \text{Subtracting (5) from (1),} & (a - 1)x = a - 1. & \\
 & \therefore x = 1. & \\
 \text{Subtracting (5) from (2),} & (a - 1)y = 2a - 2. & \\
 & \therefore y = 2. & \\
 \text{Subtracting (5) from (3),} & (a - 1)z = a^2 - a. & \\
 & \therefore z = a. &
 \end{array}$$

72. See next page.

$$\begin{array}{rcl}
 73. & \left\{ \begin{array}{l} x^2 + 3xy = 7, \\ xy + 4y^2 = 18. \end{array} \right. & \begin{array}{l} (1) \\ (2) \end{array} \\
 \text{Adding,} & x^2 + 4xy + 4y^2 = 25, & \\
 \text{whence, extracting the square root,} & x + 2y = 5 \text{ or } -5. & (3) \\
 \text{From (3),} & x = 5 - 2y \text{ or } -5 - 2y. & (4) \\
 \text{Substituting } 5 - 2y \text{ for } x \text{ in (2), } & 5y - 2y^2 + 4y^2 = 18. & (5) \\
 \text{Solving (5),} & y = 2 \text{ or } -\frac{3}{2}, & \\
 \text{whence, since } x = 5 - 2y, & x = 1 \text{ or } 14. & \\
 \text{Substituting } -5 - 2y \text{ for } x \text{ in (2), } & -5y - 2y^2 + 4y^2 = 18. & (6) \\
 \text{Solving (6),} & y = \frac{3}{2} \text{ or } -2, & \\
 \text{whence, since } x = -5 - 2y, & x = -14 \text{ or } -1. & \\
 \text{Hence,} & x = 1, 14, -14, -1; & \\
 \text{and} & y = 2, -\frac{3}{2}, \frac{3}{2}, -2. &
 \end{array}$$

$$\begin{array}{rcl}
 74. & \left\{ \begin{array}{l} x^2y + xy^2 = 6, \\ x^3 + y^3 = 9. \end{array} \right. & \begin{array}{l} (1) \\ (2) \end{array} \\
 \text{Multiplying (1) by 3,} & 3x^2y + 3xy^2 = 18. & (3) \\
 \text{Adding (2) and (3), } & x^3 + 3x^2y + 3xy^2 + y^3 = 27. & (4) \\
 \text{Extracting the cube root,} & x + y = 3. & (5) \\
 \text{Dividing (1) by (5),} & xy = 2. & (6) \\
 \text{Squaring (5),} & x^2 + 2xy + y^2 = 9. & (7) \\
 \text{Subtracting (6) } \times 4 \text{ from (7), } & x^2 - 2xy + y^2 = 1, & \\
 \text{whence,} & x - y = \pm 1. & (8) \\
 \text{From (5) and (8),} & x = 2 \text{ or } 1, & \\
 \text{and} & y = 1 \text{ or } 2. &
 \end{array}$$

Since (4) can be written in the form $(x + y)^3 - 3^3 = 0$, which can be factored, there are other roots. The other roots are imaginary and are so involved that the student must defer this part of the solution.

72.

$$\begin{cases} x^2 + xy = 24, & (1) \\ y^2 + xy = 12. & (2) \end{cases}$$

Adding,

$$x^2 + 2xy + y^2 = 36,$$

whence,

$$x + y = 6 \text{ or } -6. \quad (3)$$

Subtracting (2) from (1),

$$x^2 - y^2 = 12. \quad (4)$$

Dividing (4) by (3),

$$x - y = 2 \text{ or } -2; \quad (5)$$

that is, the corresponding values of $x + y$ and $x - y$ are 6 and 2, or -6 and -2 .

From (3) and (5),

$$x = 4 \text{ or } -4,$$

and

$$y = 2 \text{ or } -2.$$

75.

$$\begin{cases} x^2 + x = 26 - y^2 - y, & (1) \\ xy = 8. & (2) \end{cases}$$

From (1),

$$x^2 + y^2 + (x + y) - 26 = 0. \quad (3)$$

From (2),

$$2xy - 16 = 0. \quad (4)$$

Adding (4) to (3),

$$(x + y)^2 + (x + y) - 42 = 0. \quad (5)$$

Factoring (5),

$$(x + y - 6)(x + y + 7) = 0.$$

$$\therefore x + y = 6 \text{ or } -7. \quad (6)$$

Squaring (6),

$$x^2 + 2xy + y^2 = 36 \text{ or } 49. \quad (7)$$

Subtracting (2) $\times 4$ from (7),

$$x^2 - 2xy + y^2 = 4 \text{ or } 17,$$

whence,

$$x - y = \pm 2 \text{ or } \pm \sqrt{17}. \quad (8)$$

From (6) and (8),

$$x = 4 \text{ or } 2 \text{ or } \frac{1}{2}(-7 + \sqrt{17}) \text{ or } \frac{1}{2}(-7 - \sqrt{17}),$$

and

$$y = 2 \text{ or } 4 \text{ or } \frac{1}{2}(-7 - \sqrt{17}) \text{ or } \frac{1}{2}(-7 + \sqrt{17}).$$

76.

$$\begin{cases} \sqrt{xy} = 12, & (1) \\ x + y - \sqrt{x + y} = 20. & (2) \end{cases}$$

$$\text{Transposing in (2), } x + y - \sqrt{x + y} - 20 = 0. \quad (3)$$

$$\text{Factoring (3), } (\sqrt{x + y} - 5)(\sqrt{x + y} + 4) = 0.$$

$$\therefore \sqrt{x + y} = 5 \text{ or } -4. \quad (4)$$

Squaring (4),

$$x + y = 25 \text{ or } 16. \quad (5)$$

Squaring (1),

$$xy = 144. \quad (6)$$

Squaring (5),

$$x^2 + 2xy + y^2 = 625 \text{ or } 256. \quad (7)$$

Subtracting (6) $\times 4$ from (7),

$$x^2 - 2xy + y^2 = 49 \text{ or } -320,$$

whence,

$$x - y = \pm 7 \text{ or } \pm 8\sqrt{-5}. \quad (8)$$

From (5) and (8),

$$x = 16 \text{ or } 9 \text{ or } 8 + 4\sqrt{-5} \text{ or } 8 - 4\sqrt{-5},$$

and

$$y = 9 \text{ or } 16 \text{ or } 8 - 4\sqrt{-5} \text{ or } 8 + 4\sqrt{-5}.$$

77.

$$\begin{cases} x^2 - y^2 = 7, & (1) \\ x^4 - y^4 = 175. & (2) \end{cases}$$

Dividing (2) by (1),

$$x^2 + y^2 = 25. \quad (3)$$

From (3) and (1),

$$x^2 = 16,$$

and

$$y^2 = 9.$$

Therefore,

$$x = \pm 4,$$

and

$$y = \pm 3.$$

78. $\begin{cases} xy - xy^2 = -6, & (1) \\ x - xy^3 = 9. & (2) \end{cases}$

Multiplying (1) by 3, etc., $3xy(1 - y) = -18. \quad (3)$

Multiplying (2) by 2, etc., $2x(1 + y + y^2)(1 - y) = 18. \quad (4)$

Adding (3) and (4), $x(2 + 5y + 2y^2)(1 - y) = 0. \quad (5)$

When, from (5), $2y^2 + 5y + 2 = 0,$

factoring, $(y + 2)(2y + 1) = 0,$

whence, $y = -2 \text{ or } -\frac{1}{2}. \quad (6)$

Substituting these values in (1), $x = 1 \text{ or } 8. \quad (7)$

When, from (5), $1 - y = 0,$ $y = 1, \quad (8)$

and, in (2), $x = \frac{9}{0} = \infty. \quad (9)$

When, from (5), $x = 0$, substituting in (2), $y = -\frac{9}{0} = -\infty.$

Hence, $\begin{cases} x = 1, 8, \infty, 0; \\ y = -2, -\frac{1}{2}, 1, -\infty. \end{cases}$

79. $\begin{cases} xy = x + y, & (1) \\ x^2 + y^2 = 8. & (2) \end{cases}$

From (1), $2xy - 2(x + y) = 0. \quad (3)$

Adding (3) to (2), $x^2 + 2xy + y^2 - 2(x + y) = 8. \quad (4)$

Completing the square, $(x + y)^2 - 2(x + y) + 1 = 9.$

Extracting the square root, $x + y - 1 = \pm 3.$

$\therefore x + y = 4 \text{ or } -2. \quad (5)$

From (1) and (5), $2xy = 8 \text{ or } -4. \quad (6)$

Subtracting (6) from (2), $x^2 - 2xy + y^2 = 0 \text{ or } 12,$

whence, $x - y = \pm 0 \text{ or } \pm 2\sqrt{3}. \quad (7)$

From (5) and (7), $x = 2 \text{ or } 2 \text{ or } -1 + \sqrt{3} \text{ or } -1 - \sqrt{3},$

and $y = 2 \text{ or } 2 \text{ or } -1 - \sqrt{3} \text{ or } -1 + \sqrt{3}.$

80. $\begin{cases} x^2y^2 - 4xy = 5, & (1) \\ x^2 + 4y^2 = 29. & (2) \end{cases}$

Solving (1) for xy , $xy = 5 \text{ or } -1, \quad (3)$

whence, $4xy = 20 \text{ or } -4. \quad (4)$

Adding (4) to (2), $x^2 + 4xy + 4y^2 = 49 \text{ or } 25,$

whence, $x + 2y = \pm 7 \text{ or } \pm 5. \quad (5)$

Subtracting (4) from (2), $x^2 - 4xy + 4y^2 = 9 \text{ or } 33,$

whence, $x - 2y = \pm 3 \text{ or } \pm \sqrt{33}. \quad (6)$

From (5) and (6), since they are derived separately,

$x = 5, 2, -2, -5, \frac{1}{2}(5 + \sqrt{33}), \frac{1}{2}(5 - \sqrt{33}), \frac{1}{2}(-5 + \sqrt{33}), \frac{1}{2}(-5 - \sqrt{33});$

$y = 1, \frac{5}{2}, -\frac{5}{2}, -1, \frac{1}{4}(5 + \sqrt{33}), \frac{1}{4}(5 - \sqrt{33}), \frac{1}{4}(-5 + \sqrt{33}), \frac{1}{4}(-5 - \sqrt{33}).$

81. $\begin{cases} 2x^3 + 2y^3 = 9xy, & (1) \\ x + y = 3. & (2) \end{cases}$

From (1), $2(x + y)(x^2 - xy + y^2) = 9xy. \quad (3)$

Substituting (2) in (3), $6(x^2 - xy + y^2) = 9xy.$

Transposing, $6x^2 - 15xy + 6y^2 = 0. \quad (4)$

Factoring (4), $(x - 2y)(6x - 3y) = 0.$

$\therefore x = 2y \text{ or } \frac{1}{2}y. \quad (5)$

Substituting $2y$ for x in (2), $y = 1,$

whence, $x = 2.$

Substituting $\frac{1}{2}y$ for x in (2), $y = 2,$

whence, $x = 1.$

$$82. \quad \begin{cases} x^2 + xy + y^2 = 189, \\ x + \sqrt{xy} + y = 21. \end{cases} \quad (1)$$

$$\text{Dividing (1) by (2),} \quad x - \sqrt{xy} + y = 9. \quad (2)$$

$$\text{Subtracting (3) from (2),} \quad 2\sqrt{xy} = 12. \quad (3)$$

$$\therefore \sqrt{xy} = 6. \quad (4)$$

$$\text{Adding (4) to (3),} \quad x + y = 15. \quad (5)$$

$$\text{Squaring (4) and multiplying by 3,} \quad 3xy = 108. \quad (6)$$

$$\text{Subtracting (6) from (1),} \quad x^2 - 2xy + y^2 = 81, \quad (7)$$

$$\text{whence,} \quad x - y = \pm 9. \quad (7)$$

$$\text{From (5) and (7),} \quad x = 12 \text{ or } 3, \quad (7)$$

$$\text{and} \quad y = 3 \text{ or } 12.$$

$$83. \quad \begin{cases} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4, \\ x^{\frac{3}{2}} + y^{\frac{3}{2}} = 16. \end{cases} \quad (1)$$

$$\text{Dividing (2) by (1),} \quad x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{2}} = 4. \quad (2)$$

$$\text{Squaring (1),} \quad x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 16. \quad (3)$$

$$\text{Subtracting (3) from (4),} \quad 3x^{\frac{1}{2}}y^{\frac{1}{2}} = 12. \quad (4)$$

$$\therefore x^{\frac{1}{2}}y^{\frac{1}{2}} = 4. \quad (5)$$

$$\text{Subtracting (5) from (3),} \quad x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{2}} = 0, \quad (6)$$

$$\text{whence,} \quad x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 0. \quad (6)$$

$$\text{From (1) and (6),} \quad x^{\frac{1}{2}} = 2, \quad (7)$$

$$\text{and} \quad y^{\frac{1}{2}} = 2.$$

$$\text{Hence,} \quad x = 4 \text{ and } y = 8.$$

It is seen from (6) that each of the roots occurs twice.

$$84. \quad \begin{cases} \sqrt{x} - \sqrt{y} = \frac{6}{5}(x - y), \\ \sqrt{xy} = \frac{1}{5}. \end{cases} \quad (1)$$

$$\text{Dividing (1) by } \sqrt{x} - \sqrt{y}, \quad 1 = \frac{6}{5}(\sqrt{x} + \sqrt{y}). \quad (2)$$

$$\text{Multiplying (3) by } \frac{5}{6}, \quad \sqrt{x} + \sqrt{y} = \frac{5}{6}. \quad (3)$$

$$\text{Squaring (4),} \quad x + 2\sqrt{xy} + y = \frac{25}{36}. \quad (4)$$

$$\text{Subtracting (2) } \times 4 \text{ from (5),} \quad x - 2\sqrt{xy} + y = \frac{1}{36}, \quad (5)$$

$$\text{whence,} \quad \sqrt{x} - \sqrt{y} = \pm \frac{1}{6}. \quad (6)$$

$$\text{From (4) and (6),} \quad \sqrt{x} = \frac{1}{2} \text{ or } \frac{1}{3}, \text{ and } \sqrt{y} = \frac{1}{3} \text{ or } \frac{1}{2}. \quad (7)$$

$$\therefore x = \frac{1}{4} \text{ or } \frac{1}{9}, \text{ and } y = \frac{1}{9} \text{ or } \frac{1}{4}.$$

$$\text{Since } \sqrt{x} - \sqrt{y} = 0 \text{ is a factor of (1),} \quad y = x. \quad (7)$$

$$\text{Substituting (7) in (2),} \quad x = y = \pm \frac{1}{6}.$$

$$\text{Hence,} \quad \begin{cases} x = \frac{1}{4}, \frac{1}{9}, \frac{1}{6}, -\frac{1}{6}; \\ y = \frac{1}{9}, \frac{1}{4}, \frac{1}{6}, -\frac{1}{6}. \end{cases}$$

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85. Let x = number of dollars it cost to hire the carriage.

$$\text{Then,} \quad \frac{x}{2} - \frac{x}{6} = 4.$$

$$\text{Solving,} \quad x = 12.$$

Hence, the cost of hiring the carriage was \$12.

86. Let $x =$ the number.

Then, $\frac{x + 12}{12} - 12 = 12.$

Solving, $x = 276$, the number.

87. Let $x =$ number of gallons of 50-cent sirup taken.

Then $45 - x =$ number of gallons of 80-cent sirup taken;

$$\therefore 50x + 80(45 - x) = 45 \cdot 60 = 2700.$$

Solving, $x = 30,$

whence, $45 - x = 15.$

Hence, the grocer must take 30 gallons of the 50-cent sirup and 15 gallons of the 80-cent sirup.

88. Let $x =$ the number of dimes,

and $y =$ the number of quarters.

Then, $x + y = 18,$

and $10x + 25y = 300.$

Solving, $x = 10$, the number of dimes,

and $y = 8$, the number of quarters.

89. Let $x =$ number of pounds of gunpowder.

Then, $\frac{1}{2}x + 5 =$ number of pounds of saltpeter,

and $\frac{1}{5}x - 2 =$ number of pounds of sulphur,

and $\frac{1}{10}x + 1 =$ number of pounds of charcoal;

$$\therefore x = (\frac{1}{2}x + 5) + (\frac{1}{5}x - 2) + (\frac{1}{10}x + 1).$$

Solving, $x = 20,$

whence, $\frac{1}{2}x + 5 = 15$, $\frac{1}{5}x - 2 = 2$, and $\frac{1}{10}x + 1 = 3$.

Hence, the gunpowder was composed of 15 pounds of saltpeter, 2 pounds of sulphur, and 3 pounds of charcoal.

90. Let $x =$ number of miles down the river.

and $12 + 3 = 15$, number of miles per hour downstream,

$12 - 3 = 9$, number of miles per hour upstream;

$$\therefore \frac{x}{15} + \frac{x}{9} = 8.$$

Solving, $x = 45.$

Hence, the steamboat can go 45 miles down the river and return in 8 hours.

91. Let $x =$ number of cents paid per dozen.

Then, $6x = 32 \div \frac{x}{12}.$

Solving, $x = \pm 8.$

Hence, the price was 8 cents per dozen.

92. Let $x =$ number of horses he had.

Then, $\frac{1}{x - 8} =$ part of the stable room occupied by 1 horse,

and $\frac{1}{x + 8} =$ part of the stable room occupied by 1 horse after the new stable was built.

Since the stable room was increased one half,

$$\frac{1}{x - 8} = \frac{\frac{3}{2}}{x + 8}.$$

Solving, $x = 40$, the number of horses.

93. Let x = number of pounds the whole weighed.

Then, $\frac{1}{2}x - 5$ = number of pounds of copper,

$\frac{1}{3}(\frac{1}{2}x + 5) + 5$ = number of pounds of lead and of tin;

$$\therefore \frac{1}{2}x - 5 + \frac{2}{3}(\frac{1}{2}x + 5) + 10 = x.$$

Solving, $x = 50$,

whence, $\frac{1}{2}x - 5 = 20$ and $\frac{1}{3}(\frac{1}{2}x + 5) + 5 = 15$.

Hence, there were 20 pounds of copper and 15 pounds of lead and of tin.

94. Let x = number of minutes after 4 o'clock.

Then, $\frac{x}{12}$ = number of minute spaces traveled by the hour hand after 4 o'clock.

Since the minute hand must travel 50 minute spaces more than the hour hand,

$$x - \frac{x}{12} = 50.$$

Solving, $x = 54\frac{6}{11}$.

Hence, the hands form a straight line at $4:54\frac{6}{11}$ o'clock.

95. Let x = number of miles he may ride.

Then, $\frac{x}{m}$ = number of hours he rides,

and $\frac{x}{n}$ = number of hours he walks;

$$\therefore \frac{x}{m} + \frac{x}{n} = a.$$

Solving, $x = \frac{amn}{m+n}.$

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96. Let x = number of car loads he had at first.

Then, $\frac{1}{2}x - \frac{1}{2}$ = number of car loads left after 1st sale,

and $\frac{1}{2}(\frac{1}{2}x - \frac{1}{2}) - \frac{1}{2} = \frac{1}{4}x - \frac{3}{4}$ = number of car loads left after 2d sale,

$\therefore \frac{1}{8}x - \frac{7}{8} = 0.$

Solving, $x = 7.$

Hence, he had 7 car loads of grain at first.

97. See next page.

98. Let x = number of hours past noon.

Then, $x = \frac{3}{5}(12 - x).$

Solving, $x = 4\frac{1}{2}.$

Hence, the time was 4:30 P.M.

99. Let x = the numerator,

and y = the denominator.

Then, $\frac{x+3}{y+3} = \frac{5}{7},$

and $\frac{x-3}{y-3} = \frac{3}{5}.$

Solving, $x = 12,$

and $y = 18.$

Hence, the fraction is $\frac{12}{18}.$

97. Let x = number of years in A's age,
 and y = number of years in B's age.
 Then, $x - 4 = \frac{1}{2}(y - 4)$,
 and $x + 4 = \frac{2}{3}(y + 4)$.
 Solving, $x = 12$,
 and $y = 20$.

Hence, A is 12 years old, and B is 20 years old.

100. Let x = number of miles per hour he can row in still water.
 Then, $x + 2$ = number of miles per hour downstream,
 $x - 2$ = number of miles per hour upstream;
 $\therefore 2(x + 2) = 4(x - 2)$.
 Solving, $x = 6$.

Hence, his rate of rowing in still water is 6 miles an hour.

101. Let $3x$ = number of days he worked.
 Then, x = number of days he was idle;
 $\therefore \frac{5}{2} \cdot 3x - \frac{3}{2}x = 24$.

Multiplying by $\frac{2}{3}$, $5x - x = 16$.
 $\therefore x = 4$,

whence, $3x = 12$, the number of days he worked.

102. Let x = number of dollars 1st cup is worth,
 and y = number of dollars 2d cup is worth.
 Then, $x + \frac{3}{2} = \frac{15}{8}y$,
 and $y + \frac{3}{2} = \frac{11}{2}x$.
 Solving, $x = 6$,
 and $y = 4$.

Hence, the first cup is worth \$6, and the second, \$4.

103. Let x = number of bales the cave would hold,
 and y = number of casks the cave would hold.
 Then, $\frac{1}{x} =$ part of the cave occupied by 1 bale,
 and $\frac{1}{y} =$ part of the cave occupied by 1 cask.

$$\therefore \frac{13}{x} + \frac{33}{y} = 1,$$

- and $\frac{5}{x} + \frac{9}{y} = \frac{1}{3}$.
 Solving, $x = 24$,
 and $y = 72$.

Hence, the cave would hold 24 bales or 72 casks.

104. Let x = number of tons each cart can carry,
 and y = number of tons each wagon can carry.
 Then, $15x + 12y = 28$,
 and $24x + 8y = 28$.
 Solving, $x = \frac{2}{3}$,
 and $y = \frac{1}{2}$.

Hence, each cart can carry $\frac{2}{3}$ of a ton, and each wagon $1\frac{1}{2}$ tons.

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105. Let x represent the units' digit.

Then, $100x + 10x + x =$ the number;

$$\therefore 100x + 10x + x - 7 \cdot 3x = 180.$$

Solving,

$$x = 2.$$

Hence, the number is 222.

106. Let $x =$ number of days in which A can do it,

$y =$ number of days in which B can do it,

$z =$ number of days in which C can do it.

and

Then,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{m}, \quad (1)$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{n}, \quad (2)$$

and

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{p}. \quad (3)$$

Adding and dividing by 2,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{np + mp + mn}{2mnp}. \quad (4)$$

Since (4) represents the part of the work all together can do in one day, it will take them all $1 \div \frac{np + mp + mn}{2mnp}$ days, or $\frac{2mnp}{mp + np + mn}$ days, to complete the work.

Subtracting (2), (3), and (1) in succession from (4), and solving,

$$x = \frac{2mnp}{-mp + np + mn}, \text{ number of days in which A can do it,}$$

$$y = \frac{2mnp}{mp + np - mn}, \text{ number of days in which B can do it,}$$

$$\text{and } z = \frac{2mnp}{mp - np + mn}, \text{ number of days in which C can do it.}$$

107. Let $x =$ number of pounds of baggage allowed each passenger without charge.

Under the first condition $(400 - 2x)$ pounds are subject to charge at $\frac{100}{400 - 2x}$ cents per pound, and under the second condition $(400 - x)$

pounds at $\frac{150}{400 - x}$ cents per pound.

$$\text{Therefore, } \frac{100}{400 - 2x} = \frac{150}{400 - x}.$$

$$\text{Solving, } x = 100, \text{ number of pounds allowed.}$$

108. See next page.

109. Let $x =$ number of inches in each side of one of the larger tiles.

$$\text{Then, } 1000x^2 = 1440(x - 1)^2.$$

$$\text{Dividing by 40, } 25x^2 = 36(x - 1)^2.$$

$$\text{Extracting the square root, } 5x = \pm 6(x - 1),$$

whence, rejecting the negative root, $x = 6$.

Since each tile is 6 inches square, its area is $\frac{1}{4}$ of a square foot; and the area of the floor is 1000 times $\frac{1}{4}$, or 250, square feet.

108. Let $x =$ one part,
 and $y =$ the other.
 Then, $x + y = 20,$ (1)
 and $\frac{x}{y} + \frac{y}{x} = \frac{17}{4}.$ (2)
 Reducing (2), $4x^2 - 17xy + 4y^2 = 0.$
 Factoring, $(x - 4y)(4x - y) = 0,$
 whence, $y = \frac{1}{4}x$ or $4x.$ (3)
 Substituting (3) in (1), $x = 16$ or $4,$
 whence, $y = 4$ or $16.$
 Hence, the parts of 20 are 16 and 4.

110. Let $x =$ larger number,
 and $y =$ smaller number.
 Then, $x + y = 16,$
 and $x^2 - y^2 = 128.$
 Solving, $x = 12,$ larger number,
 and $y = 4,$ smaller number.

111. Let $x =$ greater number,
 and $y =$ less number.
 Then, $x + y = xy = x^2 - y^2.$

Since these equations stand for three simultaneous equations, any two of which are independent, while the third is derived from the two independent equations, we may select the equations

- $\begin{cases} x + y = xy, \\ x^2 - y^2 = xy. \end{cases}$ (1)
 Dividing (2) by (1), $x - y = 1.$ (3)
 Squaring (3), $x^2 - 2xy + y^2 = 1.$ (4)
 From (1), $4xy - 4(x + y) = 0.$ (5)
 Adding (5) to (4), $x^2 + 2xy + y^2 - 4(x + y) = 1.$
 Completing the square, $(x + y)^2 - 4(x + y) + 4 = 5.$
 Extracting the square root, $x + y - 2 = \pm \sqrt{5}.$
 $\therefore x + y = 2 \pm \sqrt{5}.$ (6)
 From (6) and (3), $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{3}{2} - \frac{1}{2}\sqrt{5},$
 and $y = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1}{2} - \frac{1}{2}\sqrt{5}.$
 Hence, the numbers are $\frac{3}{2} + \frac{1}{2}\sqrt{5}$ and $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{3}{2} - \frac{1}{2}\sqrt{5}$ and $\frac{1}{2} - \frac{1}{2}\sqrt{5}.$

112. Let $x =$ one part,
 and $y =$ the other part.
 Then, $x + y = 25,$
 and $\sqrt{x} - \sqrt{y} = 1.$
 Solving, $x = 16$ or $9,$
 and $y = 9$ or $16.$
 Hence, the parts are 16 and 9.

113. Let $x =$ greater number,
 and $y =$ less number.
 Then, $x - y = 6,$
 and $xy = 2y^3.$
 Solving, the greater number is $x = 6$ or 8 or $\frac{3}{2},$
 and the less number is $y = 0$ or 2 or $-\frac{3}{2}.$

114. Let x = number of men.
 Then, $x^2 = 4(x + 3)$.
 Solving, $x = 6$ or -2 .
 Hence, there were 6 men.

115. Let x = one number,
 and y = the other.
 Then, $xy = 8$,
 and $x^2 + y^2 = x + y + 14$.
 Solving, $x = 4, 2, \frac{1}{2}(-5 + \sqrt{-7}), \frac{1}{2}(-5 - \sqrt{-7})$,
 and $y = 2, 4, \frac{1}{2}(-5 - \sqrt{-7}), \frac{1}{2}(-5 + \sqrt{-7})$.
 Hence, the numbers are 4 and 2 or $\frac{1}{2}(-5 + \sqrt{-7})$ and $\frac{1}{2}(-5 - \sqrt{-7})$.

116. Let x = number of feet in width of walk.
 Since the lawn and the walk together form a rectangle $2x$ feet longer
 and $2x$ feet wider than the lawn,
 $(50 + 2x)(40 + 2x) - 50 \cdot 40 = 64 \cdot 9$.
 Solving, $x = 3$ or -48 .
 Hence, the walk is 3 feet wide.

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117. Let x = number of dollars the goods cost.
 Then, $\frac{x}{100}$ of x , or $\frac{x^2}{100}$ = number of dollars gained;
 $\therefore x + \frac{x^2}{100} = 56$.
 Solving, $x = 40$ or -140 .
 Hence, the cost of the goods was \$40.

118. Let x = number of miles per hour he can swim in still water.
 Then, $x + 1\frac{1}{2}$ = number of miles per hour downstream,
 and $x - 1\frac{1}{2}$ = number of miles per hour upstream;
 $\therefore \frac{1}{x - 1\frac{1}{2}} = \frac{1}{x + 1\frac{1}{2}} \times 3$.
 Solving, $x = 3$.
 Hence, he can swim 3 miles per hour in still water.

119. Let x = number of oxen he bought.
 Then, $\frac{900}{x}$ = number of dollars he paid for each;
 $\therefore \left(\frac{900}{x} + 20\right)(x - 5) = 900 + 350$.
 Solving, $x = 30$ or $-\frac{15}{2}$.
 Hence, he bought 30 oxen.

120. Let x = number of men in front at first,
 and y = number of men in depth at first.
 Then, $y = x + 5$,
 and $xy = 5(x + 845)$.
 Solving, $x = 65$ or -65 ,
 and $y = 70$ or -60 .
 Hence, the number of men was 65×70 , or 4550.

121. Let x = number of feet in length of bar,
 and y = number of pounds weight per foot.
 Then, $xy = 36$,
 and $(x + 1)(y - \frac{1}{2}) = 36$.
 Solving, $x = 8$ or -9 ,
 and $y = 4\frac{1}{2}$ or -4 .

Hence, the bar was 8 feet long and weighed $4\frac{1}{2}$ pounds to the foot.

122. Let x = number of yards in length of side of larger granary.
 and y = number of yards in length of side of smaller granary.
 Then, $x - y = 3$,
 and $x^2 - y^2 = 117$.
 Solving, $x = 5$ or -2 ,
 and $y = 2$ or -5 .

Hence, the side of the larger is 5 yards, of the smaller, 2 yards.

123. Let x = number of days.

Then,
$$x \cdot \frac{19.20}{x-2} = (x-2) \cdot \frac{30}{x}.$$

Solving,
$$x = 10 \text{ or } \frac{10}{9}.$$

Since the second value is less than 2, it is inadmissible.

Hence, the number of days was 10, A's daily wages were $\frac{30}{10}$ dollars, or \$3.
 and B's daily wages were $\frac{19.20}{10-2}$ dollars, or \$2.40.

124. Let x = number of miles an hour in usual rate.

Then,
$$\frac{300}{x-5} - \frac{300}{x} = \frac{5}{6}.$$

Solving,
$$x = 45 \text{ or } -40,$$

whence, rejecting the negative value,
$$\frac{300}{x} = 6\frac{2}{3}.$$

Hence, the train usually ran 45 miles an hour and completed the journey in $6\frac{2}{3}$ hours.

125. Let x = one number,
 and y = the other.
 Then, $x + y = xy = x^2 + y^2.$

Since these equations stand for three simultaneous equations any two of which are independent and the third derived from the two independent equations, we may select the equations

$$\begin{cases} x^2 + y^2 = x + y, \\ xy = x + y. \end{cases} \quad (1)$$

$$xy = x + y. \quad (2)$$

Adding (2) $\times 2$ to (1), $x^2 + 2xy + y^2 = 3(x + y).$ (3)

Transposing, $(x + y)^2 - 3(x + y) = 0.$

Factoring, $(x + y - 3)(x + y) = 0.$

$$\therefore x + y = 3 \text{ or } 0, \quad (4)$$

whence, $x = 3 - y \text{ or } -y. \quad (5)$

Substituting $3 - y$ for x in (2), $y = \frac{1}{2}(3 + \sqrt{-3}) \text{ or } \frac{1}{2}(3 - \sqrt{-3}),$

whence, $x = \frac{1}{2}(3 - \sqrt{-3}) \text{ or } \frac{1}{2}(3 + \sqrt{-3}).$

Substituting $-y$ for x in (2), $y = 0,$

whence, $x = 0.$

Hence, the numbers are $\frac{1}{2}(3 + \sqrt{-3})$ and $\frac{1}{2}(3 - \sqrt{-3})$ or 0 and 0.

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126. Let x = number of pounds in the first lot.

Then, $x - 20$ = number of pounds in the second lot;

$$\therefore x^2 + (x - 20)^2 = 3400.$$

Solving, $x = 50$ or -30 ,

whence, rejecting the negative value, $x - 20 = 30$.

Hence, there were 50 pounds in the first lot and 30 pounds in the second.

127. See next page.

128. Let x = number of cents per dozen in first price,

$x - 1$ = number of cents per dozen in second price.

Then, $\frac{60}{x - 1} = \frac{60}{x} + 5$.

Solving, $x = 4$ or -3 .

Hence, she sold apples for 4 cents per dozen at first.

129. Let x = number of miles per hour traveled by the train from A,

and y = number of miles per hour traveled by the train from B.

Then, $\frac{300}{x + y}$ = number of hours till they meet.

Since at the time of their meeting the distance each has yet to go is equal to the distance the other has already traveled,

$\frac{300y}{x + y}$ = number of miles the train from A has yet to travel after meeting,

and $\frac{300x}{x + y}$ = number of miles the train from B has yet to travel after meeting.

Therefore, $\frac{300y}{x + y} \div x = 9$, (1)

and $\frac{300x}{x + y} \div y = 4$. (2)

Dividing (1) by (2), $\frac{y^2}{x^2} = \frac{9}{4}$. (3)

$$\therefore \frac{y}{x} = \pm \frac{3}{2}.$$

Rejecting the negative root, $y = \frac{3}{2}x$. (4)

Substituting (4) in (1), $x = 20$, (5)

whence, $y = 30$,

and $\frac{300}{x + y} = 6$.

Hence, the trains meet 6 hours after starting, the train from A travels 20 miles per hour, and the train from B travels 30 miles per hour.

127. Let x = number of shillings paid per week.

Under the first condition A had to pay $(x - 18)$ shillings for the pasture of 4 horses. Hence, $\frac{x - 18}{4}$ shillings per horse were paid by both A and B, and the number of horses in the pasture was $x \div \frac{x - 18}{4}$, or $\frac{4x}{x - 18}$.

Similarly, under the second condition, the number of horses was $\frac{4x}{x - 20}$.

Since the second condition was due to increasing the number of horses by 2,

$$\frac{4x}{x - 20} = \frac{4x}{x - 18} + 2.$$

Reducing, $x^2 - 42x + 360 = 0$.

Solving this equation, $x = 30$ or 12.

The second value is evidently inadmissible.

Hence, the cost of hiring the pasture was 30 shillings per week.

128, 129. See preceding page.

130. Let x = number of seconds for 1 revolution.

Then, since $14\frac{2}{3}$ is contained 360 times in 5280, the number of feet in 1 mile,

$360x$ = number of seconds for 1 mile,

$\frac{360x}{3600}$, or $\frac{x}{10}$ = number of hours for 1 mile,

whence, $\frac{10}{x}$ = number of miles per hour the carriage is traveling;

also, if it takes the carriage wheel 1 second longer to make 1 revolution,

$\frac{10}{x + 1}$ = number of miles per hour the carriage would travel;

$$\therefore \frac{10}{x + 1} = \frac{10}{x} - \frac{8}{3}.$$

Solving,

$$x = \frac{3}{2} \text{ or } -\frac{5}{2},$$

whence, rejecting the negative value,

$$\frac{10}{x} = \frac{10}{\frac{3}{2}} = 6\frac{2}{3}.$$

Hence, the carriage is traveling $6\frac{2}{3}$ miles per hour.

131. Let x = number of miles per hour traveled before accident,
and y = number of hours required to make the trip at the rate of x miles per hour.

Then, xy = number of miles in the whole distance,
 $2x$ = number of miles traveled before accident,

and $\frac{xy - 2x}{\frac{3}{5}x}$ = number of hours occupied in completing trip.

$$\therefore 2 + 1 + \frac{xy - 2x}{\frac{3}{5}x} = y + 7\frac{2}{3}. \quad (1)$$

Under the second condition,

$2x + 50$ = number of miles traveled before accident,

$2 + \frac{50}{x}$ = number of hours before accident,

and $\frac{xy - 2x - 50}{\frac{3}{5}x}$ = number of hours occupied in completing trip.

$$\therefore 2 + \frac{50}{x} + 1 + \frac{xy - 2x - 50}{\frac{3}{5}x} = y + 6\frac{1}{3}. \quad (2)$$

From (1), $y = 12.$ (3)

Subtracting (2) from (1), $-\frac{50}{x} + \frac{50}{\frac{3}{5}x} = \frac{4}{3}.$ (4)

Solving (4), $x = 25.$ (5)

From (3) and (5), $xy = 300.$

Hence, the whole distance was 300 miles.

132. Let $x =$ number of acres,
and $y =$ number of dollars paid per acre.

Then, $xy = 200,$

and $(x - 5)(y + 1) = 210.$

Solving, $x = 40$ or $-25,$

and $y = 5$ or $-8.$

Hence, he rents 40 acres at \$5 per acre.

133. Let $x =$ number of miles per hour A walked,
and $y =$ number of miles per hour B walked.

Then, $\frac{30}{x} = \frac{30}{y} + 2,$ (1)

and $\frac{42}{x + \frac{1}{2}} = \frac{42}{y + \frac{1}{2}} + 2.$ (2)

From (1), $15y = 15x + xy.$ (3)

Multiplying each member of (2) by $2(x + \frac{1}{2})(y + \frac{1}{2}),$

$$84y + 42 = 84x + 42 + 4xy + 2x + 2y + 1. \quad (4)$$

Canceling $42 = 42,$ subtracting (3) $\times 4,$ and reducing,

$$22y = 26x + 1. \quad (5)$$

Multiplying (3) by 22, $15 \cdot 22y = 330x + x \cdot 22y.$ (6)

Substituting (5) in (6), $15(26x + 1) = 330x + x(26x + 1).$

$$26x^2 - 59x - 15 = 0.$$

$$(2x - 5)(13x + 3) = 0.$$

$$\therefore x = \frac{5}{2} \text{ or } -\frac{3}{13}.$$

Rejecting the negative value and substituting $\frac{5}{2}$ for x in (5),

$$y = 3.$$

Hence, A traveled $2\frac{1}{2}$ miles per hour and B 3 miles per hour.

RATIO AND PROPORTION

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PROOFS OF PRINCIPLES 2-12

2. Let $a : b = c : d.$

Prin. 1, $ad = bc.$

Solving for $a, d, b,$ and $c,$ respectively,

$$a = \frac{bc}{d}; d = \frac{bc}{a}; b = \frac{ad}{c}; \text{ and } c = \frac{ad}{b}.$$

3. Let $ad = bc.$

Dividing by $bd,$ $\frac{a}{b} = \frac{c}{d},$

or, § 310, $a : b = c : d.$

Similarly, dividing both members of $bc = ad$ by ac it is found that $b : a = d : c.$

4. Let

Prin. 1,

Dividing by cd ,

or, § 310,

$$a:b = c:d.$$

$$ad = bc.$$

$$\frac{a}{c} = \frac{b}{d},$$

$$a:c = b:d.$$

5. Let

Prin. 1,

Dividing by ac ,

or, § 310,

$$a:b = c:d.$$

$$bc = ad.$$

$$\frac{b}{a} = \frac{d}{c},$$

$$b:a = d:c.$$

6. Let

§ 310,

Adding 1 to each member,

or, § 310,

Again, dividing (3) by (2),

or, § 310,

$$a:b = c:d. \quad (1)$$

$$\frac{a}{b} = \frac{c}{d}. \quad (2)$$

$$\frac{a+b}{b} = \frac{c+d}{d}, \quad (3)$$

$$a+b:b = c+d:d. \quad (4)$$

$$\frac{a+b}{a} = \frac{c+d}{c}, \quad (5)$$

$$a+b:a = c+d:c. \quad (6)$$

7. Let

§ 310,

Subtracting 1 from each member,

or, § 310,

Again, dividing (3) by (2),

or, § 310,

$$a:b = c:d. \quad (1)$$

$$\frac{a}{b} = \frac{c}{d}. \quad (2)$$

$$\frac{a-b}{b} = \frac{c-d}{d}, \quad (3)$$

$$a-b:b = c-d:d. \quad (4)$$

$$\frac{a-b}{a} = \frac{c-d}{c}, \quad (5)$$

$$a-b:a = c-d:c. \quad (6)$$

8. Let

Prin. 6,

Prin. 7,

Dividing (2) by (3),

or, § 310,

$$a:b = c:d. \quad (1)$$

$$\frac{a+b}{b} = \frac{c+d}{d}. \quad (2)$$

$$\frac{a-b}{b} = \frac{c-d}{d}. \quad (3)$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d},$$

$$a+b:a-b = c+d:c-d.$$

The same result may be obtained by dividing (5) in the proof of Prin. 6 by (5) in the proof of Prin. 7.

9. Let

§ 310,

Raising to the n th power, § 218, 4.

or, § 310,

$$a:b = c:d. \quad (1)$$

$$\frac{a}{b} = \frac{c}{d}. \quad (2)$$

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}, \quad (3)$$

$$a^n:b^n = c^n:d^n.$$

Taking the principal n th root of each member of (2), § 229, 4,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}},$$

or, § 310,

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

10. Let

$$a : b = c : d. \quad (1)$$

§ 310,

$$\frac{a}{b} = \frac{c}{d}. \quad (2)$$

§ 165,

$$\frac{ma}{mb} = \frac{nc}{nd}, \quad (3)$$

or

$$ma : mb = nc : nd. \quad (4)$$

Again, multiplying both members of (2) by $\frac{m}{n}$,

$$\frac{ma}{nb} = \frac{mc}{nd},$$

or, § 310,

$$ma : nb = mc : nd. \quad (5)$$

Similarly, if (4) or (5) is given, (1) may be obtained.

11. Let

$$a : b = c : d \text{ and } x : y = z : w.$$

§ 310,

$$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{x}{y} = \frac{z}{w}.$$

Ax. 4,

$$\frac{ax}{by} = \frac{cz}{dw},$$

or, § 310,

$$ax : by = cz : dw.$$

Ax. 5,

$$\frac{a}{b} \div \frac{x}{y} = \frac{c}{d} \div \frac{z}{w}, \text{ or } \frac{ay}{bx} = \frac{cw}{dz},$$

or, § 310,

$$ay : bx = cw : dz.$$

12. Let

$$a : b = c : d \text{ and } c : d = e : f.$$

Then, Ax. 1, since the ratios $a : b$ and $e : f$ are each equal to $c : d$,

$$a : b = e : f.$$

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13. Given

$$a : b = c : d.$$

By alternation,

$$a : c = b : d.$$

By inversion,

$$c : a = d : b,$$

that is,

$$d : b = c : a.$$

14. Given

$$a : b = c : d.$$

By alternation,

$$a : c = b : d.$$

By inversion,

$$c : a = d : b.$$

Writing as a fractional equation,

$$\frac{c}{a} = \frac{d}{b},$$

which may be written

$$c \times \frac{1}{a} = d \times \frac{1}{b}.$$

Writing as a proportion,

$$c : d = \frac{1}{b} : \frac{1}{a}.$$

15. Given

By alternation,

or

Cubing each proportional,

$$a : b = c : d.$$

$$a : c = b : d,$$

$$b : d = a : c.$$

$$b^3 : d^3 = a^3 : c^3.$$

16. Given

Squaring each proportional,

Expressing as a fractional equation,

Dividing each member by c^2 ,

Expressing as a proportion,

$$a : b = c : d.$$

$$a^2 : b^2 = c^2 : d^2.$$

$$\frac{a^2}{b^2} = \frac{c^2}{d^2}.$$

$$\frac{a^2}{b^2 c^2} = \frac{1}{d^2}.$$

$$a^2 : b^2 c^2 = 1 : d^2.$$

17. Given

Expressing as a fractional equation,

Multiplying each member by $\frac{m}{\frac{1}{2}}$,

Expressing as a proportion,

$$a : b = c : d.$$

$$\frac{a}{b} = \frac{c}{d}.$$

$$\frac{ma}{\frac{b}{2}} = \frac{mc}{\frac{d}{2}}.$$

$$ma : \frac{b}{2} = mc : \frac{d}{2}.$$

18. Given

Expressing as a fractional equation,

Multiplying each member by $\frac{c}{d}$,

Expressing as a proportion,

$$a : b = c : d.$$

$$\frac{a}{b} = \frac{c}{d}.$$

$$\frac{ac}{bd} = \frac{c^2}{d^2}.$$

$$ac : bd = c^2 : d^2.$$

19. Given

Expressing as an integral equation,

Extracting the square root of each member,

$$a : b = c : d.$$

$$ad = bc.$$

$$\sqrt{ad} = \sqrt{bc},$$

$$\sqrt{ad} \times 1 = \sqrt{b} \times \sqrt{c}.$$

$$\sqrt{ad} : \sqrt{b} = \sqrt{c} : 1.$$

which may be written,

Expressing as a proportion,

20. Given

By composition and division,

By alternation,

21. See next page.

22. Given

By alternation,

Expressing as a fractional equation,

Multiplying the first member by $\frac{2}{3}$, and the second by $\frac{8}{12}$,

Expressing as a proportion,

By composition and division,

$$a : b = c : d.$$

$$a : c = b : d.$$

$$\frac{a}{c} = \frac{b}{d}.$$

$$\frac{2a}{3c} = \frac{8b}{12d}.$$

$$2a : 3c = 8b : 12d.$$

$$2a + 3c : 2a - 3c = 8b + 12d : 8b - 12d.$$

21. Given $a:b=c:d$. (1)

Raising each proportional to the fourth power,

$$a^4:b^4=c^4:d^4. \quad (2)$$

By division, $a^4-b^4:c^4-d^4=c^4:d^4$. (3)

Taking (1) by division, $a-b:a=c-d:c$. (4)

Dividing (3) by (4), $a^3+a^2b+ab^2+b^3:a^3=c^3+c^2d+cd^2+d^3:c^3$.

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26.
$$\frac{\sqrt{x} + \sqrt{m}}{\sqrt{x} - \sqrt{m}} = \frac{m}{n}.$$

By composition and division,

$$\frac{2\sqrt{x}}{2\sqrt{m}} = \frac{m+n}{m-n}.$$

Dividing both terms of the first ratio by 2,

$$\frac{\sqrt{x}}{\sqrt{m}} = \frac{m+n}{m-n}.$$

Squaring,

$$\frac{x}{m} = \frac{(m+n)^2}{(m-n)^2}.$$

Solving,

$$x = \frac{m(m+n)^2}{(m-n)^2}.$$

27.
$$\frac{\sqrt{x} + \sqrt{2a}}{\sqrt{x} - \sqrt{2a}} = \frac{2}{1}.$$

By composition and division,

$$\frac{2\sqrt{x}}{2\sqrt{2a}} = \frac{3}{1}.$$

Dividing both terms of the first ratio by 2,

$$\frac{\sqrt{x}}{\sqrt{2a}} = \frac{3}{1}.$$

Squaring,

$$\frac{x}{2a} = \frac{9}{1}.$$

Solving,

$$x = 18a.$$

28. See next page.

29.
$$\frac{\sqrt{x+b} + \sqrt{x-b}}{\sqrt{x+b} - \sqrt{x-b}} = a.$$

Writing $\frac{a}{1}$ for a and taking the proportion by composition and division,

$$\frac{2\sqrt{x+b}}{2\sqrt{x-b}} = \frac{a+1}{a-1}.$$

Dividing both terms of the first ratio by 2 and squaring,

$$\frac{x+b}{x-b} = \frac{a^2+2a+1}{a^2-2a+1}.$$

By composition and division,

$$\frac{2x}{2b} = \frac{2(a^2+1)}{4a}.$$

Dividing both terms of each ratio by 2,

$$\frac{x}{b} = \frac{a^2+1}{2a}.$$

Solving,

$$x = \frac{b(a^2+1)}{2a}.$$

28.

$$\frac{x + \sqrt{x-1}}{x - \sqrt{x-1}} = \frac{13}{7}.$$

By composition and division,

$$\frac{2x}{2\sqrt{x-1}} = \frac{20}{6}.$$

Dividing both terms of each ratio by 2,

$$\frac{x}{\sqrt{x-1}} = \frac{10}{3}.$$

Squaring,

$$\frac{x^2}{x-1} = \frac{100}{9}.$$

Solving,

$$x = 10 \text{ or } \frac{10}{9}.$$

30.

$$\frac{\sqrt{a} + \sqrt{a-x}}{\sqrt{a} - \sqrt{a-x}} = \frac{1}{a}.$$

By composition and division,

$$\frac{2\sqrt{a}}{2\sqrt{a-x}} = \frac{1+a}{1-a}.$$

Dividing both terms of the first ratio by 2 and squaring,

$$\frac{a}{a-x} = \frac{1+2a+a^2}{1-2a+a^2}.$$

By division,

$$\frac{x}{a} = \frac{4a}{(1+a)^2}.$$

Solving,

$$x = \frac{4a^2}{(1+a)^2}.$$

31.

$$\frac{\sqrt{ax} - b}{\sqrt{ax} + b} = \frac{3\sqrt{ax} - 2b}{3\sqrt{ax} + 5b}.$$

By composition and division,

$$\frac{2\sqrt{ax}}{2b} = \frac{6\sqrt{ax} + 3b}{7b}.$$

Dividing both consequents by b , and both terms of the first ratio by 2,

$$\frac{\sqrt{ax}}{1} = \frac{6\sqrt{ax} + 3b}{7}.$$

Solving for \sqrt{ax} ,

$$\sqrt{ax} = 3b.$$

Squaring,

$$ax = 9b^2.$$

$$\therefore x = \frac{9b^2}{a}.$$

32.

$$\frac{\sqrt{x+2} + \sqrt{x+1}}{\sqrt{x+2} + \sqrt{x-1}} = \frac{3}{1}.$$

By composition and division, dividing both terms of each ratio by 2,

$$\frac{\sqrt{x+2} + \sqrt{x}}{1} = \frac{2}{1}.$$

Therefore,

$$\sqrt{x+2} + \sqrt{x} = 2.$$

Solving,

$$x = \frac{1}{4}.$$

33.

$$\frac{\sqrt{a} + \sqrt{a+x}}{\sqrt{a} - \sqrt{a+x}} = \frac{\sqrt{b} + \sqrt{x-b}}{\sqrt{b} - \sqrt{x-b}}$$

By composition and division, dividing both terms of each ratio by 2,

$$\frac{\sqrt{a}}{\sqrt{a+x}} = \frac{\sqrt{b}}{\sqrt{x-b}}$$

Squaring,

$$\frac{a}{a+x} = \frac{b}{x-b}$$

Solving,

$$x = \frac{2ab}{a-b}$$

34.

$$\frac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = \frac{a-b}{a+b}$$

By composition and division, dividing both terms of each ratio by 2,

$$\frac{a}{\sqrt{2ax - x^2}} = \frac{a}{b}$$

Therefore,

$$\sqrt{2ax - x^2} = b.$$

Squaring,

$$2ax - x^2 = b^2.$$

Solving,

$$x = a \pm \sqrt{a^2 - b^2}.$$

35. See next page.

36. Let x = number of dollars in smaller share.Then, $35 - x$ = number of dollars in larger share;

$$\therefore \frac{x}{35-x} = \frac{3}{4}$$

By composition,

$$\frac{35}{x} = \frac{7}{3}$$

$$\therefore x = 15,$$

$$35 - x = 20.$$

whence,

Hence, the smaller share is \$15 and the larger is \$20.

37. Let

 x = greater number,

and

 y = less number.

Then,

$$\frac{x}{y} = \frac{3}{2}, \quad (1)$$

and

$$\frac{x+4}{y+4} = \frac{4}{3}. \quad (2)$$

From (1),

$$3y = 2x. \quad (3)$$

Multiplying both terms of the first ratio in (2) by 3, and both terms of the second ratio by x ,

$$\frac{3x+12}{3y+12} = \frac{4x}{3x}. \quad (4)$$

Substituting (3) in (4),

$$\frac{3x+12}{2x+12} = \frac{4x}{3x}. \quad (5)$$

Taking (5) by composition and division,

$$\frac{5x+24}{x} = \frac{7x}{x}.$$

$$\therefore 5x+24 = 7x.$$

$$x = 12, \text{ greater number.} \quad (6)$$

$$y = 8, \text{ less number.}$$

Substituting (6) in (3),

35.

$$\frac{\sqrt{x+1} + \sqrt{x-2}}{\sqrt{x+1} - \sqrt{x-2}} = \frac{\sqrt{x-3} + \sqrt{x-4}}{\sqrt{x-3} - \sqrt{x-4}}.$$

By composition and division, dividing both terms of each ratio by 2,

$$\frac{\sqrt{x+1}}{\sqrt{x-2}} = \frac{\sqrt{x-3}}{\sqrt{x-4}}.$$

Squaring,

$$\frac{x+1}{x-2} = \frac{x-3}{x-4}.$$

By composition and division,

$$\frac{2x-1}{3} = \frac{2x-7}{1}.$$

Solving,

$$x = 5.$$

38. Let
and x = greater part, y = less part.

Then,

$$x + y = 16, \quad (1)$$

and

$$\frac{xy}{x^2 + y^2} = \frac{3}{10}. \quad (2)$$

Multiplying each ratio in (2) by 2 and taking the proportion by inversion,

$$\frac{x^2 + y^2}{2xy} = \frac{5}{3}.$$

By composition and division, $\frac{x^2 + 2xy + y^2}{x^2 - 2xy + y^2} = \frac{8}{2} = \frac{4}{1}.$ Substituting 16 for $x + y$, etc.,

$$x - y = \sqrt{\frac{16^2}{4}} = 8. \quad (3)$$

From (1) and (3),
and $x = 12$, greater part, $y = 4$, less part.

39. Let

Then,

 x = smaller part. $25 - x$ = larger part ;

$$\therefore \frac{26 - x}{x - 1} = \frac{4}{1}.$$

By composition,

$$\frac{25}{x - 1} = \frac{5}{1}.$$

$$\therefore x - 1 = 5,$$

whence,
and $x = 6$, smaller part, $25 - x = 19$, larger part.40. Let
and x = one number, y = the other.

Then,

$$x + y = 4, \quad (1)$$

and

$$\frac{x^2 + 2xy + y^2}{x^2 + y^2} = \frac{8}{5}. \quad (2)$$

Substituting 16 for $x^2 + 2xy + y^2$ in (2) and changing $\frac{8}{5}$ to $\frac{16}{10}$,

$$\frac{16}{x^2 + y^2} = \frac{16}{10}.$$

$$\therefore x^2 + y^2 = 10, \quad (3)$$

Taking (2) by division,

$$\frac{2xy}{x^2 + y^2} = \frac{3}{5}. \quad (4)$$

Substituting (3) in (4),

$$\frac{2xy}{10} = \frac{3}{5}$$

$$\therefore 2xy = 6. \quad (5)$$

Subtracting (5) from (3),

$$x^2 - 2xy + y^2 = 4,$$

whence,

$$x - y = \pm 2. \quad (6)$$

From (1) and (6),

$$x = 3 \text{ or } 1,$$

and

$$y = 1 \text{ or } 3.$$

Hence, the numbers are 3 and 1.

41. Let x = number of gallons in larger cask,
and y = number of gallons in smaller cask.

Then,

$$\frac{x - 34}{y - 8} = \frac{5}{4}, \quad (1)$$

and

$$\frac{\frac{1}{2}x + 8}{\frac{1}{2}y + 6} = \frac{5}{3}. \quad (2)$$

From (1),

$$4x - 5y = 96. \quad (3)$$

From (2),

$$3x - 5y = 12. \quad (4)$$

Subtracting (4) from (3),

$$x = 84. \quad (5)$$

Substituting (5) in (4),

$$y = 48.$$

Hence, the capacity of the larger cask was 84 gallons, of the smaller, 48 gallons.

VARIATION

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4. Let c stand for the circumference and d for the diameter of any circle, and let k be the constant ratio of c to d .

Then, $c \propto d$, or $c = kd$. (1)

Substituting 3.1416 for c and 1 for d , (1) becomes

$$3.1416 = k.$$

Substituting 3.1416 for k and 100 for d in (1),

$$c = 3.1416 \times 100 = 314.16.$$

Hence, the circumference is 314.16 feet.

5. Let a represent the area and d the diameter of any circle, and let k represent the constant ratio of a to d^2 .

Then, $a \propto d^2$, or $a = kd^2$. (1)

Substituting 78.54 for a and 10 for d , (1) becomes

$$78.54 = k \times 100.$$

$$\therefore k = .7854.$$

Substituting .7854 for k and 20 for d in (1),

$$a = .7854 \times 20^2 = 314.16.$$

Hence, the area is 314.16 square feet.

6. Let d represent the distance in feet and t the time in seconds of any falling body, and let k represent the constant ratio of d to t^2 .

Then, $d \propto t^2$, or $d = kt^2$. (1)

Substituting 64.32 for d and 2 for t , (1) becomes

$$64.32 = k \cdot 4.$$

$$\therefore k = 16.08.$$

Substituting 16.08 for k and 5 for t in (1),

$$d = 16.08 \times 25 = 402.$$

Hence, the stone will fall 402 feet in 5 seconds.

7. Let a represent the area, b the base, and h the altitude of any triangle, and let k represent the constant ratio of a to $b \times h$.

Then, $a \propto bh$, or $a = kbh$. (1)

Substituting 36 for a , 12 for b , and 6 for h , (1) becomes

$$36 = k \times 12 \times 6.$$

$$\therefore k = \frac{1}{2}, \text{ the constant ratio.}$$

Substituting $\frac{1}{2}$ for k , 8 for b , and 10 for h in (1),

$$a = \frac{1}{2} \times 8 \times 10 = 40.$$

Hence, the area of the triangle is 40 square inches.

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8. Let w represent the weight in pounds, l the length in yards, and a the area in square inches of the cross section of a wrought iron bar, and let k represent the constant ratio of w to $l \times a$.

Then, $w \propto la$, or $w = kla$. (1)

Substituting 10 for w , 1 for l , and 1 for a , (1) becomes

$$10 = k \times 1 \times 1.$$

$$\therefore k = 10.$$

Substituting 10 for k , 12 for l , and 16 for a in (1),

$$w = 10 \times 12 \times 16 = 1920.$$

Hence, the wrought iron bar weighs 1920 pounds.

9. Adopting the notation of Ex. 8, and substituting $\frac{1}{2}$ for k , 8 for l , and 144 for a in (1),

$$w = \frac{1}{2} \times 8 \times 144 = 960.$$

Hence, the wooden beam weighs 960 pounds.

10. Adopting the notation of Ex. 8, and substituting $\frac{1}{4}$ for k , $\frac{2}{3}$ for l , and 8 for a in (1),

$$w = \frac{1}{4} \times \frac{2}{3} \times 8 = 4\frac{2}{3}.$$

Hence, the brick weighs $4\frac{2}{3}$ pounds.

11. Let a represent the number of men, t the number of days, and w the number of days work in the piece of work.

Then, $w \propto at$, or $w = at$, k being 1. (1)

Substituting 10 for a and 20 for t , (1) becomes

$$w = 10 \times 20 = 200.$$

Substituting 200 for w and 25 for a in (1),

$$200 = 25 t.$$

$$\therefore t = 8.$$

Hence, 25 men can do the work in 8 days.

12. Adopting the notation of Ex. 11, and substituting ab for w , and c for t in (1), if x = the number of men required,

$$ab = xc.$$

$$\therefore x = \frac{ab}{c}.$$

13. Let W represent the larger weight in pounds and D its distance in feet from the fulcrum; also let w represent the smaller weight and d its distance from the fulcrum, in the same units.

$$\text{Then,} \quad D : d = \frac{1}{W} : \frac{1}{w},$$

$$\text{or} \quad D : d = w : W. \quad (1)$$

$$\S 327, \text{ Prin. 1,} \quad WD = wd. \quad (2)$$

Let x = number of feet the heavier boy has.

Then, $\frac{17}{2} - x$ = number of feet the lighter boy has.

$$\text{Substituting } x \text{ for } D, \frac{17}{2} - x \text{ for } d, 90 \text{ for } W, \text{ and } 80 \text{ for } w \text{ in (2),}$$

$$90x = 80\left(\frac{17}{2} - x\right). \quad (3)$$

$$\text{Solving (3),} \quad x = 4,$$

$$\text{whence,} \quad \frac{17}{2} - x = 4\frac{1}{2}.$$

Hence, the heavier boy has 4 feet of the board and the lighter boy has $4\frac{1}{2}$ feet.

14. Let x = number of feet the greater weight has.

Then, $4 - x$ = number of feet the less weight has.

$$\text{By (2), Ex. 13,} \quad 100x = 60(4 - x).$$

$$\text{Solving,} \quad x = 1\frac{1}{2},$$

$$\text{whence,} \quad 4 - x = 2\frac{1}{2}.$$

Hence, the point of the stick resting on his shoulder is $1\frac{1}{2}$ feet from the 100-pound end, or $2\frac{1}{2}$ feet from the 60-pound end.

15. Let W represent the weight in pounds of a body near the earth's surface and d its distance in miles from the earth's center.

$$\text{Then,} \quad W \propto \frac{1}{d^2} \text{ or } W = \frac{k}{d^2}, \quad (1)$$

k being the constant ratio between W and $\frac{1}{d^2}$.

Since a brick 4000 miles from the earth's center weighs 4 pounds, substituting 4 for W and 4000 for d , (1) becomes

$$4 = \frac{k}{(4000)^2}.$$

$$\therefore k = 4 \times (4000)^2.$$

Substituting $4 \times (4000)^2$ for k and 8000 for d in (1), the weight of the brick 8000 miles from the earth's center is

$$W = \frac{4 \times (4000)^2}{(8000)^2} = 1.$$

Hence, the brick would weigh 1 pound.

16. Adopting the notation of Ex. 6, $d = kt^2$. (1)

Substituting 31.5 for d and 1.4 for t , (1) becomes

$$31.5 = k \times 1.96.$$

$$\therefore k = \frac{31.5}{1.96}.$$

$$\text{Substituting } \frac{31.5}{1.96} \text{ for } k \text{ and } 3 \text{ for } t \text{ in (1), } d = \frac{\frac{31.5}{1.96} \times 9}{1.96} = 144\frac{9}{14}.$$

Hence, the height of the tower is $144\frac{9}{14}$ feet.

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17. Adopting the notation of Ex. 5, $a = kd^2$.

$$\therefore k = \frac{a}{d^2}.$$

Since a represents the area of any circle and d its diameter, while k is the same for all circles, if the area of the first circular grass plot is represented by $1\frac{1}{2}$ and that of the second circular grass plot by x , we have

$$k = \frac{1\frac{1}{2}}{45^2} = \frac{1}{1350}, \quad (1)$$

and

$$k = \frac{x}{60^2} = \frac{x}{3600}, \quad (2)$$

the unit area being the area grazed over by the horse in 1 day.

Comparing the values of k ,

$$\frac{x}{3600} = \frac{1}{1350}.$$

Solving,

$$x = 2\frac{2}{3}.$$

Hence, it would take him $2\frac{2}{3}$ days.

18. Let d represent the distance in feet and u the illumination from a source of light upon an object d feet away.

Then,

$$u \propto \frac{1}{d^2}, \text{ or } u = \frac{k}{d^2}. \quad (1)$$

It is evident from (1) that if the first member, u , is divided by 4, the second member must be divided by 4 by changing d^2 to $4d^2$, that is, to $(2d)^2$, since k cannot be changed. That is, the distance of the screen from the lantern must be doubled.

Hence, the screen must be moved 10 feet farther away.

19. Let l represent the length in inches of a pendulum, and n the number of times the pendulum oscillates in any given time, as 1 second.

Then,

$$n \propto \frac{1}{\sqrt{l}}, \text{ or } n = \frac{k}{\sqrt{l}}. \quad (1)$$

Substituting 1 for n and 39.1 for l in (1), $1 = \frac{k}{\sqrt{39.1}}$.

$$\therefore k = \sqrt{39.1}. \quad (2)$$

Substituting $\sqrt{39.1}$ for k and 2 for n in (1),

$$2 = \frac{\sqrt{39.1}}{\sqrt{l}}.$$

Solving,

$$l = \frac{39.1}{4} = 9.775.$$

Hence, a pendulum that oscillates twice a second must be 9.775 inches long.

20. From (1) and (2), Ex. 19,

$$\sqrt{l} = \frac{k}{n} = \frac{\sqrt{39.1}}{n}.$$

Substituting $\frac{1}{3}$ for n ,

$$\sqrt{l} = \frac{\sqrt{39.1}}{\frac{1}{3}}.$$

Solving,

$$l = 39.1 \times 9 = 351.9.$$

Hence, a pendulum that oscillates once in 3 seconds must be 351.9 inches, or 29.325 feet long.

21. Given

$$x \propto \frac{y}{z}, \text{ or } x = \frac{ky}{z}, \quad (1)$$

and $x = 2$ when $y = 12$ and $z = 2$.

Substituting these values in (1) to obtain the value of k ,

$$2 = \frac{12k}{2}.$$

$$\therefore k = \frac{1}{3}. \quad (2)$$

From (1) and (2),

$$x = \frac{y}{3z}. \quad (3)$$

Substituting 84 for y and 7 for z in (3), $x = \frac{84}{21} = 4$.

22. See next page.

23. Given

$$x \propto \frac{yz}{w^2}, \text{ or } x = \frac{k yz}{w^2}, \quad (1)$$

and $x = 30$ when $y = 3$, $z = 5$, and $w = 4$.

Substituting these values in (1) to obtain k , $30 = \frac{15k}{16}$.

$$\therefore k = 32. \quad (2)$$

From (1) and (2),

$$x = \frac{32 yz}{w^2}.$$

24. Given $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, to prove that $x \propto z$.

Let m be the constant ratio of x to $\frac{1}{y}$, and n the constant ratio of y to $\frac{1}{z}$.

Then,

$$x = \frac{m}{y}, \quad (1)$$

and

$$y = \frac{n}{z}. \quad (2)$$

From (2),

$$\frac{n}{z} = y. \quad (3)$$

Multiplying (1) by (3),

$$\frac{nx}{z} = m.$$

$$\therefore x = \frac{m}{n} z. \quad (4)$$

Since m and n are constants, $\frac{m}{n}$ is a constant.

Hence, (4) may be written

$$x \propto z.$$

25. Given $x \propto y$ and $z \propto y$, to prove that $(x \pm z) \propto y$.

Let m be the constant ratio of x to y , and n the constant ratio of z to y .

Then,

$$x = my, \quad (1)$$

and

$$z = ny. \quad (2)$$

Adding (2) to (1),

$$x + z = (m + n)y. \quad (3)$$

Subtracting (2) from (1),

$$x - z = (m - n)y. \quad (4)$$

Since m and n are constants, $m + n$ and $m - n$ are constants.

Hence,

$$x \pm z \propto y.$$

22. Given $x \propto \frac{y}{z}$, or $x = \frac{ky}{z}$, (1)

and $x = 60$ when $y = 24$ and $z = 2$.

Substituting these values in (1) to obtain the value of k ,

$$60 = \frac{24k}{2}.$$

$$\therefore k = 5. \quad (2)$$

From (1) and (2),

$$x = \frac{5y}{z},$$

whence, the value of y is

$$y = \frac{xz}{5}. \quad (3)$$

Substituting 20 for x and 7 for z in (3), $y = \frac{20 \times 7}{5} = 28.$

26. Let V represent the volume in cubic inches and R the radius in inches, of a sphere.

Then, $V \propto R^3$, or $V = kR^3$. (1)

Let a , b , and c represent the volumes, respectively, of the spheres whose radii are 6 inches, 8 inches, and 10 inches.

Then, by (1), $a = k \times 6^3 = 216k$, (2)

$$b = k \times 8^3 = 512k, \quad (3)$$

and $c = k \times 10^3 = 1000k$. (4)

Adding, the volume of all is $a + b + c = 1728k$. (5)

But (5) may be written $a + b + c = k \times 12^3$.

Hence, by (1), the radius of the resulting sphere is 12 inches.

27. Let V represent the volume in cubic feet, H the altitude in feet, and D the diameter of the base, in feet, of any cone.

Then, $V \propto D^2H$, or $V = kD^2H$. (1)

By (1), $P = k \times 25 \times 10 = 250k$, (2)

and $R = k \times 100 \times 5 = 500k$. (3)

Adding, $P + R = S = 750k$. (4)

Let d represent the diameter of the base, in feet, of S .

Then, by (1), $S = k \times d^2 \times 30 = 30d^2k$. (5)

From (5) and (4), $30d^2k = 750k$. (6)

Solving (6) for d , $d = \pm 5$.

Hence, the diameter of the base of S is 5 feet.

PROGRESSIONS

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11. $s = 1 + 2 + 3 + \dots + 12$
 $= \frac{n}{2}(a + l) = 6 \times 13 = 78.$

12. $l = a + (n - 1)d = 16\frac{1}{12} + 9 \times 32\frac{1}{6} = 305\frac{7}{12}.$
 $s = \frac{n}{2}(a + l) = 5 \times 321\frac{2}{3} = 1608\frac{1}{3}.$

Hence, the body will fall $1608\frac{1}{3}$ feet in 10 seconds.

$$13. \quad s = 8 + 16 + 24 + \cdots 240 \\ = \frac{n}{2}(a + l) = 15 \times 248 = 3720.$$

Hence, she must walk 3720 feet.

$$14. \quad l = a + (n - 1)d = 5 + 29 \times 5 = 150. \\ s = \frac{n}{2}(a + l) = 15 \times 155 = 2325.$$

Hence, he was paid \$1.50 the thirtieth day and earned \$23.25 in 30 days.

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3. In the series 2, 6, 10, \cdots 66, $a = 2$, $l = 66$, and $d = 4$; and n is to be found.

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = \frac{66 - 2}{4} + 1 = 17.$$

4. In the series 1, 6, 11, \cdots 61, $a = 1$, $l = 61$, and $d = 5$; and s is to be found.

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = \frac{61 - 1}{5} + 1 = 13. \\ s = n \left(\frac{a + l}{2} \right) = 13 \times 31 = 403.$$

5. In the series $-1, 2, 5, \cdots$, $a = -1$, $d = 3$, and $s = 221$ are given; and n is to be found.

$$\text{From } l = a + (n - 1)d = -1 + 3(n - 1) \text{ we obtain} \\ l = 3n - 4. \quad (1)$$

$$\text{From } s = \frac{n}{2}(a + l), \text{ or } 221 = \frac{n}{2}(-1 + l) \text{ we obtain} \\ l = \frac{442 + n}{n}. \quad (2)$$

Eliminating l and solving (1) and (2) for n ,
 $n = 13 \text{ or } -11\frac{1}{3}.$

Hence, the given series has 13 terms.

6. In the series 2, 9, 16, \cdots 86, $a = 2$, $l = 86$, and $d = 7$; and n and s are to be found.

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = \frac{86 - 2}{7} + 1 = 13. \\ s = n \left(\frac{a + l}{2} \right) = 13 \times 44 = 572.$$

7. See next page.

8. In the series $\cdots 22, 27, 32, \cdots$, we have $d = 5$, $s = 714$, and $n = 17$; and a and l are to be found.

$$\text{From } l = a + (n - 1)d, \quad l - a = 16 \times 5 = 80, \quad (1)$$

$$\text{and from } s = \frac{n}{2}(a + l), \quad l + a = \frac{2s}{n} = \frac{1428}{17} = 84. \quad (2)$$

$$\text{From (2) and (1),} \quad a = 2, \\ \text{and} \quad l = 82.$$

7. In the series $-10, -8\frac{1}{2}, -7, \dots$ to 10 terms, $a = -10$, $d = \frac{3}{2}$, and $n = 10$; and l and s are to be found.

$$l = a + (n - 1)d = -10 + 9 \times \frac{3}{2} = 3\frac{1}{2}.$$

$$s = \frac{n}{2}(a + l) = 5(-10 + 3\frac{1}{2}) = -32\frac{1}{2}.$$

9. Given $s = 113\frac{2}{3}$, $a = \frac{1}{3}$, and $d = 2$, to find n .

$$\text{From } s = \frac{n}{2}(a + l), \quad n = \frac{2s}{a + l} = \frac{227\frac{1}{3}}{\frac{1}{3} + l} = \frac{682}{1 + 3l}. \quad (1)$$

Substituting $a + (n - 1)d = \frac{1}{3} + 2n - 2$ for l in (1),

$$n = \frac{682}{1 + 1 + 6n - 6} = \frac{682}{6n - 4} = \frac{341}{3n - 2}. \quad (2)$$

Clearing (2) of fractions,

$$3n^2 - 2n = 341. \quad (3)$$

Solving (3),

$$n = 11 \text{ or } -\frac{31}{3}.$$

Rejecting the negative value,

$$n = 11.$$

10. In the series $-16, -11, -6, \dots$ 34, $a = -16$, $l = 34$, and $d = 5$; and s is required.

$$s = n \left(\frac{a + l}{2} \right) = n \left(\frac{-16 + 34}{2} \right) = 9n. \quad (1)$$

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = \frac{34 + 16}{5} + 1 = 11. \quad (2)$$

$$\text{Substituting 11 for } n \text{ in (1),} \quad s = 99.$$

11. In the series $\dots -1, 3, 7, \dots$ 23, $d = 4$, $l = 23$, and $n = 16$; and s is required.

$$s = \frac{n}{2}(a + l) = 8a + 184. \quad (1)$$

$$\text{From } l = a + (n - 1)d, \quad a = l - (n - 1)d = 23 - 15 \times 4 = -37. \quad (2)$$

$$\text{Substituting } -37 \text{ for } a \text{ in (1),} \quad s = -296 + 184 = -112.$$

12. Given $d = 2$, $s = 300$, and $n = 20$, to find a and l .

$$\text{From } s = \frac{n}{2}(a + l), \quad a + l = \frac{2s}{n} = \frac{600}{20} = 30. \quad (1)$$

$$\text{From } l = a + (n - 1)d, \quad a - l = -(n - 1)d = -19 \times 2 = -38. \quad (2)$$

and

$$\begin{aligned} a &= -4, \\ l &= 34. \end{aligned}$$

13. In the series $1, 5, 9, \dots$ l , $a = 1$, $d = 4$, and $l = l$; and n is required.

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = \frac{l - 1}{4} + 1 = \frac{l + 3}{4}.$$

Hence, the number of terms is $\frac{l + 3}{4}$, which is known when l is known.

14. Given $a = x$, $l = y$, and $n = b$, to find s .

$$s = \frac{n(a + l)}{2} = \frac{b(x + y)}{2}$$

$$2. \text{ From } l = a + (n - 1)d, \quad d = \frac{l - a}{n - 1} = \frac{6 - 1}{11 - 1} = \frac{5}{10} = \frac{1}{2}.$$

Hence, the A. P. is $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 5, 5\frac{1}{2}, 6$.

3. From $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{2-24}{12-1} = -2$.

Hence, the A. P. is 24, 22, 20, 18, ..., 2.

4. From $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{-14-10}{9-1} = -3$.

Hence, the A. P. is 10, 7, 4, 1, -2, -5, -8, -11, -14.

5. From $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{2-(-1)}{8-1} = \frac{3}{7}$.

Hence, the A. P. is -1, $-\frac{4}{7}$, $-\frac{1}{7}$, $\frac{2}{7}$, $\frac{5}{7}$, $1\frac{1}{7}$, $1\frac{4}{7}$, 2.

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6. From $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{20-15}{16-1} = \frac{1}{3}$.

Hence, the A. P. is 15, $15\frac{1}{3}$, $15\frac{2}{3}$, 16, $16\frac{1}{3}$, ..., 20.

7. From $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{(a+b)-(a-b)}{5-1} = \frac{b}{2}$.

Hence, the A. P. is $a-b$, $a-b+\frac{b}{2}$, $a-b+b$, $a-b+\frac{3b}{2}$, $a+b$;

or $a-b$, $\frac{2a-b}{2}$, a , $\frac{2a+b}{2}$, $a+b$.

8. From $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{l-a}{(m+2)-1} = \frac{l-a}{m+1}$.

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2. Let the series be $x-y$, x , $x+y$.

Then, $3x = 18$, (1)

and $x(x-y)(x+y) = 120$. (2)

From (1), $x = 6$. (3)

Substituting (3) in (2), $6(6-y)(6+y) = 120$. (4)

$$36 - y^2 = 20.$$

Solving (4), $y = \pm 4$.

Forming the series from $x = 6$ and $y = \pm 4$, the terms are

$$2, 6, 10, \text{ or } 10, 6, 2;$$

that is, the numbers are 2, 6, and 10.

3. Let the series be $x-y$, x , $x+y$.

Then, $3x = 21$, (1)

and $(x-y)^2 + x^2 + (x+y)^2 = 155$. (2)

From (1), $x = 7$. (3)

From (2), $3x^2 + 2y^2 = 155$. (4)

Substituting (3) in (4), $147 + 2y^2 = 155$. (5)

Solving (5), $y = \pm 2$.

Forming the series from $x = 7$ and $y = \pm 2$, the terms are

$$5, 7, 9, \text{ or } 9, 7, 5;$$

that is, the numbers are 5, 7, and 9.

4. Let the series be $x - y, x, x + y$.

Then, $(x - y)^2 + x^2 + (x + y)^2 = 93,$ (1)

and $x + y = 4(x - y).$ (2)

From (1), $3x^2 + 2y^2 = 93.$ (3)

From (2), $y = \frac{3}{2}x.$ (4)

Solving (3) and (4), $x = 5$ or $-5,$

and $y = 3$ or $-3.$

Forming the series from $x = 5$ and $y = 3$, and from $x = -5$ and $y = -3$, the terms are 2, 5, 8, or $-2, -5, -8$.

5. Let the series be $x - y, x, x + y$.

Then, $x^2 - y^2 = x^2 - 4,$ (1)

and $3x = 24.$ (2)

From (1), $y = \pm 2.$

From (2), $x = 8.$

Forming the series from $x = 8$ and $y = \pm 2$, the terms are

$$6, 8, 10, \text{ or } 10, 8, 6;$$

that is, the numbers are 6, 8, and 10.

6. Let the series be $x - 3y, x - y, x + y, x + 3y$.

Then, $4x = 14,$ (1)

and $x^2 - y^2 = 12.$ (2)

Solving (1) and (2), $x = \frac{7}{2},$

and $y = \pm \frac{1}{2}.$

Forming the series from $x = \frac{7}{2}$ and $y = \pm \frac{1}{2}$, the terms are

$$2, 3, 4, 5, \text{ or } 5, 4, 3, 2;$$

that is, the numbers are 2, 3, 4, and 5.

7. Let the series be $x - 3y, x - 2y, x - y, x, x + y, x + 2y, x + 3y$.

Then, $7x = 98,$ (1)

and $(x - 3y)^2 + (x - 2y)^2 + (x - y)^2 + x^2 + (x + y)^2 + (x + 2y)^2 + (x + 3y)^2 = 1484.$ (2)

From (1), $x = 14.$ (3)

From (2), $x^2 + 4y^2 = 212.$ (4)

Solving (3) and (4), $y = \pm 2.$

Forming the series from $x = 14$ and $y = \pm 2$, the terms are

$$8, 10, 12, 14, 16, 18, 20, \text{ or } 20, 18, 16, 14, 12, 10, 8;$$

that is, the numbers are 8, 10, 12, 14, 16, 18, and 20.

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8. Let the series be $x - 2y, x - y, x, x + y, x + 2y$.

Then, $5x = 15,$ (1)

and $x^2 - 4y^2 = x^2 - y^2 - 3.$ (2)

From (1), $x = 3.$

From (2), $y = \pm 1.$

Forming the series from $x = 3$ and $y = \pm 1$, the terms are

$$1, 2, 3, 4, 5 \text{ or } 5, 4, 3, 2, 1;$$

that is, the numbers are 1, 2, 3, 4, and 5.

9. Let the series be $x - y, x, x + y$.

$$\text{Then, } \frac{100(x - y) + 10x + (x + y)}{3x} = 20\frac{1}{2}, \quad (1)$$

$$\text{and } 100(x - y) + 10x + (x + y) + 594 = 100(x + y) + 10x + (x - y). \quad (2)$$

$$\text{From (1), } x = 2y.$$

$$\text{From (2), } y = 3.$$

$$\therefore x = 6.$$

Forming the series from $x = 6$ and $y = 3$, the terms are 3, 6, 9.

Hence, the number is 369.

10. In the series $1, 3, 5, \dots, 99$, $a = 1$, $d = 2$, and $l = 99$; and s is to be found.

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = 50.$$

$$s = \frac{n}{2}(a + l) = 25 \times 100 = 2500.$$

$$11. \text{ Given } al = 70, \quad (1)$$

$$\text{and in } s = \frac{n}{2}(a + l), \text{ since } n = 10, \text{ and } s = 95,$$

$$a + l = 19. \quad (2)$$

$$\text{Solving (1) and (2), } a = 14 \text{ or } 5,$$

$$\text{and } l = 5 \text{ or } 14.$$

Hence, the extremes are 5 and 14.

12. Let l = number of logs in the bottom layer.

Then, in the series, $1, 2, 3, \dots, l$, $a = 1$, $d = 1$, and $s = 55$; and l is to be found.

$$\text{From } l = a + (n - 1)d, \quad n = \frac{l - a}{d} + 1 = \frac{l - 1}{1} + 1 = l. \quad (1)$$

$$\text{From } s = \frac{n}{2}(a + l), \quad n = \frac{2s}{a + l} = \frac{110}{1 + l}. \quad (2)$$

$$\text{Eliminating } n, \quad l = \frac{110}{1 + l}. \quad (3)$$

$$\text{Solving (3) for } l, \quad l = 10 \text{ or } -11.$$

Hence, there were 10 logs in the bottom layer.

13. Let n = number of yards in depth of well.

Then, in the series $1\frac{1}{2}, 1\frac{3}{4}, 2, \dots$, $a = 1\frac{1}{2}$, $d = \frac{1}{4}$, and $s = 19$; and n is to be found.

$$\text{From } s = \frac{n}{2}(a + l), \quad l = \frac{2s}{n} - a = \frac{38}{n} - 1\frac{1}{2}. \quad (1)$$

$$\text{Also, } l = a + (n - 1)d = 1\frac{1}{2} + \frac{n - 1}{4}. \quad (2)$$

$$\text{Eliminating } l, \quad 1\frac{1}{2} + \frac{n - 1}{4} = \frac{38}{n} - 1\frac{1}{2}. \quad (3)$$

$$\text{Solving (3) for } n, \quad n = 8 \text{ or } -19.$$

Hence, the well was 8 yards deep.

14. Given, in an A. P., $al = 93$, (1)

$$(a + d) + (l - d) = 34, \quad (2)$$

$$n = 15, \quad (3)$$

and the formulae, $l = a + (n - 1)d$, (4)

and $s = \frac{n}{2}(a + l)$, (5)

to write the progression.

From (2), $a + l = 34$. (6)

Solving (1) and (6), $a = 31$ or 3 , (7)

and $l = 3$ or 31 . (8)

Substituting (7), (8), and (3) in (4), $d = -2$ or 2 .

Hence, the progression is $31, 29, 27, \dots, 3$; or $3, 5, 7, \dots, 31$.

15. Let $x =$ number of means required.

Then, $5 + dx + d = 37$, (1)

and $\frac{5 + d}{37 - d} = \frac{3}{11}$. (2)

From (2), $d = 4$.

Substituting 4 for d in (1), $4x = 28$,

$$\therefore x = 7.$$

16. Let $x =$ number of means required.

Then, from $s = \frac{n}{2}(a + l)$, $116 = \frac{n}{2}(4 + 25)$,

$$\therefore n = 8.$$

Since the number of means is 2 less than the number of terms,

$$x = n - 2 = 6.$$

17. Let $a, a + d, a + 2d, \dots$ be an A. P., and let m be any common multiplier of its terms.

It is to be proved that $ma, m(a + d),$ and $m(a + 2d), \dots$ are in arithmetical progression.

$$m(a + d) - ma = m(a + d - a) = md.$$

$$m(a + 2d) - m(a + d) = m[a + 2d - (a + d)] = md.$$

Since the successive terms of the series $ma, m(a + d), m(a + 2d), \dots$ have a common difference md , the series is an A. P.

18. Let $x, x + 1, x + 2, x + 3, \dots$ be any consecutive integers.

It is to be proved that $(x + 1)^2 - x^2, (x + 2)^2 - (x + 1)^2, (x + 3)^2 - (x + 2)^2, \dots$ are in arithmetical progression, and that the common difference is 2.

$$(x + 1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1.$$

$$(x + 2)^2 - (x + 1)^2 = x^2 + 4x + 4 - (x^2 + 2x + 1) = 2x + 3.$$

$$(x + 3)^2 - (x + 2)^2 = x^2 + 6x + 9 - (x^2 + 4x + 4) = 2x + 5.$$

But $2x + 1, 2x + 3, 2x + 5,$ are three terms of an A. P. whose common difference is 2.

Since to x may be assigned any integral value or any number of successive integral values, the proof given above is general.

19. In the series $1, 3, 5, \dots$ to n terms, $a = 1, d = 2, l = a + (n - 1)d = 1 + 2(n - 1) = 2n - 1$; and s is to be found.

$$s = n \left(\frac{a + l}{2} \right) = n \left(\frac{1 + 2n - 1}{2} \right) = n \cdot \frac{2n}{2} = n^2.$$

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$$12. \quad l = ar^{n-1} = a^{10}b \times \left(\frac{b}{a}\right)^{10} = a^9b^{11}.$$

$$13. \quad l = ar^{n-1} = 2 \times \left(\frac{1}{\sqrt{2}}\right)^{n-1} = 2^{1-\frac{n-1}{2}} = 2^{\frac{3-n}{2}}.$$

$$14. \quad l = ar^{n-1} = \frac{1}{4} \times 2^9 = 128.$$

Hence, he received \$128 the tenth day.

$$15. \quad l = ar^{n-1} = 200 \times 2^6 = 12800.$$

Hence, he will have \$12800 at the end of the sixth year.

$$16. \quad l = ar^{n-1} = 76 \times 2^{5-1} = 1216.$$

Hence, the population will be 1216 millions in the year 2000.

$$17. \quad l = ar^{n-1} = 512 \times \left(\frac{5}{4}\right)^{6-1} = 1562\frac{1}{2}.$$

Hence, his salary the sixth year was \$1562.50.

$$18. \quad l = ar^{n-1} = 20736 \times \left(\frac{3}{2}\right)^{5-1} = 50625.$$

Hence, the population at the end of 40 years was 50625.

$$19. \quad l = ar^{n-1} = 2 \times 3^9 = 39366.$$

Hence, the last bushel cost him \$393.66.

$$20. \quad l = ar^{n-1} = 1 \times 150^4, \text{ number of grains, 4th year.}$$

$$\frac{150 \times 150 \times 150 \times 150}{150 \times 75} = 45000, \text{ number of bushels harvested the}$$

fourth year.

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$$2. \quad s = \frac{ar^n - a}{r - 1} = \frac{2^8 - 1}{1} = 255.$$

$$3. \quad s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r} = \frac{1 - \left(\frac{1}{2}\right)^8}{1 - \frac{1}{2}} = \frac{255}{128}.$$

$$4. \quad s = \frac{ar^n - a}{r - 1} = \frac{\left(\frac{3}{2}\right)^{10} - 1}{\frac{3}{2} - 1} = \frac{58025}{512}.$$

$$5. \quad s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r} = \frac{2 - 2\left(-\frac{1}{3}\right)^7}{1 + \frac{1}{3}} = \frac{2 + \frac{2}{3^{1+7}}}{\frac{4}{3}} = \frac{1094}{729}.$$

$$6. \quad s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r} = \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{12}}{1 + \frac{1}{2}} = -\frac{1365}{4096}.$$

$$7. \quad s = \frac{ar^n - a}{r - 1} = \frac{(2x)^7 - 1}{2x - 1} = \frac{128x^7 - 1}{2x - 1}.$$

$$8. \quad s = \frac{ar^n - a}{r - 1} = \frac{(-2x)^7 - 1}{-2x - 1} = \frac{128x^7 + 1}{2x + 1}.$$

$$9. \quad s = \frac{ar^n - a}{r - 1} = \frac{(x^2)^n - 1}{x^2 - 1} = \frac{x^{2n} - 1}{x^2 - 1}.$$

$$10. \quad s = \frac{ar^n - a}{r - 1} = \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

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$$11. \quad s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r} = \frac{1 - (\frac{1}{3})^n}{1 - \frac{1}{3}} = \frac{3^n - 1}{3^n \times \frac{2}{3}} = \frac{3^n - 1}{3^{n-1} \times 2} = \frac{1}{2} \left(\frac{3^n - 1}{3^{n-1}} \right).$$

$$12. \quad s = \frac{rl - a}{r - 1} = \frac{3 \times 729 - 1}{2} = 1093.$$

$$13. \quad s = \frac{rl - a}{r - 1} = \frac{2 \times 192 - 3}{1} = 381.$$

$$14. \quad s = \frac{a - rl}{1 - r} = \frac{7 - (-2)(-224)}{1 - (-2)} = \frac{7 - 448}{3} = -147.$$

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$$3. \quad s = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

$$4. \quad s = \frac{a}{1 - r} = \frac{3}{1 - \frac{1}{4}} = \frac{3}{\frac{3}{4}} = 4.$$

$$5. \quad s = \frac{a}{1 - r} = \frac{1}{1 - (-\frac{1}{3})} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$$

$$6. \quad .407407 \dots = .407 + .000407 + .000000407 + \dots$$

$$s = \frac{a}{1 - r} = \frac{.407}{1 - \frac{1}{1000}} = \frac{407}{1000} \div \frac{999}{1000} = \frac{407}{999} = \frac{11}{27}.$$

$$7. \quad .363636 \dots = .36 + .0036 + .000036 + \dots$$

$$s = \frac{a}{1 - r} = \frac{.36}{1 - \frac{1}{100}} = \frac{36}{100} \div \frac{99}{100} = \frac{36}{99} = \frac{4}{11}.$$

$$8. \quad 1.94444 = (1 + \frac{9}{10}) + (\frac{4}{100} + \frac{4}{1000} + \frac{4}{10000} + \dots).$$

$$s = \frac{19}{10} + \frac{a}{1 - r} = \frac{19}{10} + \frac{.04}{1 - \frac{1}{10}}$$

$$= \frac{19}{10} + \frac{4}{90} = \frac{175}{90} = 1\frac{7}{18}.$$

$$2. \quad \text{Substituting } 625 \text{ for } l, 1 \text{ for } a, \text{ and } 5 \text{ for } n \text{ in } l = ar^{n-1},$$

$$625 = r^4,$$

$$\therefore r = \pm 5.$$

Hence, the series is either 1, 5, 25, 125, 625, or 1, -5, 25, -125, 625.

$$3. \quad \text{Substituting } \frac{2048}{81} \text{ for } l, \frac{9}{2} \text{ for } a, \text{ and } 7 \text{ for } n \text{ in } l = ar^{n-1},$$

$$\frac{2048}{81} = \frac{9}{2} r^6,$$

$$r^6 = \frac{4096}{9^6} = \frac{4^6}{3^6}.$$

$$\therefore r = \pm \frac{4}{3}.$$

Hence, the series is either $4\frac{1}{2}, 6, 8, \frac{32}{3}, \frac{128}{9}, \frac{512}{27}, \frac{2048}{81};$
 or $4\frac{1}{2}, -6, 8, -\frac{32}{3}, \frac{128}{9}, -\frac{512}{27}, \frac{2048}{81}.$

4. Substituting $\frac{6}{4}\frac{4}{3}$ for l , $\frac{3}{1}\frac{4}{6}$ for a , and 6 for n in $l = ar^{n-1}$,

$$\frac{6}{4}\frac{4}{3} = \frac{3}{1}\frac{4}{6} r^5.$$

$$r = \sqrt[5]{\frac{64 \cdot 16}{49 \cdot 343}} = \sqrt[5]{\frac{4^3 \cdot 4^2}{7^2 \cdot 7^3}} = \sqrt[5]{\frac{4^5}{7^5}} = \frac{4}{7}.$$

Hence, the series is $\frac{3}{1}\frac{4}{6}$, $\frac{4}{4}$, 7, 4, $\frac{1}{7}$, $\frac{6}{4}\frac{4}{3}$.

5. Substituting 5120 for a , 5 for l , and 6 for n in $l = ar^{n-1}$,

$$5120 r^5 = 5.$$

$$\therefore r = \frac{1}{10}.$$

Hence, the series is 5120, 1280, 320, 80, 20, 5.

6. Substituting 1 for l , $4\sqrt{2}$ for a , and 6 for n in $l = ar^{n-1}$,

$$1 = 4\sqrt{2} r^5 = (2)^{\frac{1}{2}} r^5.$$

$$\therefore r = \frac{1}{(2)^{\frac{1}{2}}} = \frac{1}{2} \sqrt{2}.$$

Hence, the series is $4\sqrt{2}$, 4, $2\sqrt{2}$, 2, $\sqrt{2}$, 1.

7. Substituting b^6 for l , a^6 for a , and 7 for n in $l = ar^{n-1}$,

$$b^6 = a^6 r^6.$$

$$\therefore r = \pm \frac{b}{a}.$$

Hence, the series is either a^6 , a^5b , a^4b^2 , a^3b^3 , a^2b^4 , ab^5 , b^6 ;

or

$$a^6, -a^5b, a^4b^2, -a^3b^3, a^2b^4, -ab^5, b^6.$$

8. Substituting $\frac{1}{8}\sqrt{2}$ for l , -2 for a , and 8 for n in $l = ar^{n-1}$,

$$\frac{1}{8}\sqrt{2} = -2 r^7.$$

$$r^7 = -\frac{1}{16}\sqrt{2} = -(2)^{-4} (2)^{\frac{1}{2}} = -(2)^{-\frac{7}{2}}$$

$$\therefore r = -(2)^{-\frac{1}{2}} = -\frac{1}{2}\sqrt{2}.$$

Hence, the series is -2 , $\sqrt{2}$, -1 , $\frac{1}{2}\sqrt{2}$, $-\frac{1}{2}$, $\frac{1}{4}\sqrt{2}$, $-\frac{1}{4}$, $\frac{1}{8}\sqrt{2}$.

9. Substituting $-y$ for l , x for a , and 6 for n in $l = ar^{n-1}$,

$$-y = x r^5.$$

$$\therefore r = \sqrt[5]{-\frac{y}{x}} = -\frac{y^{\frac{1}{5}}}{x^{\frac{1}{5}}}.$$

Hence, the series is x , $-x^{\frac{4}{5}}y^{\frac{1}{5}}$, $x^{\frac{3}{5}}y^{\frac{2}{5}}$, $-x^{\frac{2}{5}}y^{\frac{3}{5}}$, $x^{\frac{1}{5}}y^{\frac{4}{5}}$, $-y$.

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1. Given r , l , and s , to find a .

Formula III,

$$s = \frac{rl - a}{r - 1}. \quad (1)$$

Solving for a ,

$$a = rl - s(r - 1). \quad (2)$$

2. By (2), Ex. 1, $a = rl - s(r - 1) = 5 \times 625 - 775 \times 4 = 25$.

3. Since the last term of an infinite decreasing geometrical series may be neglected, or counted as 0, by (2), Ex. 1,

$$a = rl - s(r - 1) = \frac{1}{10} \times 0 - \frac{1}{3} \left(\frac{1}{10} - 1 \right) = \frac{2}{15}.$$

4. Find l in terms of a , r , and s .

Formula III,

$$s = \frac{rl - a}{r - 1}. \quad (1)$$

Solving for l ,

$$l = \frac{a + s(r - 1)}{r}. \quad (2)$$

5. By (2), Ex. 4, $l = \frac{5 + 155 \times 1}{2} = 80$.

6. By (2), Ex. 4, $l = \frac{\frac{1}{8} + \frac{155}{8}(1 + \sqrt{2})(\sqrt{2} - 1)}{\sqrt{2}} = \frac{\frac{1}{8} + \frac{155}{8}}{\sqrt{2}} = \sqrt{2}$. (1)

Substituting $\sqrt{2}$ for l , $\frac{1}{8}$ for a , and $\sqrt{2}$ for r in $l = ar^{n-1}$,

$$\sqrt{2} = \frac{1}{8}(\sqrt{2})^{n-1}, \text{ or } 2^{\frac{1}{2}} = 2^{-3} \cdot 2^{\frac{n-1}{2}} = 2^{\frac{n-7}{2}}.$$

$$\therefore \frac{1}{2} = \frac{1}{2}(n - 7), \text{ whence, } n = 8.$$

7. Deduce the formula for r in terms of a , l , and s .

Formula III,

$$s = \frac{rl - a}{r - 1}. \quad (1)$$

Solving for r ,

$$r = \frac{s - a}{s - l}. \quad (2)$$

8. By (2), Ex. 7, $r = \frac{s - a}{s - l} = \frac{665 - 32}{665 - 243} = \frac{3}{2}$.

Hence, the series is 32, 48, 72, 108, 162, 243.

9. By (2), Ex. 7, $r = \frac{s - a}{s - l} = \frac{700}{525} = \frac{4}{3}$.

Since the sum is 525 greater than the last term and 700 greater than the first term, the last term is 175 greater than the first term.

$$\therefore l = 81 + 175 = 256.$$

Hence, the progression is 81, 108, 144, 192, 256.

10. Deduce the formula for r in terms of a , n , and l .

Formula I,

$$l = ar^{n-1}. \quad (1)$$

Solving for r ,

$$r = \sqrt[n-1]{\frac{l}{a}}. \quad (2)$$

11. By (2), Ex. 10, $r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[5]{\frac{729}{3}} = \sqrt[5]{243} = 3$.

Hence, the series is 3, 9, 27, 81, 243, 729.

12. Find l in terms of r , n , and s .

Formula I,

$$l = ar^{n-1}. \quad (1)$$

Formula II,

$$s = \frac{a(r^n - 1)}{r - 1}. \quad (2)$$

From (1),

$$\frac{l}{a} = r^{n-1}. \quad (3)$$

From (2),

$$a = \frac{s(r - 1)}{r^n - 1}. \quad (4)$$

Multiplying (3) by (4),

$$l = \frac{r^{n-1}s(r - 1)}{r^n - 1}. \quad (5)$$

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13. By (5), Ex. 12, $l = \frac{r^{n-1}s(r-1)}{r^n - 1} = \frac{2^{11} \times 4095 \times 1}{2^{12} - 1} = 2^{11} = 2048$.

14. $s = 100 [1 + \frac{4}{5} + (\frac{4}{5})^2 + \dots]$.

The series in brackets is an infinite decreasing geometrical progression whose ratio is $\frac{4}{5}$ and first term 1.

$$\therefore s = 100 \times \frac{1}{1 - \frac{4}{5}} = 100 \times 5 = 500.$$

Hence, the sled will go 500 feet.

15. At first the contents of the cask is all vinegar; then $\frac{2}{3}$ vinegar; then $\frac{2}{3}$ of $\frac{2}{3}$, or $\frac{4}{9}$ vinegar; etc. Hence, the part of the contents of the cask which will be vinegar after the sixth time is equal to the seventh term of the geometrical progression $1, \frac{2}{3}, \frac{4}{9}, \dots$.

$$7\text{th term} = l = ar^{n-1} = 1(\frac{2}{3})^6 = \frac{64}{729},$$

which is less than $\frac{1}{10}$.

Hence, the contents of the cask will be more than $\frac{9}{10}$ water.

16. Whole distance passed through $= 200 + 80 + 32 + \dots$.

In the series $200, 80, 32, \dots$, $a = 200$, $r = \frac{2}{5}$, and the number of terms is infinite. Let s represent the whole distance passed through, in feet.

Then,
$$s = \frac{a}{1 - r} = \frac{200}{1 - \frac{2}{5}} = \frac{200}{\frac{3}{5}} = 333\frac{1}{3}.$$

17. The amount is $1(1+r)$, or $1+r$, at the end of 1 year; $(1+r)(1+r)$, or $(1+r)^2$, at the end of 2 years; $(1+r)^2(1+r)$, or $(1+r)^3$, at the end of 3 years; etc.

Hence, the amounts $1+r, (1+r)^2, (1+r)^3, \dots$ for 1, 2, 3, \dots years, respectively, form a geometrical progression whose ratio is $1+r$.

18. Let a, ar, ar^2, \dots

be a geometrical progression, and m any common multiplier of its terms.

Then, ma, mar, mar^2, \dots

also is a geometrical progression having the same ratio as the given geometrical progression, since $mr \div m = r$, $mr^2 \div mr = r$, etc.

19. Let the series be x^2, xy, y^2 .

Then, $x^2 + xy + y^2 = 19, \quad (1)$

and $x^4 + x^2y^2 + y^4 = 133. \quad (2)$

Dividing (2) by (1), $x^2 - xy + y^2 = 7. \quad (3)$

Subtracting (3) from (1), $2xy = 12. \quad (4)$

Adding (4) $\div 2$ to (1), $x^2 + 2xy + y^2 = 25, \quad (5)$

whence, $x + y = \pm 5. \quad (5)$

Subtracting (4) $\div 2$ from (3), $x^2 - 2xy + y^2 = 1, \quad (6)$

whence, $x - y = \pm 1. \quad (6)$

From (5) and (6), $x = 3 \text{ or } 2 \text{ or } -2 \text{ or } -3, \quad (7)$

and $y = 2 \text{ or } 3 \text{ or } -3 \text{ or } -2. \quad (8)$

From (7) and (8), $x^2 = 9 \text{ or } 4$, $xy = 6$, and $y^2 = 4 \text{ or } 9$.

Hence, the numbers are 4, 6, and 9.

20. Let the series be x^2 , xy , y^2 .

Then,

$$x^3y^3 = 8, \quad (1)$$

and

$$x^4 + x^2y^2 + y^4 = 21. \quad (2)$$

From (1),

$$xy = 2, \text{ or } y = \frac{2}{x}. \quad (3)$$

Substituting (3) in (2),

$$x^4 + 4 + \frac{16}{x^4} = 21.$$

Clearing of fractions, etc., $x^8 - 17x^4 + 16 = 0$.

Factoring, $(x-1)(x+1)(x-2)(x+2)(x^2+1)(x^2+4) = 0$. (4)

From (4), $x = 1, x = -1, x = 2, x = -2, x^2 = -1, x^2 = -4$.

$$\therefore x^2 = 1 \text{ or } 4 \text{ or } -1 \text{ or } -4. \quad (5)$$

Substituting (5) in the square of (3),

$$y^2 = 4 \text{ or } 1 \text{ or } -4 \text{ or } -1. \quad (6)$$

From (5), (3), and (6) the numbers are found to be

$$1, 2, \text{ and } 4, \text{ or } -1, 2, \text{ and } -4.$$

21. See next page.

22. Let the series be x^2 , xy , y^2 .

Then,

$$x^3y^3 = 64, \quad (1)$$

and

$$x^6 + x^3y^3 + y^6 = 584. \quad (2)$$

Adding (1) to (2),

$$x^6 + 2x^3y^3 + y^6 = 648. \quad (3)$$

Extracting the square root,

$$x^3 + y^3 = \pm 18\sqrt{2}. \quad (4)$$

Subtracting (1) $\times 3$ from (2), $x^6 - 2x^3y^3 + y^6 = 392$.

(5)

Extracting the square root,

$$x^3 - y^3 = \pm 14\sqrt{2}. \quad (6)$$

From (4) and (6),

$$x^3 = 16\sqrt{2}, 2\sqrt{2}, -2\sqrt{2}, -16\sqrt{2}, \quad (7)$$

and

$$y^3 = 2\sqrt{2}, 16\sqrt{2}, -16\sqrt{2}, -2\sqrt{2}. \quad (8)$$

Since $x^3 = \pm 16\sqrt{2} = (\pm 2)^{\frac{3}{2}}$, $x = (\pm 2)^{\frac{1}{2}}$ and $x^2 = (\pm 2)^{\frac{1}{2}} = \pm 8$.

Since $x^3 = \pm 2\sqrt{2} = (\pm 2)^{\frac{3}{2}}$, $x = (\pm 2)^{\frac{1}{2}}$ and $x^2 = (\pm 2)^{\frac{1}{2}} = \pm 2$.

Similarly, the values of y^2 are found to be 2 or -2 or 8 or -8 .

From (1), the value of xy is found to be 4.

Hence, the numbers are 2, 4, and 8, or $-2, 4$, and -8 .

23. Let the series be $\frac{x^2}{y}$, x , y , $\frac{y^2}{x}$.

Then,

$$\frac{x^2}{y} + x = 15, \quad (1)$$

and

$$y + \frac{y^2}{x} = 60. \quad (2)$$

Dividing (2) by (1),

$$\frac{y^2}{x^2} = 4.$$

$$\therefore y = 2x \text{ or } -2x. \quad (3)$$

Substituting $2x$ for y in (1),

$$x = 10. \quad (4)$$

Substituting $-2x$ for y in (1),

$$x = 30. \quad (5)$$

Substituting (4) and (5) in (3),

$$y = 20 \text{ or } -60.$$

Forming the series from $x = 10$ and $y = 20$, and from $x = 30$ and $y = -60$, the numbers are 5, 10, 20, and 40, or $-15, 30, -60$, and 120.

21. Let x and y represent the numbers.

Then,

$$\sqrt{xy} = 4, \quad (1)$$

and

$$x + y = 10. \quad (2)$$

Adding (1) $\times 2$ to (2),

$$x + 2\sqrt{xy} + y = 18,$$

whence,

$$\sqrt{x} + \sqrt{y} = \pm 3\sqrt{2}. \quad (3)$$

Subtracting (1) $\times 2$ from (2),

$$x - 2\sqrt{xy} + y = 2,$$

whence,

$$\sqrt{x} - \sqrt{y} = \pm \sqrt{2}. \quad (4)$$

Multiplying (3) by (4),

$$x - y = 6 \text{ or } -6. \quad (5)$$

From (2) and (5),

$$x = 8 \text{ or } 2,$$

and

$$y = 2 \text{ or } 8.$$

Hence, the numbers are 2 and 8.

24. Let the series be x^2, xy, y^2 .

Then,

$$x^2 + y^2 = 130, \quad (1)$$

and

$$x^2 y^2 = 625. \quad (2)$$

From (1),

$$y^2 = 130 - x^2. \quad (3)$$

Substituting (3) in (2),

$$x^2 (130 - x^2) = 625.$$

$$x^4 - 130x^2 + 625 = 0.$$

$$(x^2 - 5)(x^2 - 125) = 0;$$

$$\therefore x^2 = 5 \text{ or } 125. \quad (4)$$

Substituting (4) in (3),

$$y^2 = 125 \text{ or } 5. \quad (5)$$

Extracting the square root in (2),

$$xy = \pm 25.$$

Hence, the numbers are 5, 25, and 125, or 5, -25, and 125.

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25. See next page.

26. Given a, b , and c in geometrical progression.

To prove that $\frac{1}{a}, \frac{1}{b}$, and $\frac{1}{c}$ are in geometrical progression.

Since a, b , and c are in geometrical progression,

$$\frac{c}{b} = \frac{b}{a}. \quad (1)$$

Squaring (1),

$$\frac{c^2}{b^2} = \frac{b^2}{a^2} \quad (2)$$

Dividing (1) by (2),

$$\frac{b}{c} = \frac{a}{b}. \quad (3)$$

But in the series $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, the fraction $\frac{b}{c}$ is the ratio of the third term

to the second term, and the fraction $\frac{a}{b}$ is the ratio of the second term to the first term. Since, by (3), these ratios are equal, the series is a geometrical progression.

27. Let

and

x = larger number,

$x - 24$ = smaller number.

Then,

$$\frac{x + x - 24}{2} = \sqrt{x(x - 24)} + 6.$$

Solving,

$x = 27$, larger number,

whence,

$x - 24 = 3$, smaller number.

25. Let x = number of dollars first receives,
and y = number of dollars third receives.
Then, \sqrt{xy} = number of dollars second receives;

$$\therefore x + \sqrt{xy} + y = 700, \quad (1)$$

$$\text{and} \quad x - y = 300. \quad (2)$$

Put $u + v$ for x and $u - v$ for y .

$$(1) \text{ becomes } 2u + \sqrt{u^2 - v^2} = 700. \quad (3)$$

$$(2) \text{ becomes } 2v = 300. \quad (4)$$

$$\therefore v = 150. \quad (5)$$

$$\text{Substituting (5) in (3),} \quad u = 683\frac{1}{3} \text{ or } 250. \quad (6)$$

The first value of u must be rejected since $683\frac{1}{3} + 150 > 700$.

$x = u + v = 400$, $y = u - v = 100$, and $\sqrt{xy} = \sqrt{400 \times 100} = 200$.

Hence, the first receives \$400, the second \$200, and the third \$100.

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2. The reciprocals of the terms form the A. P. 2, 4, 6, ...

10th term of A. P. = $l = a + (n - 1)d = 2 + 9 \times 2 = 20$.

\therefore 10th term of H. P. = $\frac{1}{20}$.

$$3. \text{ From } l = a + (n - 1)d, d = \frac{l - a}{n - 1} = \frac{\frac{1}{2} - \frac{2}{3}}{8 - 1} = -\frac{1}{12}.$$

The A. P. is $\frac{2}{3}, \frac{7}{12}, \frac{1}{2}, \frac{5}{12}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}$.

Hence, the H. P. is $1\frac{1}{2}, 1\frac{2}{3}, 2, 2\frac{2}{3}, 3, 4, 6, 12$.

$$4. \text{ From } l = a + (n - 1)d, d = \frac{l - a}{n - 1} = \frac{\frac{1}{5} - \frac{1}{2}}{4 - 1} = -\frac{1}{10}.$$

The A. P. is $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{1}{5}$.

Hence, the H. P. is $2, 2\frac{1}{2}, 3\frac{1}{3}, 5$.

$$5. \text{ From } l = a + (n - 1)d, d = \frac{l - a}{n - 1} = \frac{\frac{2}{5} - \frac{2}{5}}{9 - 1} = \frac{1}{25}.$$

The A. P. is $\frac{2}{25}, \frac{3}{25}, \frac{4}{25}, \frac{1}{5}, \frac{6}{25}, \frac{7}{25}, \frac{8}{25}, \frac{9}{25}, \frac{2}{5}$.

Hence, the H. P. is $12\frac{1}{2}, 8\frac{1}{3}, 6\frac{1}{4}, 5, 4\frac{1}{5}, 3\frac{1}{6}, 2\frac{1}{7}, 2\frac{1}{2}$.

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$$6. \text{ From } l = a + (n - 1)d, d = \frac{l - a}{n - 1} = \frac{\frac{1}{5} - \frac{1}{b}}{5 - 1} = \frac{b - a}{4ab}.$$

The A. P. is $\frac{1}{b}, \frac{1}{b} + \frac{b - a}{4ab}, \frac{1}{b} + \frac{b - a}{2ab}, \frac{1}{b} + \frac{3(b - a)}{4ab}, \frac{1}{b} + \frac{b - a}{ab};$

or $\frac{1}{b}, \frac{3a + b}{4ab}, \frac{a + b}{2ab}, \frac{a + 3b}{4ab}, \frac{1}{a}.$

Hence, the H. P. is $b, \frac{4ab}{3a + b}, \frac{2ab}{a + b}, \frac{4ab}{a + 3b}, a.$

7. The n th term of A. P. = $a + (n - 1)d = 5 + (n - 1)3 = 3n + 2$.

Hence, the n th term of the H. P. is $\frac{1}{3n + 2}.$

8. The 3d and 4th terms of the corresponding A. P. are $\frac{2}{3}$ and $\frac{3}{15}$.

Since $d = \frac{2}{15}$, the 2d term is $\frac{2}{3} - \frac{2}{15} = \frac{4}{15}$, the 1st term is $\frac{4}{15} - \frac{2}{15} = \frac{2}{15}$, and the terms following the 4th are $\frac{2}{15} + \frac{2}{15} = \frac{4}{15}$, $\frac{4}{15} + \frac{2}{15} = \frac{6}{15} = \frac{2}{5}$, etc.

\therefore the A. P. is $\frac{2}{15}, \frac{4}{15}, \frac{2}{3}, \frac{8}{15}, \frac{2}{3}, \frac{4}{5}, \dots$

Hence, the H. P. is $7\frac{1}{2}, 3\frac{3}{4}, 2\frac{1}{2}, 1\frac{7}{8}, 1\frac{1}{2}, 1\frac{1}{4}, \dots$

17. See next page.

18. Let a and b represent the numbers.

$$\text{Then, } A = \frac{a+b}{2} = 5, \quad (1)$$

$$\text{and } H = \frac{2ab}{a+b} = \frac{16}{5}. \quad (2)$$

$$\text{From (1), } a+b = 10. \quad (3)$$

$$\text{Multiplying (2) by (3), } 2ab = 32. \quad (4)$$

$$\text{Solving (3) and (4) for } a \text{ and } b, \quad a = 8 \text{ or } 2,$$

$$\text{and } b = 2 \text{ or } 8.$$

Hence, the numbers are 2 and 8.

19. Let a and b represent the numbers, $a > b$.

$$\text{Then, } a-b = 2, \quad (1)$$

$$\text{and } A-H = \frac{a+b}{2} - \frac{2ab}{a+b} = \frac{1}{3}. \quad (2)$$

$$\text{Clearing (2) of fractions, } 3(a^2 - 2ab + b^2) = 2(a+b).$$

$$\therefore a+b = \frac{3}{2}(a-b)^2.$$

$$\text{Substituting 2 for } a-b, \quad a+b = 6. \quad (3)$$

$$\text{From (3) and (1), } a = 4, \text{ larger number,}$$

$$\text{and } b = 2, \text{ smaller number.}$$

20. Given a, b , and c in H. P.

It is to be proved that $a-b : b-c = a : c$.

By the definition of harmonical progression,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

Uniting terms,

$$\frac{b-c}{bc} = \frac{a-b}{ab}.$$

Multiplying by b ,

$$\frac{b-c}{c} = \frac{a-b}{a}.$$

Expressing as a proportion,

$$b-c : c = a-b : a.$$

By alternation and inversion, $a-b : b-c = a : c$.

21. Given a, b, c , and d in H. P.

It is to be proved that

$$ab : cd = b-a : d-c.$$

By the definition of a H. P.,

$$\frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

Uniting terms,

$$\frac{c-d}{cd} = \frac{b-c}{bc} = \frac{a-b}{ab}.$$

Equating the 1st and 3d fractions and changing signs,

$$\frac{d-c}{cd} = \frac{b-a}{ab}.$$

Expressing as a proportion,

$$d-c : cd = b-a : ab.$$

Alternating the extremes,

$$ab : cd = b-a : d-c.$$

17. The 5th and 11th terms of the corresponding A. P. are 12 and 24.

Since 11th term = 5th term + 6 times the common difference, the common difference is $(24 - 12) \div 6$, or 2. Hence, the first term of the A. P. is equal $12 - 4$ times 2, or $12 - 8$, or 4.

Hence, the first term of the H. P. is $\frac{1}{4}$.

22. Given a , b , and c in H. P.

It is to be proved that
$$\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}.$$

Since b is the harmonical mean between a and c , $b = \frac{2ac}{a+c}$.

To find the value of $\frac{1}{b-a} + \frac{1}{b-c}$ in terms of a and c , substitute this value of b for b in the expression $\frac{1}{b-a} + \frac{1}{b-c}$.

$$\begin{aligned} \text{Then, } \frac{1}{b-a} + \frac{1}{b-c} &= \frac{1}{\frac{2ac}{a+c} - a} + \frac{1}{\frac{2ac}{a+c} - c} \\ &= \frac{c+a}{ac-a^2} + \frac{c+a}{ac-c^2} = \frac{c+a}{a(c-a)} - \frac{c+a}{c(c-a)} \\ &= \frac{c+a}{c-a} \left(\frac{1}{a} - \frac{1}{c} \right) = \frac{c+a}{c-a} \times \frac{c-a}{ac} \\ &= \frac{c+a}{ac} = \frac{1}{a} + \frac{1}{c}. \end{aligned}$$

23. Given

$$b-a : c-b = a : x.$$

(1)

It is to be proved that
that
and that

$x = a$, if a , b , and c are in A. P. ;
 $x = b$, if a , b , and c are in G. P. ;
 $x = c$, if a , b , and c are in H. P.

Solving (1) for x ,

$$x = \frac{a(c-b)}{b-a}. \quad (2)$$

Substituting $\frac{a+c}{2}$ for b in (2), when a , b , and c are in A. P.,

$$x = \frac{a \left(c - \frac{a+c}{2} \right)}{\frac{a+c}{2} - a} = \frac{2ac - a^2 - ac}{a+c-2a} = \frac{ac-a^2}{c-a} = a.$$

Substituting \sqrt{ac} for b in (2), when a , b , and c are in G. P.,

$$x = \frac{a(c-\sqrt{ac})}{\sqrt{ac}-a} = \frac{ac-a\sqrt{ac}}{\sqrt{ac}-a} = \sqrt{ac} = b.$$

Substituting $\frac{2ac}{a+c}$ for b in (2), when a , b , and c are in H. P.,

$$x = \frac{a \left(c - \frac{2ac}{a+c} \right)}{\frac{2ac}{a+c} - a} = \frac{a^2c + ac^2 - 2a^2c}{2ac - a^2 - ac} = \frac{ac^2 - a^2c}{ac - a^2} = c.$$

24. § 380, $G = \sqrt{AH} = \sqrt{\frac{1}{2} \times \frac{7}{3}} = \sqrt{\frac{7}{6}} = \pm \frac{\sqrt{42}}{6}$.

Hence, the geometrical mean is either $\frac{\sqrt{42}}{6}$ or $-\frac{\sqrt{42}}{6}$.

$$\begin{aligned} 25. \quad \frac{2(x+xy)(xy+xy^2)}{(x+xy)+(xy+xy^2)} &= \frac{2x(1+y)xy(1+y)}{(x+xy)(1+y)} = \frac{2x^2y(1+y)}{x+xy} \\ &= \frac{2xy(x+xy)}{x+xy} = 2xy. \end{aligned}$$

Since $2xy$, the middle term, is equal to twice the product of the first and third terms divided by their sum, by § 379 the numbers are in harmonical progression.

26. If $b+c$, $c+a$, and $a+b$ are in H. P.,

$$\frac{1}{a+b} - \frac{1}{c+a} = \frac{1}{c+a} - \frac{1}{b+c}.$$

Uniting terms,
$$\frac{c-b}{(a+b)(c+a)} = \frac{b-a}{(b+c)(c+a)}.$$

Multiplying by $(c+a)$,
$$\frac{c-b}{b+a} = \frac{b-a}{c+b}.$$

Clearing of fractions, $c^2 - b^2 = b^2 - a^2$;
that is, a^2 , b^2 , and c^2 are in A. P.

IMAGINARY AND COMPLEX NUMBERS

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10. $\sqrt{-4} + \sqrt{-49} = 2\sqrt{-1} + 7\sqrt{-1} = 9\sqrt{-1}.$

11. $\sqrt{-9} + \sqrt{-64} = 3\sqrt{-1} + 8\sqrt{-1} = 11\sqrt{-1}.$

12. $2\sqrt{-4} + 3\sqrt{-1} = 2 \cdot 2\sqrt{-1} + 3\sqrt{-1} = 7\sqrt{-1}.$

13. $\sqrt{-12} + 4\sqrt{-3} = 2\sqrt{-3} + 4\sqrt{-3} = 6\sqrt{-3}.$

14. $5\sqrt{-18} - \sqrt{-72} = 5 \cdot 3\sqrt{-2} - 6\sqrt{-2} = 9\sqrt{-2}.$

15. $3\sqrt{-20} - \sqrt{-80} = 3 \cdot 2\sqrt{-5} - 4\sqrt{-5} = 2\sqrt{-5}.$

16. $(\sqrt{-a} + 3\sqrt{-b}) + (\sqrt{-a} - 3\sqrt{-b}) = 2\sqrt{-a}.$

17. $(\sqrt{-9xy} - \sqrt{-xy}) - (\sqrt{-4xy} + \sqrt{-xy})$
 $= 3\sqrt{-xy} - \sqrt{-xy} - 2\sqrt{-xy} - \sqrt{-xy} = -\sqrt{-xy}.$

18. $\sqrt{-x^2} + \sqrt{-4x^2} - \sqrt{-x^3} + 3x\sqrt{-x}$
 $= x\sqrt{-1} + 2x\sqrt{-1} - x\sqrt{-x} + 3x\sqrt{-x} = 3x\sqrt{-1} + 2x\sqrt{-x}$
 $= x(3\sqrt{-1} + 2\sqrt{-x}).$

19. $\sqrt{-16} - 3\sqrt{-4} + \sqrt{-18} + \sqrt{-50} + \sqrt{-25}$
 $= 4\sqrt{-1} - 3 \cdot 2\sqrt{-1} + 3\sqrt{-2} + 5\sqrt{-2} + 5\sqrt{-1}$
 $= 3\sqrt{-1} + 8\sqrt{-2}.$

$$\begin{aligned}
 20. \quad & \sqrt{-8} + a\sqrt{-2} - \sqrt{-98} - 5\sqrt{-2a^2} \\
 &= 2\sqrt{-2} + a\sqrt{-2} - 7\sqrt{-2} - 5a\sqrt{-2} \\
 &= (-5 - 4a)\sqrt{-2} = -(4a + 5)\sqrt{-2}.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & \sqrt{-16a^2x^2} + \sqrt{-a^2x^2} - \sqrt{-9a^2x^2} \\
 &= 4ax\sqrt{-1} + ax\sqrt{-1} - 3ax\sqrt{-1} = 2ax\sqrt{-1}.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & \sqrt{1-5} - 3\sqrt{1-10} + 2\sqrt{5-30} \\
 &= 2\sqrt{-1} - 3 \cdot 3\sqrt{-1} + 2 \cdot 5\sqrt{-1} = 3\sqrt{-1}.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & 3\sqrt{-5} \times 2\sqrt{-15} = 3\sqrt{5}\sqrt{-1} \times 2\sqrt{15}\sqrt{-1} \\
 &= 6\sqrt{5}\sqrt{5}\sqrt{3}(-1) = -6 \times 5\sqrt{3} = -30\sqrt{3}.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & 4\sqrt{-27} \times \sqrt{-12} = 4\sqrt{27}\sqrt{-1} \times \sqrt{12}\sqrt{-1} \\
 &= 4 \times 3\sqrt{3} \times 2\sqrt{3}(-1) = -24 \times 3 = -72.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & 2\sqrt{-8} \times 5\sqrt{-3} = 2 \times 2\sqrt{2}\sqrt{-1} \times 5\sqrt{3}\sqrt{-1} \\
 &= 20\sqrt{2}\sqrt{3}(-1) = -20\sqrt{6}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & 8\sqrt{-1} \times \sqrt{-b^3} = 8\sqrt{-1} \times b\sqrt{b}\sqrt{-1} = 8b\sqrt{b}(-1) \\
 &= -8b\sqrt{b}.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \sqrt{-125} \times \sqrt{-108} = 5\sqrt{5}\sqrt{-1} \times 6\sqrt{3}\sqrt{-1} = 30\sqrt{15}(-1) \\
 &= -30\sqrt{15}.
 \end{aligned}$$

$$30. \quad \sqrt{-100} \times \sqrt{-30} = 10\sqrt{-1}\sqrt{30}\sqrt{-1} = -10\sqrt{30}.$$

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$$\begin{aligned}
 31. \quad & (\sqrt{-6} + \sqrt{-3})(\sqrt{-6} - \sqrt{-3}) \\
 &= (\sqrt{-6})^2 - (\sqrt{-3})^2 \\
 &= -6 - (-3) = -6 + 3 = -3.
 \end{aligned}$$

$$\begin{aligned}
 32. \quad & (\sqrt{-ab} + \sqrt{-a})(\sqrt{-ab} - \sqrt{-a}) = (\sqrt{-ab})^2 - (\sqrt{-a})^2 \\
 &= -ab - (-a) = a - ab.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad & (\sqrt{-xy} + \sqrt{-x})(\sqrt{-xy} + \sqrt{-x}) \\
 &= (\sqrt{-xy})^2 + 2\sqrt{-xy}\sqrt{-x} + (\sqrt{-x})^2 \\
 &= -xy + 2\sqrt{x^2y}(\sqrt{-1})^2 + (-x) \\
 &= -xy - 2x\sqrt{y} - x = -x(y + 2\sqrt{y} + 1).
 \end{aligned}$$

$$\begin{aligned}
 34. \quad & \sqrt{-50} - \sqrt{-12} = 5\sqrt{-2} - 2\sqrt{-3} \\
 & \sqrt{-8} - \sqrt{-75} = 2\sqrt{-2} - 5\sqrt{-3} \\
 & \quad \quad \quad -20 \quad + \quad 4\sqrt{6} \\
 & \quad \quad \quad -30 \quad + \quad 25\sqrt{6} \\
 & \quad \quad \quad -50 \quad + \quad 29\sqrt{6}
 \end{aligned}$$

$$35. \frac{\sqrt{-a} + \sqrt{-b} + \sqrt{-c}}{\sqrt{-a} + \sqrt{-b} - \sqrt{-c}}$$

$$\frac{-a - \sqrt{ab} - \sqrt{ac}}{-a - \sqrt{ab} - \sqrt{ac}}$$

$$\frac{-b - \sqrt{bc}}{-b - \sqrt{bc} + c}$$

$$\frac{-a - 2\sqrt{ab} - b + c}{-b + c}$$

$$39. \sqrt{-18} \div \sqrt{-3} = \sqrt{6}\sqrt{-3} \div \sqrt{-3} = \sqrt{6}.$$

$$40. \frac{\sqrt{27}}{\sqrt{-3}} = \frac{3\sqrt{3}}{\sqrt{3}\sqrt{-1}} = \frac{3}{\sqrt{-1}} = \frac{3\sqrt{-1}}{-1} = -3\sqrt{-1}.$$

$$41. \frac{14\sqrt{-5}}{2\sqrt{-7}} = \frac{7\sqrt{5}\sqrt{-1}}{\sqrt{7}\sqrt{-1}} = \sqrt{7}\sqrt{5} = \sqrt{35}.$$

$$42. \frac{-\sqrt{-a^2}}{\sqrt{-b^2}} = \frac{-a\sqrt{-1}}{b\sqrt{-1}} = -\frac{a}{b}.$$

$$43. \frac{1}{\sqrt{-1}} = \frac{1\sqrt{-1}}{-1} = -\sqrt{-1}.$$

$$44. \frac{(\sqrt{-1})^4 - \sqrt{-1}}{\sqrt{-1}} = (\sqrt{-1})^3 - 1 = (-1)\sqrt{-1} - 1 = -(\sqrt{-1} + 1).$$

$$45. \frac{\sqrt{-3} + (\sqrt{-1})^2}{\sqrt{-1}} = \frac{\sqrt{3}\sqrt{-1} + (\sqrt{-1})^2}{\sqrt{-1}} = \sqrt{3} + \sqrt{-1}.$$

$$46. \frac{\sqrt{8} + 3\sqrt{14}}{\sqrt{-2}} = \frac{(2 + 3\sqrt{7})\sqrt{2}\sqrt{-1}}{\sqrt{2}(-1)} = -(2 + 3\sqrt{7})\sqrt{-1}.$$

$$47. \frac{\sqrt{12} + \sqrt{3}}{\sqrt{-3}} = \frac{(2 + 1)\sqrt{3}\sqrt{-1}}{\sqrt{3}(-1)} = -3\sqrt{-1}.$$

$$48. \frac{-2}{\sqrt{-1}} = \frac{-2\sqrt{-1}}{-1} = 2\sqrt{-1}.$$

$$49. \frac{(\sqrt{-1})^5}{\frac{1}{3}\sqrt{-1}} = 3(\sqrt{-1})^4 = 3(+1) = 3.$$

$$50. \frac{(\sqrt{-1})^8}{(\sqrt{-1})^{15}} = \frac{1}{(\sqrt{-1})^{12}} = \frac{1}{\{(\sqrt{-1})^4\}^3} = \frac{1}{1^3} = 1.$$

$$51. \frac{\sqrt{4ab}}{\sqrt{-bc}} = \frac{2\sqrt{ab}\sqrt{c}\sqrt{-1}}{\sqrt{b \cdot c}(-1)} = -\frac{2\sqrt{-ac}}{c}.$$

$$52. \frac{\sqrt{-20} - \sqrt{-2}}{2\sqrt{-1}} = \frac{(2\sqrt{5} - \sqrt{2})\sqrt{-1}}{2\sqrt{-1}} = \sqrt{5} - \frac{1}{2}\sqrt{2}.$$

53. $\frac{\sqrt{-16} - \sqrt{-6}}{2\sqrt{-2}} = \frac{(\sqrt{8} - \sqrt{3})\sqrt{-2}}{2\sqrt{-2}} = \frac{2\sqrt{2} - \sqrt{3}}{2} = \sqrt{2} - \frac{1}{2}\sqrt{3}.$
54. $\frac{(\sqrt{-1})^{14}}{-\frac{1}{2}\sqrt{-1}} = -2(\sqrt{-1})^{13} = -2(\sqrt{-1})^{12}\sqrt{-1} = -2\sqrt{-1}.$
55. $(\sqrt{-1})^{10} \div (\sqrt{-1})^{-2} = (\sqrt{-1})^{12} = ((\sqrt{-1})^4)^3 = (+1)^3 = 1.$
56. $\frac{\sqrt{-a^2 + b}\sqrt{-1}}{\sqrt{-ab}} = \frac{(a+b)\sqrt{-1}}{\sqrt{ab}\sqrt{-1}} = \frac{a+b}{ab}\sqrt{ab}.$
57. $\frac{\sqrt{-4}}{\sqrt{-2} \cdot \sqrt{-2} \cdot \sqrt{-1}} = \frac{2\sqrt{-1}}{-2\sqrt{-1}} = -1.$

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2. $(5 + \sqrt{-4}) + (\sqrt{-9} - 3) = (5 - 3) + (2\sqrt{-1} + 3\sqrt{-1})$
 $= 2 + 5\sqrt{-1}.$
3. $(2 - \sqrt{-16}) + (3 + \sqrt{-4}) = (2 + 3) + (-4\sqrt{-1} + 2\sqrt{-1})$
 $= 5 - 2\sqrt{-1}.$
4. $(3 - \sqrt{-8}) + (4 + \sqrt{-18}) = (3 + 4) + (-2\sqrt{-2} + 3\sqrt{-2})$
 $= 7 + \sqrt{-2}.$
5. $(\sqrt{-20} - \sqrt{16}) + (\sqrt{-45} + \sqrt{4}) = (-4 + 2) + (2\sqrt{-5} + 3\sqrt{-5})$
 $= -2 + 5\sqrt{-5}.$
6. $(4 + \sqrt{-25}) - (2 + \sqrt{-4}) = (4 - 2) + (5\sqrt{-1} - 2\sqrt{-1})$
 $= 2 + 3\sqrt{-1}.$
7. $(3 - 2\sqrt{-5}) - (2 - 3\sqrt{-5}) = (3 - 2) + (-2\sqrt{-5} + 3\sqrt{-5})$
 $= 1 + \sqrt{-5}.$
8. $(2 - 2\sqrt{-1} + 3) - (\sqrt{16} - \sqrt{-16}) = (5 - 4) + (-2\sqrt{-1} + 4\sqrt{-1})$
 $= 1 + 2\sqrt{-1}.$
9. $\sqrt{-49} - 2 - 3\sqrt{-4} - \sqrt{-1} + 6 = (6 - 2) + 7\sqrt{-1} - 6\sqrt{-1} - \sqrt{-1} = 4.$
12. $(2 + 3\sqrt{-1})(1 + \sqrt{-1}) = 2 + (2 + 3)\sqrt{-1} + 3(-1)$
 $= 2 - 3 + 5\sqrt{-1} = -1 + 5\sqrt{-1}.$
13. $(5 - \sqrt{-1})(1 - 2\sqrt{-1}) = 5 - (10 + 1)\sqrt{-1} + 2(-1)$
 $= 5 - 2 - 11\sqrt{-1} = 3 - 11\sqrt{-1}.$
14. $(\sqrt{2} + \sqrt{-2})(\sqrt{8} - \sqrt{-8}) = \sqrt{2}(1 + \sqrt{-1}) \times \sqrt{8}(1 - \sqrt{-1})$
 $= 4(1 + \sqrt{-1})(1 - \sqrt{-1}) = 4 \times 2 = 8.$

$$15. (2 + 3i)^2 = 2^2 + 2 \cdot 6i + 3^2 i^2 = 4 + 12i - 9 = -5 + 12i.$$

$$16. (2 - 3i)^2 = 2^2 - 2 \cdot 6i + 3^2 i^2 = 4 - 12i - 9 = -5 - 12i.$$

$$17. (a - bi)^2 = a^2 - 2abi + b^2 i^2 = a^2 - 2abi - b^2.$$

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$$\begin{aligned} 18. (1 + \sqrt{-3})(1 + \sqrt{-3})(1 + \sqrt{-3}) &= (1 + \sqrt{-3})^2 (1 + \sqrt{-3}) \\ &= (1 + 2\sqrt{-3} - 3)(1 + \sqrt{-3}) \\ &= -2(1 - \sqrt{-3})(1 + \sqrt{-3}) \\ &= -2[1^2 - (\sqrt{-3})^2] \\ &= -2(1 + 3) = -8. \end{aligned}$$

19.

$$\begin{aligned} (-1 + \sqrt{-3})(-1 + \sqrt{-3})(-1 + \sqrt{-3}) &= (-1 + \sqrt{-3})^2 (-1 + \sqrt{-3}) \\ &= (1 - 2\sqrt{-3} - 3)(-1 + \sqrt{-3}) \\ &= 2(-1 - \sqrt{-3})(-1 + \sqrt{-3}) \\ &= 2[(-1)^2 - (\sqrt{-3})^2] \\ &= 2(1 + 3) = 8. \end{aligned}$$

20. If each factor given in Ex. 19 is divided by 2, the result must be divided by 2^3 , or by 8. Therefore,

$$(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})(-\frac{1}{2} + \frac{1}{2}\sqrt{-3}) = 1.$$

$$22. \frac{3}{1 - \sqrt{-2}} = \frac{3(1 + \sqrt{-2})}{(1 - \sqrt{-2})(1 + \sqrt{-2})} = \frac{3(1 + \sqrt{-2})}{1 + 2} = 1 + \sqrt{-2}.$$

$$23. \frac{2}{1 + \sqrt{-1}} = \frac{2(1 - \sqrt{-1})}{(1 + \sqrt{-1})(1 - \sqrt{-1})} = \frac{2(1 - \sqrt{-1})}{1 + 1} = 1 - \sqrt{-1}.$$

$$24. \frac{4 + \sqrt{4}}{2 - \sqrt{-2}} = \frac{(4 + \sqrt{4})(2 + \sqrt{-2})}{(2 - \sqrt{-2})(2 + \sqrt{-2})} = \frac{6(2 + \sqrt{-2})}{4 + 2} = 2 + \sqrt{-2}.$$

$$\begin{array}{l} 25. \frac{6 + \sqrt{-3} + 3}{6 - 2\sqrt{-3}} \quad \left| \frac{3 - \sqrt{-3}}{2 + \sqrt{-3}} \right. \\ \quad \frac{3\sqrt{-3} + 3}{3\sqrt{-3} + 3} \\ 26. \frac{6 + 4\sqrt{-5} + 10}{6 - 2\sqrt{-5}} \quad \left| \frac{3 - \sqrt{-5}}{2 + 2\sqrt{-5}} \right. \\ \quad \frac{6\sqrt{-5} + 10}{6\sqrt{-5} + 10} \end{array}$$

$$\begin{aligned} 27. \frac{a^2 + b^2}{a - b\sqrt{-1}} &= \frac{a^2 - b^2(\sqrt{-1})^2}{a - b\sqrt{-1}} = \frac{(a + b\sqrt{-1})(a - b\sqrt{-1})}{a - b\sqrt{-1}} \\ &= a + b\sqrt{-1}. \end{aligned}$$

Or

$$\begin{array}{l} \frac{a^2}{a^2 - ab\sqrt{-1}} + \frac{b^2}{ab\sqrt{-1} + b^2} \left| \frac{a - b\sqrt{-1}}{a + b\sqrt{-1}} \right. \\ \quad \frac{ab\sqrt{-1} + b^2}{ab\sqrt{-1} + b^2} \end{array}$$

$$28. \frac{a+bi}{ai+b} = \frac{(a+bi)(ai-b)}{(ai+b)(ai-b)} = \frac{a^2i - ab - ab - b^2i}{-a^2 - b^2} = \frac{2ab - (a^2 - b^2)i}{a^2 + b^2}.$$

$$29. \frac{(1+i)^2}{1-i} = \frac{1+2i-1}{1-i} = \frac{2i}{1-i} = \frac{2i(1+i)}{(1-i)(1+i)} = \frac{2(i-1)}{1+1} = i-1.$$

$$31. \quad 4 + 2\sqrt{-21} = 7 + 2\sqrt{7(-3)} - 3 = (\sqrt{7} + \sqrt{-3})^2.$$

$$\therefore \sqrt{4 + 2\sqrt{-21}} = \sqrt{7} + \sqrt{-3}.$$

$$32. \quad 1 + 2\sqrt{-6} = 3 + 2\sqrt{3(-2)} - 2 = (\sqrt{3} + \sqrt{-2})^2.$$

$$\therefore \sqrt{1 + 2\sqrt{-6}} = \sqrt{3} + \sqrt{-2}.$$

$$33. \quad 6 - 2\sqrt{-7} = 7 - 2\sqrt{7(-1)} - 1 = (\sqrt{7} - \sqrt{-1})^2.$$

$$\therefore \sqrt{6 - 2\sqrt{-7}} = \sqrt{7} - \sqrt{-1}.$$

$$34. \quad 9 + 2\sqrt{-22} = 11 + 2\sqrt{11(-2)} - 2 = (\sqrt{11} + \sqrt{-2})^2.$$

$$\therefore \sqrt{9 + 2\sqrt{-22}} = \sqrt{11} + \sqrt{-2}.$$

$$35. \quad 12\sqrt{-1} - 5 = 2\sqrt{-36} + 4 - 9 = 4 + 2\sqrt{4(-9)} - 9.$$

$$\therefore \sqrt{12\sqrt{-1} - 5} = \sqrt{4 + 2\sqrt{4(-9)} - 9} = \sqrt{4} + \sqrt{-9} = 2 + 3\sqrt{-1}.$$

$$36. \quad b^2 + 2ab\sqrt{-1} - a^2 = b^2 + 2b \cdot a\sqrt{-1} + (a\sqrt{-1})^2 = (b + a\sqrt{-1})^2.$$

$$\therefore \sqrt{b^2 + 2ab\sqrt{-1} - a^2} = b + a\sqrt{-1}.$$

$$37. \text{ Substituting } -1 + \sqrt{-1} \text{ for } x \text{ in } x^2 + 2x + 2 = 0,$$

$$(-1 + \sqrt{-1})^2 + 2(-1 + \sqrt{-1}) + 2 = 1 - 2\sqrt{-1} - 1 - 2 + 2\sqrt{-1} + 2$$

$$= 0, \text{ or } 0 = 0.$$

$$\text{Substituting } -1 - \sqrt{-1} \text{ for } x \text{ in } x^2 + 2x + 2 = 0,$$

$$(-1 - \sqrt{-1})^2 + 2(-1 - \sqrt{-1}) + 2 = 1 + 2\sqrt{-1} - 1 - 2 - 2\sqrt{-1} + 2$$

$$= 0, \text{ or } 0 = 0.$$

Hence, $-1 + \sqrt{-1}$ and $-1 - \sqrt{-1}$ are roots of the equation $x^2 + 2x + 2 = 0$.

$$38. \quad \left(\frac{1}{2} + \frac{1}{2}\sqrt{-3}\right)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\frac{1}{2}\sqrt{-3} + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{-3}\right)^2 + \left(\frac{1}{2}\sqrt{-3}\right)^3$$

$$= \frac{1}{8} + \frac{3}{8}\sqrt{-3} - \frac{9}{8} - \frac{3}{8}\sqrt{-3} = -1.$$

INEQUALITIES

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4.
Transposing, Prin. 2,
Dividing by 6, Prin. 4,

$$6x - 5 > 13.$$

$$6x > 18.$$

$$x > 3.$$

5.
Transposing, Prin. 2,
Changing signs, Prin. 3,

$$5x - 1 < 6x + 4.$$

$$-x < 5.$$

$$x > -5.$$

6.
Uniting terms,
Multiplying by $\frac{2}{5}$, Prin. 4,

$$3x - \frac{1}{2}x < 30.$$

$$\frac{5}{2}x < 30.$$

$$x < 12.$$

7. $4x + 1 < 6x - 11.$
 Transposing, Prin. 2, $-2x < -12.$
 Dividing by -2 , Prin. 4, $x > 6.$

8. $\begin{cases} 4x - 11 > \frac{1}{3}x, & (1) \\ 20 - 2x > 10. & (2) \end{cases}$
 Multiplying (1) by 3, Prin. 4, $12x - 33 > x.$
 Transposing, Prin. 2, $11x > 33.$
 Dividing by 11, Prin. 4, $x > 3.$
 Transposing in (2), Prin. 2, $-2x > -10.$
 Dividing by -2 , Prin. 4, $x < 5.$

Hence, x is greater than 3 and less than 5; that is, x can have all values that lie between 3 and 5.

9. $\begin{cases} 3 - 4x < 7, & (1) \\ 5x + 10 < 20. & (2) \end{cases}$
 Transposing in (1), Prin. 2, $-4x < 4.$
 Dividing by -4 , Prin. 4, $x > -1.$
 Dividing (2) by 5, Prin. 4, $x + 2 < 4.$
 Transposing, Prin. 2, $x < 2.$
 Hence, x is greater than -1 and less than 2.

10. $x + \frac{2x}{3} + \frac{5x}{6} > 25 \text{ and } < 30.$
 Multiplying by 6, Prin. 4, $6x + 4x + 5x > 150 \text{ and } < 180.$
 Dividing by 15, Prin. 4, $15x > 150 \text{ and } < 180.$
 $x > 10 \text{ and } < 12.$

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12. $x^2 + 3x > 10.$
 Transposing, Prin. 2, $x^2 + 3x - 10 > 0.$
 Factoring, $(x - 2)(x + 5) > 0.$

Since $(x - 2)(x + 5)$ is positive, both factors are positive or both factors are negative. If both factors are positive, $x > 2$. If both factors are negative, $x < -5$.

Hence, x can have all values except 2 and -5 and those which lie between 2 and -5 .

13. $x^2 + 8x > 20.$
 Transposing, Prin. 2, $x^2 + 8x - 20 > 0.$
 Factoring, $(x - 2)(x + 10) > 0.$

Since $(x - 2)(x + 10)$ is positive, both factors are positive or both factors are negative. If both factors are positive, $x > 2$. If both factors are negative, $x < -10$.

Hence, x can have all values except 2 and -10 and those which lie between 2 and -10 .

14. $x^2 + 5x > 24.$
 Transposing, Prin. 2, $x^2 + 5x - 24 > 0.$
 Factoring, $(x - 3)(x + 8) > 0.$

Since $(x - 3)(x + 8)$ is positive, both factors are positive or both factors are negative. If both factors are positive, $x > 3$. If both factors are negative, $x < -8$.

Hence, x can have all values except 3 and -8 and those which lie between 3 and -8 .

15.

Changing signs, Prin. 3,

$$(x-2)(3-x) > 0.$$

$$(x-2)(x-3) < 0.$$

Since $(x-2)(x-3)$ is negative, the factors have unlike signs. Hence, $(x-2)$, the greater factor, is positive, and $x-3$, the less factor, is negative; that is, x can have all values between 2 and 3, but no others.

16.

Transposing, Prin. 2,

$$x^2 > 9x - 18$$

$$x^2 - 9x + 18 > 0.$$

Factoring,

$$(x-3)(x-6) > 0.$$

Since $(x-3)(x-6)$ is positive, both factors are positive or both factors are negative. If both factors are positive, $x > 6$. If both factors are negative, $x < 3$.

Hence, x can have all values except 6 and 3 and those which lie between 6 and 3.

17.

Transposing, Prin. 2,

$$x^2 + 40x > 3(4x - 25).$$

Factoring,

$$x^2 + 40x > 12x - 75.$$

$$x^2 + 28x + 75 > 0.$$

$$(x+3)(x+25) > 0.$$

Since $(x+3)(x+25)$ is positive, both factors are positive or both factors are negative. If both factors are positive, $x > -3$. If both factors are negative, $x < -25$.

Hence, x can have all values except -3 and -25 and intermediate values.

18.

Transposing, Prin. 2,

$$x^2 + bx > ax + ab.$$

$$x^2 - ax + bx - ab > 0.$$

Factoring,

$$(x-a)(x+b) > 0.$$

Since $(x-a)(x+b)$ is positive, both factors are positive or both factors are negative. If both factors are positive, $x > a$. If both factors are negative, $x < -b$.

Hence, x can have all values except a and $-b$ and those which lie between a and $-b$.

19. See next page.

20.

Subtracting (2) from (1), Prin. 1,

$$\begin{cases} 2x - 3y < 2, & (1) \\ 2x + 5y < 25. & (2) \end{cases}$$

Dividing (3) by -8 , Prin. 4,

$$-8y < -23. \quad (3)$$

Multiplying (1) by 5, Prin. 4,

$$y > 2\frac{1}{8}. \quad (4)$$

Multiplying (2) by 3,

$$10x - 15y < 10. \quad (5)$$

Adding (6) to (5), Prin. 1,

$$6x + 15y = 75. \quad (6)$$

Dividing (7) by 16, Prin. 4,

$$16x < 85. \quad (7)$$

Hence, the positive integral values are $x = 5, y = 3$.

21.

Subtracting (2) from (1), Prin. 6,

$$\begin{cases} 3x + 2y = 42, & (1) \\ 3x - \frac{2}{7}y > 16. & (2) \end{cases}$$

Multiplying (3) by $\frac{7}{15}$, Prin. 4,

$$\frac{1}{7}y < 26. \quad (3)$$

Multiplying (2) by 14, Prin. 4,

$$y < 12\frac{1}{5}. \quad (4)$$

Adding (1) to (5), Prin. 1,

$$42x - 2y > 224. \quad (5)$$

Dividing (6) by 45, Prin. 4,

$$45x > 266. \quad (6)$$

$$x > 5\frac{1}{3}.$$

Hence, the positive integral values are $x = 6, y = 12; x = 8, y = 9; x = 10, y = 6; x = 12, y = 3$.

19.

$$(x-3)(5-x) > 0.$$

Changing signs, Prin. 3,

$$(x-3)(x-5) < 0.$$

Since $(x-3)(x-5)$ is negative, the factors have unlike signs. Hence, $x-3$, the greater factor, is positive, and $x-5$, the less factor, is negative; that is, x can have all values between 3 and 5, but no others.

22.

$$\begin{cases} x+y=10, & (1) \\ 4x < 3y. & (2) \end{cases}$$

From (1),

$$y=10-x. \quad (3)$$

Substituting (3) in (2),

$$4x < 30-3x. \quad (4)$$

Transposing in (4), Prin. 2,

$$7x < 30. \quad (5)$$

Dividing (5) by 7, Prin. 4,

$$x < 4\frac{2}{7}. \quad (6)$$

From (1),

$$x=10-y. \quad (7)$$

Substituting (7) in (2),

$$40-4y < 3y. \quad (8)$$

Transposing in (8), Prin. 2,

$$-7y < -40. \quad (9)$$

Dividing (9) by -7 , Prin. 4,

$$y > 5\frac{4}{7}.$$

Hence, the positive integral values are $x=4$, $y=6$; $x=3$, $y=7$; $x=2$, $y=8$; $x=1$, $y=9$.

23.

$$\begin{cases} y=3x+4, & (1) \\ 25 < 2y+3x. & (2) \end{cases}$$

Substituting (1) in (2),

$$25 < 6x+8+3x. \quad (3)$$

Transposing in (3), Prin. 2,

$$-9x < -17. \quad (4)$$

Dividing (4) by -9 , Prin. 4,

$$x > 1\frac{8}{9}. \quad (5)$$

Subtracting (1) from (2), Prin. 1,

$$25-y < 2y-4. \quad (6)$$

Transposing in (6), Prin. 2,

$$-3y < -29. \quad (7)$$

Dividing (7) by -3 , Prin. 4,

$$y > 9\frac{2}{3}.$$

Hence, the positive integral values are $x=4$, $y=16$; $x=5$, $y=19$; $x=6$, $y=22$; etc.

24.

$$\begin{cases} y-x > 9, & (1) \\ \frac{7x}{20} + \frac{y}{15} = 9. & (2) \end{cases}$$

Multiplying (2) by 60,

$$21x+4y=540. \quad (3)$$

Multiplying (1) by 4, Prin. 4,

$$-4x+4y > 36. \quad (4)$$

Subtracting (4) from (3), Prin. 6,

$$25x < 504. \quad (5)$$

Dividing (5) by 25, Prin. 4,

$$x < 20\frac{4}{25}. \quad (6)$$

Multiplying (1) by 21, Prin. 4,

$$-21x+21y > 189. \quad (7)$$

Adding (3) to (7), Prin. 1,

$$25y > 729. \quad (8)$$

Dividing (8) by 25, Prin. 4,

$$y > 29\frac{4}{25}.$$

Hence, the positive integral values are $x=20$, $y=30$; $x=16$, $y=51$; $x=12$, $y=72$; $x=8$, $y=93$; $x=4$, $y=114$.

25. See next page.

$$\begin{aligned} 26. \quad \frac{a+b}{a+2b} - \frac{a+2b}{a+3b} &= 1 - \frac{b}{a+2b} - \left(1 - \frac{b}{a+3b}\right) \\ &= \frac{b}{a+3b} - \frac{b}{a+2b} = \frac{-b^2}{(a+3b)(a+2b)}. \end{aligned}$$

Hence, $\frac{a+2b}{a+3b} > \frac{a+b}{a+2b}$, a and b being positive.

27. Since $(a-b)^2$ is positive,

$$a^2 - 2ab + b^2 > 0, \text{ if } a \text{ and } b \text{ are}$$

unequal.

Transposing $-2ab$, Prin. 2,

$$a^2 + b^2 > 2ab.$$

25.

Subtracting (2) from (1), Prin. 1,

Transposing in (3), Prin. 2,

Multiplying (1) by 2, Prin. 4,

Subtracting (2) from (5), Prin. 1,

Canceling $2y = 2y$, Prin. 1,

$$\begin{cases} x > y + 4, & (1) \\ x - 2y = 8. & (2) \end{cases}$$

$$2y > y - 4. \quad (3)$$

$$y > -4. \quad (4)$$

$$2x > 2y + 8. \quad (5)$$

$$x + 2y > 2y.$$

$$x > 0.$$

Hence, the positive integral values are $y = 1, x = 10; y = 2, x = 12; y = 3, x = 14$; etc.

26, 27. See preceding page.

28. If a, b , and c are unequal, since $(a - b)^2$, $(a - c)^2$, and $(b - c)^2$ are positive,

$$a^2 - 2ab + b^2 > 0,$$

$$a^2 - 2ac + c^2 > 0,$$

and

$$b^2 - 2bc + c^2 > 0.$$

Adding, Prin. 5, and dividing by 2, Prin. 4,

$$a^2 + b^2 + c^2 - ab - ac - bc > 0.$$

Transposing, Prin. 2,

$$a^2 + b^2 + c^2 > ab + ac + bc.$$

29. If a and b are unequal, $(a - b)^2$ is positive.

$$\therefore a^2 - 2ab + b^2 > 0.$$

Transposing, Prin. 2,

$$a^2 - ab + b^2 > ab.$$

If a and b are positive, multiplying by $a + b$, Prin. 4,

$$a^3 + b^3 > a^2b + ab^2.$$

30.

$$\frac{a^3 + b^3}{a^2 + b^2} - \frac{a^2 + b^2}{a + b} = \frac{ab(a^2 + b^2 - 2ab)}{(a^2 + b^2)(a + b)}. \quad (1)$$

If a and b are unequal and positive, $a^2 + b^2 - 2ab$, or $(a - b)^2$, is positive, ab is positive, and $(a^2 + b^2)(a + b)$ is positive.

Hence, the second member of (1) is positive, and

$$\frac{a^3 + b^3}{a^2 + b^2} - \frac{a^2 + b^2}{a + b} > 0; \quad \therefore \frac{a^3 + b^3}{a^2 + b^2} > \frac{a^2 + b^2}{a + b}.$$

31. Except when $2a = 3b$, $(2a - 3b)^2$ is positive; that is,

$$4a^2 - 12ab + 9b^2 > 0.$$

Transposing $-12ab$, Prin. 2,

$$4a^2 + 9b^2 > 12ab.$$

Dividing by $12ab$, if a and b are positive, Prin. 4,

$$\frac{a}{3b} + \frac{3b}{4a} > 1.$$

32. Except when $a = 3b$, $(a - 3b)^2$ is positive; that is,

$$a^2 - 6ab + 9b^2 > 0.$$

Subtracting b^2 from each member, Prin. 1,

$$a^2 - 6ab + 8b^2 > -b^2.$$

Factoring,

$$(a - 2b)(a - 4b) > -b^2.$$

Changing signs, Prin. 3,

$$(a - 2b)(4b - a) < b^2.$$

33. It has been proved in Ex. 28 that $a^2 + b^2 + c^2 > ab + ac + bc$.Transposing, Prin. 2, $a^2 + b^2 + c^2 - ab - ac - bc > 0$.Multiplying by $a + b + c$, if a, b , and c are positive, Prin. 4,

$$a^3 + b^3 + c^3 - 3abc > 0.$$

Transposing $-3abc$, Prin. 2,

$$a^3 + b^3 + c^3 > 3abc.$$

34. Let a represent any positive real number, except 1.

It is to be proved that $a + \frac{1}{a} > 2$.

Except when $a = 1$, $(a - 1)^2$ is positive; that is,
 $a^2 - 2a + 1 > 0$.

Transposing $-2a$, Prin. 2, $a^2 + 1 > 2a$.

Dividing by a , Prin. 4, $a + \frac{1}{a} > 2$.

VARIABLES AND LIMITS

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1. Prin. 5, $\lim. (x + y + z) = \lim. x + \lim. y + \lim. z$
 $= a + 2 + 0 = a + 2$.
2. Prin. 6, $\lim. xy = \lim. x \cdot \lim. y = a \cdot 2 = 2a$,
 and $\lim. x^2 = \lim. x \cdot \lim. x = a \cdot a = a^2$.
 \therefore Prin. 5, 4, $\lim. (axy - x^2) = a \cdot 2a - a^2 = 2a^2 - a^2 = a^2$.
3. Prin. 5, 4, $\lim. \left(\frac{x}{2} - \frac{y}{a} \right) = \frac{\lim. x}{2} - \frac{\lim. y}{a} = \frac{a}{2} - \frac{2}{a} = \frac{a^2 - 4}{2a}$.
4. Prin. 5, $\lim. (x - \frac{1}{2}y + ax - y^2) = \lim. x - \lim. \frac{1}{2}y + \lim. ax - \lim. y^2$
 Prin. 4, 6, $= a - \frac{1}{2} \times 2 + a \times a - 2 \times 2$
 $= a - 1 + a^2 - 4$
 $= a^2 + a - 5$.
5. Prin. 5, $\lim. [(x+y)x - (x-y)z] = \lim. [(x+y)x] - \lim. [(x-y)z]$
 Prin. 6, $= \lim. (x+y) \lim. x - \lim. (x-y) \lim. z$
 Since $\lim. x = a$ and $\lim. z = 0$, $= \lim. (x+y) \cdot a = a \lim. (x+y)$
 Prin. 5, $= a(\lim. x + \lim. y)$
 $= a(a + 2) = a^2 + 2a$.
6. $\lim. \left(\frac{a + x + z}{x - y} + \frac{x + y}{a} \right)$
 Prin. 5, $= \lim. \left(\frac{a + x + z}{x - y} \right) + \lim. \left(\frac{x + y}{a} \right)$
 Prin. 7, 4, $= \frac{\lim. (a + x + z)}{\lim. (x - y)} + \frac{1}{a} \lim. (x + y)$
 Prin. 3, $= \frac{a + \lim. (x + z)}{\lim. (x - y)} + \frac{1}{a} \lim. (x + y)$
 Prin. 5, $= \frac{a + \lim. x + \lim. z}{\lim. x - \lim. y} + \frac{\lim. x + \lim. y}{a}$
 $= \frac{a + a}{a - 2} + \frac{a + 2}{a} = \frac{3a^2 - 4}{a^2 - 2a}$.
7. Prin. 7, $\lim. \left[\frac{x^2 - 5x + 8}{x - 2} \right]_{x \div 3} = \frac{\lim. (x^2 - 5x + 8)_{x \div 3}}{\lim. (x - 2)_{x \div 3}}$
 $= \frac{9 - 15 + 8}{1} = 2$.
 Prin. 6, 4, 5, 3,

$$\begin{aligned}
 8. \quad & \text{Lim. } \left[\frac{x^3 + 4x + 1}{x^4 + x^2 + 1} \right]_{x \doteq -1} \\
 \text{Prin. 7,} \quad & = \frac{\text{lim. } (x^3 + 4x + 1)_{x \doteq -1}}{\text{lim. } (x^4 + x^2 + 1)_{x \doteq -1}} \\
 \text{Prin. 6, 4, 5, 3,} \quad & = \frac{-1 - 4 + 1}{1 + 1 + 1} = -\frac{4}{3}.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \text{Lim. } \left[\frac{x^2 - 3x + 4}{x + 6} \right]_{x \doteq 2} \\
 \text{Prin. 7,} \quad & = \frac{\text{lim. } (x^2 - 3x + 4)_{x \doteq 2}}{\text{lim. } (x + 6)_{x \doteq 2}} \\
 \text{Prin. 6, 4, 5, 3,} \quad & = \frac{4 - 6 + 4}{2 + 6} = \frac{1}{4}.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \text{Lim. } \left[\frac{1 + x^n}{1 + x} \right]_{x \doteq 1} \\
 \text{Prin. 7,} \quad & = \frac{\text{lim. } (1 + x^n)_{x \doteq 1}}{\text{lim. } (1 + x)_{x \doteq 1}} \\
 \text{Prin. 3, 6,} \quad & = \frac{1 + 1^n}{1 + 1} = \frac{1 + 1}{1 + 1} = 1.
 \end{aligned}$$

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$$\begin{aligned}
 1. \quad \text{Lim. } \left[\frac{x^3 - 8}{x^2 - 4} \right]_{x \doteq 2} &= \text{lim. } \left[\frac{x^2 + 2x + 4}{x + 2} \right]_{x \doteq 2} \\
 &= \frac{4 + 4 + 4}{4} = 3.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \text{Lim. } \left[\frac{1 - x^5}{1 - x} \right]_{x \doteq 1} &= \text{lim. } (1 + x + x^2 + x^3 + x^4)_{x \doteq 1} \\
 &= 1 + 1 + 1 + 1 + 1 = 5.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \text{Lim. } \left[\frac{x^4 - a^4}{x^3 + a^3} \right]_{x \doteq -a} &= \text{lim. } \left[\frac{x^3 - ax^2 + a^2x - a^3}{x^2 - ax + a^2} \right]_{x \doteq -a} \\
 &= \frac{-a^3 - a^3 - a^3 - a^3}{a^2 + a^2 + a^2} = -\frac{4}{3}a.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \text{Lim. } \left[\frac{x^3 + b^3}{x^2 - b^2} \right]_{x \doteq -b} &= \text{lim. } \left[\frac{x^2 - bx + b^2}{x - b} \right]_{x \doteq -b} \\
 &= \frac{b^2 + b^2 + b^2}{-b - b} = \frac{3b^2}{-2b} = -\frac{3}{2}b.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \text{Lim. } \left[\frac{x^2 - 5x + 6}{x^2 - 4} \right]_{x \doteq 2} &= \text{lim. } \left[\frac{x - 3}{x + 2} \right]_{x \doteq 2} \\
 &= \frac{2 - 3}{2 + 2} = -\frac{1}{4}.
 \end{aligned}$$

$$6. \quad \text{Lim.} \left[\frac{x^4 - 3x^3 + 5x^2 - 2x - 1}{x^3 - 8x^2 + 7} \right]_{x \doteq 1} = \text{lim.} \left[\frac{x^3 - 2x^2 + 3x + 1}{x^2 - 7x - 7} \right]_{x \doteq 1} \\ = \frac{1 - 2 + 3 + 1}{1 - 7 - 7} = -\frac{3}{13}.$$

$$8. \quad \text{Lim.} \left[\frac{1 + x^2 + x^4 + x^6}{1 - x^2 - x^4 - 2x^6} \right]_{x \doteq 0} = \frac{1}{1} = 1. \\ \text{Lim.} \left[\frac{1 + x^2 + x^4 + x^6}{1 - x^2 - x^4 - 2x^6} \right]_{x \doteq \infty} = \frac{x^6}{-2x^6} = -\frac{1}{2}.$$

$$9. \quad \text{Lim.} \left[\frac{5x^3 - x^2 + 4x + 2}{2x^3 + 3x^2 - x + 1} \right]_{x \doteq 0} = \frac{2}{1} = 2. \\ \text{Lim.} \left[\frac{5x^3 - x^2 + 4x + 2}{2x^3 + 3x^2 - x + 1} \right]_{x \doteq \infty} = \frac{5x^3}{2x^3} = \frac{5}{2}.$$

$$10. \quad \text{Lim.} \left[\frac{4x^4 - 3x^3 + x + 1}{2x^4 - x^3 - x^2 + x + 1} \right]_{x \doteq 0} = \frac{1}{1} = 1. \\ \text{Lim.} \left[\frac{4x^4 - 3x^3 + x + 1}{2x^4 - x^3 - x^2 + x + 1} \right]_{x \doteq \infty} = \frac{4x^4}{2x^4} = 2.$$

$$11. \quad \text{Lim.} \left[\frac{2x^4 - 3x^3 + 2x^2 - 2}{x^4 - 2x^2 + x + 1} \right]_{x \doteq 0} = \frac{-2}{1} = -2. \\ \text{Lim.} \left[\frac{2x^4 - 3x^3 + 2x^2 - 2}{x^4 - 2x^2 + x + 1} \right]_{x \doteq \infty} = \frac{2x^4}{x^4} = 2.$$

$$12. \quad \text{Lim.} \left[\frac{5x + 10}{x^2 + 2x + 2} \right]_{x \doteq 0} = \frac{10}{2} = 5. \\ \text{Lim.} \left[\frac{5x + 10}{x^2 + 2x + 2} \right]_{x \doteq \infty} = \frac{\text{lim.} (5x + 10)_{x \doteq \infty}}{\text{lim.} (x^2 + 2x + 2)_{x \doteq \infty}} \\ = \frac{5x}{x^2} = \frac{5}{x} = \frac{5}{\infty} = 0.$$

$$13. \quad \text{Lim.} \left[\frac{3x - 4}{x^2 - x - 8} \right]_{x \doteq 0} = \frac{-4}{-8} = \frac{1}{2}. \\ \text{Lim.} \left[\frac{3x - 4}{x^2 - x - 8} \right]_{x \doteq \infty} = \frac{\text{lim.} (3x - 4)_{x \doteq \infty}}{\text{lim.} (x^2 - x - 8)_{x \doteq \infty}} \\ = \frac{3x}{x^2} = \frac{3}{x} = \frac{3}{\infty} = 0.$$

$$14. \quad \text{Lim.} \left[\frac{2x^2 - 4x + 1}{2x - 1} \right]_{x \doteq 0} = \frac{1}{-1} = -1. \\ \text{Lim.} \left[\frac{2x^2 - 4x + 1}{2x - 1} \right]_{x \doteq \infty} = \frac{\text{lim.} (2x^2 - 4x + 1)_{x \doteq \infty}}{\text{lim.} (2x - 1)_{x \doteq \infty}} \\ = \frac{2x^2}{2x} = x = \infty.$$

$$\begin{aligned}
 15. \quad \text{Lim.} \quad \left[\frac{4x^3 + 5x^2 + 2x}{2x^3 + x + 1} \right]_{x \doteq 0} &= \frac{\text{lim. } (4x^3 + 5x^2 + 2x)_{x \doteq 0}}{\text{lim. } (2x^3 + x + 1)_{x \doteq 0}} \\
 &= \frac{2x}{1} = 2x = 0. \\
 \text{Lim.} \quad \left[\frac{4x^3 + 5x^2 + 2x}{2x^3 + x + 1} \right]_{x \doteq \infty} &= \frac{4x^3}{2x^3} = 2.
 \end{aligned}$$

INTERPRETATION OF RESULTS

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1. Let x = number of years that will elapse before B will be half as old as A.

$$\begin{aligned}
 \text{Then,} \quad & 25 + x = \frac{1}{2}(40 + x). \\
 \text{Solving,} \quad & x = -10.
 \end{aligned}$$

The negative result indicates that B was half as old as A 10 years ago.

2. Let
and

Then,
and

Solving,

$$\begin{aligned}
 x &= \text{greater number,} \\
 y &= \text{less number.} \\
 x + y &= 6, & (1) \\
 x - y &= 10, & (2) \\
 x &= 8 \text{ and } y = -2.
 \end{aligned}$$

Since y proves to be negative while x is positive, (1) is explained arithmetically as the difference of two numbers, 8 and 2, and (2) as the sum of these numbers.

3. Let

$\frac{x}{y}$ represent the fraction.

Then,

$$\frac{x+1}{y} = \frac{3}{5}, \quad (1)$$

and

$$\frac{x}{y+1} = \frac{2}{3}. \quad (2)$$

Solving,

$$x = -16 \text{ and } y = -25.$$

The fraction $\frac{-16}{-25}$ has no arithmetical signification. But if $-x$ is substituted for x and $-y$ for y in (1) and (2),

$$\frac{x-1}{y} = \frac{3}{5},$$

and

$$\frac{x}{y-1} = \frac{2}{3},$$

in which $x = 16$ and $y = 25$.

Hence, the problem is made arithmetically reasonable by changing "added to" to "subtracted from," the fraction being $\frac{16}{25}$.

4. Let

x = number of apples he bought.

Then,

$$\frac{24}{x+4} = \frac{24}{x} - 1. \quad (1)$$

$$x^2 + 4x - 96 = 0. \quad (2)$$

$$\therefore x = 8 \text{ or } -12.$$

The interpretation of the result, - 12 apples bought, is 12 apples *sold*. Substituting $-x$ for x in (1), and changing signs,

$$\frac{24}{x-4} = \frac{24}{x} + 1, \quad (3)$$

in which $x = 12$ or -8 .

The result $x = -12$ in (1) or $x = 12$ in (3), indicates that if, instead of 12 apples for 24 cents, 4 apples *less* than 12 apples are sold for 24 cents, the price (or cost) of each is 1 cent *more*.

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5. Let x = number of shillings earned daily by the man,
 and y = number of shillings earned daily by his son.
 Then, $7x + 3y = 22$,
 and $5x + y = 18$.
 Solving, $x = 4$ and $y = -2$.

The negative result evidently means that the son caused an expense of 2 shillings daily.

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1. Let x = the number.
 Then, $x^2 - 2x = x^2 - 2x + 1 - 1$.
 $0x = 0$.
 $\therefore x = \frac{0}{0}$, indeterminate.

Hence, any number satisfies the conditions of the problem.

2. Let x = number of years in the son's age.
 Then, $x + 30$ = number of years in the father's age;
 $\therefore x + x + 30 = 2(x + 30) - 30$.
 $0x = 0$.
 $\therefore x = \frac{0}{0}$, indeterminate.

3. Let $x - 1$, x , and $x + 1$ represent the integers.
 Then, $x - 1 + x + 1 = 2x$.
 $0x = 0$.
 $\therefore x = \frac{0}{0}$.

Hence, any three consecutive integers fulfill this condition.

4. Let x = number of sheep in first flock at first.
 Then, $400 - x$ = number of sheep in second flock at first.
 $\therefore \frac{3}{2}x + 2(400 - x) = 2(x - 30) + \frac{5}{2}(400 - x - 56)$.
 Solving, $x = \frac{0}{0}$.

Hence, there may have been any number of sheep, less than 400, in the first flock and the remainder in the second flock.

5. Let x = number of dollars in monthly salary.
 Then, $4(10x - 600) = 5(8x - 480)$.
 Solving, $x = \frac{0}{0}$, indeterminate.

INDETERMINATE EQUATIONS

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4. $5x + 3y = 49.$
 Transposing $5x$, $3y = 49 - 5x.$
 When $5x = 5, 10, 15, 20, 25, 30, 35, 40, 45,$
 $3y = 44, 39, 34, 29, 24, 19, 14, 9, 4.$
 Hence, $x = 2, y = 13; x = 5, y = 8; x = 8, y = 3.$

5. $3x + 2y = 5. \quad (1)$
 Transposing $3x$, $2y = 5 - 3x. \quad (2)$

It is evident from (2) that x cannot be greater than 1, if y is a positive integer. Since x can have no positive integral value less than 1, the only positive integral value of x is $x = 1$.

When $x = 1$, substituting, $y = 1$.

6. $2x + 7y = 48. \quad (1)$
 Solving for x , $x = 24 - \frac{7y}{2}. \quad (2)$

It is evident from (1) that $y < 7$, and from (2) that y is even. Substituting 2, 4, 6, successively, for y in (2), $x = 17, 10, 3$.

7. $8x + 5y = 80.$
 It is evident that $x < 10$, and that x is a multiple of 5, since $8x = 5(16 - y)$. Hence, $x = 5$.
 When $x = 5$, substituting, $y = 8$.

8. $12x + 5y = 61. \quad (1)$
 Transposing $12x$, $5y = 61 - 12x. \quad (2)$
 It is evident from (2) that x cannot be greater than 4, if x and y are positive integers.

When $x = 1, 2, 3, 4,$
 $5y = 49, 37, 25, 13.$

Hence, the only possible positive integral values are $x = 3, y = 5$.

9. $6x + 7y = 72. \quad (1)$
 Solving for x , $x = 12 - \frac{7y}{6}. \quad (2)$

From (1), $y < 10$. From (2), y is divisible by 6.
 Therefore, $y = 6. \quad (3)$
 Substituting (3) in (2), $x = 5.$

10. $5x + 9y = 75. \quad (1)$
 Solving for x , $x = 15 - \frac{9y}{5}. \quad (2)$
 From (1), $y < 8$. From (2), y is divisible by 5.
 Therefore, $y = 5. \quad (3)$
 Substituting (3) in (2), $x = 6.$

11. $6x + 9y = 100.$
 Dividing by 3, $2x + 3y = \frac{100}{3}.$

Since $2x + 3y$ is equal to a fraction, x and y cannot have integral values at the same time; that is, the given equation is not satisfied by integral values of x and y .

$$12. \quad 2x = 9 + 3y, \quad (1)$$

$$\text{Dividing by 3,} \quad \frac{2x}{3} = 3 + y. \quad (2)$$

From (1), if $y = 1$, $x = 6$. From (2), x is divisible by 3.

Hence, the least positive integral values of x and y are

$$x = 6, y = 1.$$

$$13. \quad 5y = 2x + 7. \quad (1)$$

$$\text{Dividing by 2,} \quad 2y + \frac{y}{2} = x + 3 + \frac{1}{2}. \quad (2)$$

Collecting integral and fractional terms,

$$\frac{y-1}{2} = x + 3 - 2y. \quad (3)$$

$$\text{Since } x + 3 - 2y \text{ is integral,} \quad \frac{y-1}{2} = w, \text{ an integer.} \quad (4)$$

$$\text{Solving for } y, \quad y = 2w + 1. \quad (5)$$

The least integral value of y may be obtained by substituting the least positive integral value for w in (5).

$$\text{Substituting 1 for } w \text{ in (5),} \quad y = 3. \quad (6)$$

$$\text{Substituting 3 for } y \text{ in (1),} \quad x = 4.$$

$$14. \quad 7x - 2y = 6. \quad (1)$$

$$\text{Dividing by 2,} \quad 3x + \frac{x}{2} - y = 3. \quad (2)$$

$$\text{Collecting integral and fractional terms,} \quad \frac{x}{2} = y - 3x + 3. \quad (3)$$

Since $y - 3x + 3$ is integral, x is divisible by 2.

$$\text{Substituting 2 for } x \text{ in (1),} \quad y = 4.$$

Hence, the least integral values of x and y are $x = 2$, $y = 4$.

$$15. \quad 5x - 3y = 1. \quad (1)$$

$$\text{Dividing by 3,} \quad 2x - \frac{x}{3} - y = \frac{1}{3}. \quad (2)$$

Collecting integral and fractional terms, and changing signs,

$$\frac{x+1}{3} = 2x - y, \text{ an integer.} \quad (3)$$

$$\text{Put} \quad \frac{x+1}{3} = w, \text{ an integer.} \quad (4)$$

$$\text{Solving for } x, \quad x = 3w - 1. \quad (5)$$

The least integral value of x may be obtained by substituting the least positive integral value for w in (5).

$$\text{Substituting 1 for } w \text{ in (5),} \quad x = 2. \quad (6)$$

$$\text{Substituting 2 for } x \text{ in (1),} \quad y = 3.$$

$$17. \quad \begin{cases} x + y + z = 8, \\ x - y + 2z = 6. \end{cases} \quad (1)$$

$$2x + 3z = 14. \quad (2)$$

$$3z = 14 - 2x = 2(7 - x). \quad (3)$$

$$\text{By (4), } z \text{ is an even number not greater than 4.} \quad (4)$$

Substituting 2 for z in (3), $x = 4$; and substituting 4 for z , $x = 1$.

Substituting 4 for x and 2 for z in (1), $y = 2$; and substituting 1 for x and 4 for z in (1), $y = 3$. Hence, $x = 4$ or 1; $y = 2$ or 3; $z = 2$ or 4.

$$18. \quad \begin{cases} 2x + 3y + z = 15, & (1) \\ 3x + y - z = 8. & (2) \end{cases}$$

$$\text{Adding (1) and (2),} \quad 5x + 4y = 23. \quad (3)$$

$$\text{Solving (3) for positive integers,} \quad x = 3 \text{ and } y = 2. \quad (4)$$

$$\text{Substituting (4) in (2),} \quad z = 3.$$

$$19. \quad \begin{cases} 3x + 2y = 17, & (1) \\ y + 2z = 14. & (2) \end{cases}$$

From (2), y is divisible by 2. From (1), $y < 8$.

Of the values $y = 2, 4, 6$, only $y = 4$ will make x a positive integer in (1).

Substituting 4 for y in (1), $x = 3$. Substituting 4 for y in (2), $z = 5$.

Hence, $x = 3$, $y = 4$, and $z = 5$.

$$20. \quad \begin{cases} y + z = 7, & (1) \\ 3x - z = 7. & (2) \end{cases}$$

$$\text{Adding (1) and (2),} \quad 3x + y = 14. \quad (3)$$

$$\text{From (1),} \quad y < 7. \quad (4)$$

$$\text{Subtracting (4) from (3),} \quad 3x > 7. \quad (5)$$

$$\therefore x > 2\frac{1}{3}. \quad (6)$$

$$\text{From (3),} \quad y = 14 - 3x. \quad (7)$$

$$\text{From (6),} \quad x < 5. \quad (8)$$

$$\text{From (5) and (7),} \quad x = 3 \text{ or } 4. \quad (9)$$

$$\text{Substituting (8) in (6),} \quad y = 5 \text{ or } 2.$$

$$\text{Substituting (9) in (1),} \quad z = 2 \text{ or } 5.$$

21. Let $11x$ and $6y$ represent the parts of 100.

$$\text{Then,} \quad 11x + 6y = 100. \quad (1)$$

$$\text{Dividing (1) by 6,} \quad 2x - \frac{x}{6} + y = 16 + \frac{4}{6}.$$

Collecting integral and fractional terms,

$$2x + y - 16 = \frac{x + 4}{6}.$$

$$\text{Since } 2x + y - 16 \text{ is integral,} \quad \frac{x + 4}{6} = w, \text{ an integer.}$$

$$\text{Solving for } x, \quad x = 6w - 4.$$

$$\text{When } w = 1, 2, 3, \text{ etc.,} \quad x = 2, 8, 14, \text{ etc.}$$

$$\text{Since in (1), } x \text{ cannot be greater than 8,} \quad x = 2 \text{ or } 8,$$

$$\text{whence, substituting in (1),} \quad y = 13 \text{ or } 2.$$

Hence, the parts of 100 are 22 and 78 or 88 and 12.

22. See next page.

23. Let
and

$$\text{Then,} \quad \begin{aligned} x &= \text{number of cows,} \\ y &= \text{number of sheep.} \end{aligned}$$

$$\text{Dividing (1) by 6,} \quad 45x + 6y = 300. \quad (1)$$

$$\frac{15x}{2} + y = 50. \quad (2)$$

$$\text{From (1),} \quad x < 7.$$

$$\text{From (2),} \quad x \text{ is divisible by 2.}$$

Substituting 2, 4, and 6, successively, for x in (2), the corresponding values of y are

$y = 35, 20, 5$.
Hence, he can buy 2 cows and 35 sheep, or 4 cows and 20 sheep, or 6 cows and 5 sheep, for exactly \$300.

22. Let $5x$ and $2y$ be the parts of 19.

Then, $5x + 2y = 19$ (1)
 From (1), $x < 4$.

Trying 1, 2, and 3, for values of x , it is found that y has the integral values $y = 7$ when $x = 1$ and $y = 2$ when $x = 3$.

Hence, 19 pounds may be weighed with 1 5-pound weight and 7 2-pound weights; or with 3 5-pound weights and 2 2-pound weights.

24. Let x = number of cents each apple costs,
 and y = number of cents each orange costs.
 Then, $9x + 5y = 52$.
 Transposing $9x$, $5y = 52 - 9x$.
 When $9x = 9, 18, 27, 36, 45$,
 $5y = 43, 34, 25, 16, 7$.
 $\therefore x = 3$ and $y = 5$.

Hence, the apples cost 3 cents each and the oranges 5 cents each.

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25. Let x = number of pounds of 7-cent sugar,
 and y = number of pounds of 5-cent sugar.
 Then, $7x + 5y = 125$. (1)

By (1), $x < 18$, and since $x = \frac{5(25 - y)}{7}$, x is divisible by 5.

Substituting 5, 10 and 15, successively, for x in (1), the corresponding values of y are $y = 18, 11$, and 4.

Hence, the grocer may have sold 5 pounds of 7-cent sugar and 18 pounds of 5-cent sugar; or 10 pounds of 7-cent sugar and 11 pounds of 5-cent sugar; or 15 pounds of 7-cent sugar and 4 pounds of 5-cent sugar.

26. Let x = number of sheep,
 y = number of hogs,
 and z = number of cows.
 Then, $x + y + z = 9$, (1)
 and $3x + 6y + 35z = 100$. (2)
 Subtracting (1) $\times 3$ from (2), $3y + 32z = 73$. (3)
 From (3), $z < 3$.
 From (3), if $z = 1$, y is fractional; if $z = 2$, $y = 3$. (4)
 Substituting (4) in (1), $x = 4$.
 Hence, he sold 4 sheep, 3 hogs, and 2 cows.

27. Let x = number of yards at 5 cents a yard,
 y = number of yards at 7 cents a yard,
 and z = number of yards at 10 cents a yard.
 $x + y + z = 14$, (1)
 $5x + 7y + 10z = 93$. (2)
 Subtracting (1) $\times 5$ from (2), $2y + 5z = 23$. (3)
 Solving (3) in positive integers, $y = 9$ or 4, (4)
 and $z = 1$ or 3. (5)
 Substituting (4) and (5) in (1), $x = 4$ or 7.

28. Let $x, y,$ and z represent the three quotients.
 Then, $5x, 6y,$ and $7z$ represent the three parts of 74.
 $\therefore x + y + z = 12,$ (1)
 $5x + 6y + 7z = 74.$ (2)
 and Subtracting (1) $\times 5$ from (2), $y + 2z = 14.$ (3)
 Solving (3) in positive integers, $z = 1, 2, 3, 4, 5, 6,$
 and $y = 12, 10, 8, 6, 4, 2.$
 Since, from (1), $y + z < 12$, rejecting the first two values of z and y ,
 $z = 3, 4, 5, 6,$ (4)
 and $y = 8, 6, 4, 2.$ (5)
 Substituting (4) and (5) in (1), $x = 1, 2, 3, 4.$ (6)
 Forming the parts of 74 from the quotients in (4), (5) and (6), we have 5, 48, and 21; 10, 36, and 28; 15, 24, and 35; 20, 12, and 42.

29. Let $x =$ number of half-dollars,
 $y =$ number of quarters,
 $z =$ number of dimes.
 and $x + y + z = 30,$ (1)
 Then, $50x + 25y + 10z = 650.$ (2)
 and Subtracting (1) $\times 10$ from (2), $40x + 15y = 350.$ (3)
 Dividing by 5, $8x + 3y = 70.$ (4)
 Solving (3) in positive integers, $x = 2, 5, 8,$ (5)
 and $y = 18, 10, 2.$ (6)
 Substituting (4) and (5) in (1), $z = 10, 15, 20.$
 Hence, the purse may have contained 2 half-dollars, 18 quarters, and 10 dimes; or 5 half-dollars, 10 quarters, and 15 dimes; or 8 half-dollars, 2 quarters, and 20 dimes.

30. Let $x =$ number of pigs,
 $y =$ number of sheep,
 $z =$ number of ducks.
 and $x + y + z = 100,$ (1)
 Then, $6x + 4y + \frac{1}{2}z = 99.$ (2)
 and Subtracting (1) from (2) $\times 2$, $11x + 7y = 98.$ (3)
 Solving (3) in positive integers, $x = 7$ and $y = 3.$ (4)
 Substituting (4) in (1), $z = 90.$
 Hence, he bought 7 pigs, 3 sheep, and 90 ducks.

31. Let x and y represent the integral parts of the quotients when the number is divided by 25 and 33, respectively.

Then, the number $= 25x + 1 = 33y + 2.$

Dividing by 25, etc., $x = y + \frac{8y + 1}{25}.$

Put $\frac{8y + 1}{25} = w$, an integer.

Solving for y , $y = \frac{25w - 1}{8}.$

When $w = 1$, $y = \frac{25 - 1}{8} = 3.$

Hence, the number is $33 \times 3 + 2$, or 101.

32. See next page.

33. Let u, v, w, x, y , and z represent the integral parts of the quotients when the number is divided by 2, 3, 4, 5, 6, and 7, respectively.

Then, the number $= 2u + 1 = 3v + 1 = 4w + 1 = 5x + 1 = 6y + 1 = 7z$.
 $\therefore 2u = 3v = 4w = 5x = 6y = 7z - 1$.

Since $7z - 1 = 2u = 3v = 4w = 5x = 6y$, $7z - 1$ is a common multiple of 2, 3, 4, 5, and 6, and, therefore, is a multiple of 60.

Let $7z - 1 = 60m$.

Then, $z = \frac{60m + 1}{7}$.

Giving m the values 1, 2, 3, ... in succession, the least integral value of m corresponding to an integral value of z is $m = 5$, whence $z = 43$.

Hence, the number $= 7z = 7 \times 43 = 301$.

34. Let x and y represent the greatest integral number of times 4 eggs and 5 eggs, respectively, were contained in the whole number of eggs.

Then, the number $= 4x + 1 = 5y + 3$. (1)

Dividing by 4, etc., $x = y + \frac{y + 2}{4}$. (2)

Put $\frac{y + 2}{4} = w$, an integer. (3)

Solving for y , $y = 4w - 2$. (4)

Substituting (4) in (1), the number $= 20w - 7$. (5)

When $w = 1, 2, 3, 4, 5, \dots$, the number $= 13, 33, 53, 73, 93, \dots$.

Since the number of eggs was between 6 and 7 dozen, there were 73 eggs. Since the grocer paid for 78 eggs, he lost 5 eggs.

35. Let $x =$ whole number of marbles.

Then, $\frac{3}{4}(x - 1)$, or $\frac{3x - 3}{4} =$ number left by A,

$\frac{3}{4}\left(\frac{3x - 7}{4}\right)$, or $\frac{9x - 21}{16} =$ number left by B,

and $\frac{3}{4}\left(\frac{9x - 37}{16}\right)$, or $\frac{27x - 111}{64} =$ number left by C.

Let $y =$ number D takes.

Then, $4y + 1 =$ number left by C.

$\therefore \frac{27x - 111}{64} = 4y + 1$. (1)

Solving (1) for x , $x = 9y + 6 + \frac{13(y + 1)}{27}$. (2)

It is evident from (2) that the smallest integral value of y that will make x integral is obtained when $y + 1 = 27$.

$\therefore y = 26$. (3)

Substituting (3) in (2), $x = 253$. (4)

Therefore, A has $\frac{1}{4}$ of $252 + 26$, or 89; B has $\frac{1}{4}$ of $188 + 26$, or 73; C has $\frac{1}{4}$ of $140 + 26$, or 61; and D has 26.

32. Let x and y represent the integral parts of the quotients when the number is divided by 10 and 11, respectively.

Then, the number $= 10x + 3 = 11y + 6$.

Dividing by 10, etc., $x = y + \frac{y+3}{10}$.

Put $\frac{y+3}{10} = w$, an integer.

Solving for y , $y = 10w - 3$.

When $w = 1$, $y = 7$.

Hence the number is $11 \times 7 + 6$, or 83.

BINOMIAL THEOREM

2.

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$$\begin{aligned} \left(\frac{b}{2} + bx\right)^5 &= \frac{b^5}{32} (1 + 2x)^5 \\ &= \frac{b^5}{32} [1^5 + 5 \cdot 1^4 \cdot 2x + 10 \cdot 1^3 \cdot (2x)^2 + 10 \cdot 1^2 \cdot (2x)^3 + 5 \cdot 1 \cdot (2x)^4 + (2x)^5] \\ &= \frac{1}{32} b^5 + \frac{5}{16} b^5 x + \frac{5}{4} b^5 x^2 + \frac{5}{2} b^5 x^3 + \frac{5}{2} b^5 x^4 + b^5 x^5. \end{aligned}$$

$$3. (b-n)^7 = b^7 - 7b^6n + 21b^5n^2 - 35b^4n^3 + 35b^3n^4 - 21b^2n^5 + 7bn^6 - n^7.$$

$$\begin{aligned} 4. (1+a^{-1})^4 &= 1^4 + 4 \cdot 1^3 \cdot a^{-1} + 6 \cdot 1^2 \cdot (a^{-1})^2 + 4 \cdot 1 \cdot (a^{-1})^3 + (a^{-1})^4 \\ &= 1 + 4a^{-1} + 6a^{-2} + 4a^{-3} + a^{-4}. \end{aligned}$$

$$\begin{aligned} 5. (2-3x)^6 &= 2^6 - 6 \cdot 2^5 \cdot 3x + 15 \cdot 2^4 \cdot (3x)^2 - 20 \cdot 2^3 \cdot (3x)^3 + 15 \cdot 2^2 \cdot (3x)^4 \\ &\quad - 6 \cdot 2 \cdot (3x)^5 + (3x)^6 \\ &= 64 - 576x + 2160x^2 - 4320x^3 + 4860x^4 - 2916x^5 + 729x^6. \end{aligned}$$

$$\begin{aligned} 6. (x^2-x)^8 &= [x(x-1)]^8 = x^8(x-1)^8 \\ &= x^8(x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1) \\ &= x^{16} - 8x^{15} + 28x^{14} - 56x^{13} + 70x^{12} - 56x^{11} + 28x^{10} - 8x^9 + x^8. \end{aligned}$$

$$\begin{aligned} 7. (x+x^{-1})^6 &= [x^{-1}(x^2+1)]^6 = x^{-6}(x^2+1)^6 \\ &= x^{-6}(x^{12} + 6x^{10} + 15x^8 + 20x^6 + 15x^4 + 6x^2 + 1) \\ &= x^6 + 6x^4 + 15x^2 + 20 + 15x^{-2} + 6x^{-4} + x^{-6}. \end{aligned}$$

$$\begin{aligned} 8. (2a + \sqrt{x})^3 &= (2a)^3 + 3(2a)^2\sqrt{x} + 3(2a)(\sqrt{x})^2 + (\sqrt{x})^3 \\ &= 8a^3 + 12a^2\sqrt{x} + 6ax + x\sqrt{x}. \end{aligned}$$

$$\begin{aligned} 9. (a + a\sqrt{a})^4 &= [a(1 + \sqrt{a})]^4 = a^4(1 + \sqrt{a})^4 \\ &= a^4(1^4 + 4 \cdot 1^3 \cdot \sqrt{a} + 6 \cdot 1^2 \cdot (\sqrt{a})^2 + 4 \cdot 1 \cdot (\sqrt{a})^3 + (\sqrt{a})^4) \\ &= a^4(1 + 4\sqrt{a} + 6a + 4a\sqrt{a} + a^2) \\ &= a^4 + 4a^4\sqrt{a} + 6a^5 + 4a^5\sqrt{a} + a^6. \end{aligned}$$

10.

$$\begin{aligned} \left(1 + \frac{2}{x^2}\right)^5 &= 1^5 + 5 \cdot 1^4 \cdot \frac{2}{x^2} + 10 \cdot 1^3 \left(\frac{2}{x^2}\right)^2 + 10 \cdot 1^2 \left(\frac{2}{x^2}\right)^3 + 5 \cdot 1 \left(\frac{2}{x^2}\right)^4 + \left(\frac{2}{x^2}\right)^5 \\ &= 1 + \frac{10}{x^2} + \frac{40}{x^4} + \frac{80}{x^6} + \frac{80}{x^8} + \frac{32}{x^{10}}. \end{aligned}$$

11.

$$\begin{aligned} \left(\frac{a}{x} - \frac{x}{a}\right)^5 &= \left(\frac{a}{x}\right)^5 - 5 \left(\frac{a}{x}\right)^4 \frac{x}{a} + 10 \left(\frac{a}{x}\right)^3 \left(\frac{x}{a}\right)^2 - 10 \left(\frac{a}{x}\right)^2 \left(\frac{x}{a}\right)^3 + 5 \frac{a}{x} \left(\frac{x}{a}\right)^4 - \left(\frac{x}{a}\right)^5 \\ &= \frac{a^5}{x^5} - 5 \frac{a^3}{x^3} + 10 \frac{a}{x} - 10 \frac{x}{a} + 5 \frac{x^3}{a^3} - \frac{x^5}{a^5}. \end{aligned}$$

12.

$$\begin{aligned} \left(\frac{1}{x} - \frac{a}{y}\right)^3 &= \left(\frac{1}{x}\right)^3 - 3 \left(\frac{1}{x}\right)^2 \frac{a}{y} + 3 \frac{1}{x} \left(\frac{a}{y}\right)^2 - \left(\frac{a}{y}\right)^3 \\ &= \frac{1}{x^3} - 3 \frac{a}{x^2 y} + 3 \frac{a^2}{x y^2} - \frac{a^3}{y^3}. \end{aligned}$$

13.

$$\begin{aligned} (\sqrt[3]{a^2} + \sqrt[4]{b^3})^3 &= (\sqrt[3]{a^2})^3 + 3 (\sqrt[3]{a^2})^2 \sqrt[4]{b^3} + 3 \sqrt[3]{a^2} (\sqrt[4]{b^3})^2 + (\sqrt[4]{b^3})^3 \\ &= a^2 + 3 a \sqrt[3]{a} \sqrt[4]{b^3} + 3 \sqrt[3]{a^2} \sqrt[4]{b^3} + b^2 \sqrt[4]{b} \\ &= a^2 + 3 a^{1/3} \sqrt[4]{a^3 b^3} + 3 b \sqrt[6]{a^4 b^3} + b^2 \sqrt[4]{b}. \end{aligned}$$

14.

$$\begin{aligned} (2\sqrt{2} - \sqrt[3]{3})^6 &= (2^{\frac{3}{2}} - 3^{\frac{1}{3}})^6 \\ &= (2^{\frac{3}{2}})^6 - 6 (2^{\frac{3}{2}})^5 \cdot 3^{\frac{1}{3}} + 15 (2^{\frac{3}{2}})^4 (3^{\frac{1}{3}})^2 - 20 (2^{\frac{3}{2}})^3 (3^{\frac{1}{3}})^3 + 15 (2^{\frac{3}{2}})^2 (3^{\frac{1}{3}})^4 \\ &\quad - 6 \cdot 2^{\frac{3}{2}} (3^{\frac{1}{3}})^5 + (3^{\frac{1}{3}})^6 \\ &= 2^9 - 6 \cdot 2^7 \cdot 2^{\frac{5}{2}} \cdot 3^{\frac{1}{3}} + 15 \cdot 2^6 \cdot 3^{\frac{2}{3}} - 20 \cdot 2^4 \cdot 2^{\frac{3}{2}} \cdot 3 + 15 \cdot 2^3 \cdot 3 \cdot 3^{\frac{4}{3}} \\ &\quad - 6 \cdot 2 \cdot 2^{\frac{1}{2}} \cdot 3 \cdot 3^{\frac{5}{3}} + 3^2 \\ &= 512 - 768 \sqrt[6]{72} + 960 \sqrt[3]{9} - 960 \sqrt{2} + 360 \sqrt[3]{3} - 36 \sqrt[6]{648} + 9 \\ &= 521 - 768 \sqrt[6]{72} + 960 \sqrt[3]{9} - 960 \sqrt{2} + 360 \sqrt[3]{3} - 36 \sqrt[6]{648}. \end{aligned}$$

15.

$$\begin{aligned} \left(\sqrt{2} + \frac{1}{\sqrt{x}}\right)^3 &= (\sqrt{2})^3 + 3 (\sqrt{2})^2 \cdot \frac{1}{\sqrt{x}} + 3 \sqrt{2} \left(\frac{1}{\sqrt{x}}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^3 \\ &= 2\sqrt{2} + \frac{6}{\sqrt{x}} + \frac{3\sqrt{2}}{x} + \frac{1}{x\sqrt{x}}. \end{aligned}$$

16.

$$\begin{aligned} (x^{\frac{n-1}{n}} - x^{\frac{1}{n}})^4 &= (x^{\frac{n-1}{n}})^4 - 4 (x^{\frac{n-1}{n}})^3 \cdot x^{\frac{1}{n}} + 6 (x^{\frac{n-1}{n}})^2 (x^{\frac{1}{n}})^2 \\ &\quad - 4 x^{\frac{n-1}{n}} (x^{\frac{1}{n}})^3 + (x^{\frac{1}{n}})^4 \\ &= x^{\frac{4n-4}{n}} - 4 x^{\frac{3n-2}{n}} + 6 x^2 - 4 x^{\frac{n+2}{n}} + x^{\frac{4}{n}}. \end{aligned}$$

17.

$$\begin{aligned} (ax^{-2} - b\sqrt{x})^6 &= (ax^{-2})^6 - 6 (ax^{-2})^5 b\sqrt{x} + 15 (ax^{-2})^4 (b\sqrt{x})^2 \\ &\quad - 20 (ax^{-2})^3 (b\sqrt{x})^3 + 15 (ax^{-2})^2 (b\sqrt{x})^4 - 6 ax^{-2} (b\sqrt{x})^5 + (b\sqrt{x})^6 \\ &= a^6 x^{-12} - 6 a^5 b x^{-\frac{13}{2}} + 15 a^4 b^2 x^{-7} - 20 a^3 b^3 x^{-\frac{9}{2}} + 15 a^2 b^4 x^{-2} \\ &\quad - 6 a b^5 x^{\frac{1}{2}} + b^6 x^3. \end{aligned}$$

$$\begin{aligned}
 18. \quad & \left(\frac{\sqrt[3]{a}}{\sqrt[3]{b}} - \frac{\sqrt[3]{b}}{\sqrt[3]{a^3}} \right)^6 = \left(\frac{a^2 - b^{\frac{5}{3}}}{a^6 \sqrt[3]{a^2 b^2}} \right)^6 = \frac{1}{a^6 \cdot a^2 b^2} (a^2 - b^{\frac{5}{3}})^6 \\
 & = \frac{1}{a^8 b^2} [(a^2)^6 - 6(a^2)^5 b^{\frac{5}{3}} + 15(a^2)^4 (b^{\frac{5}{3}})^2 - 20(a^2)^3 (b^{\frac{5}{3}})^3 + 15(a^2)^2 (b^{\frac{5}{3}})^4 \\
 & \quad - 6a^2 (b^{\frac{5}{3}})^5 + (b^{\frac{5}{3}})^6] \\
 & = \frac{1}{a^8 b^2} (a^{12} - 6a^{10} b^{\frac{5}{3}} + 15a^8 b^{\frac{10}{3}} - 20a^6 b^{\frac{15}{3}} + 15a^4 b^{\frac{20}{3}} - 6a^2 b^{\frac{25}{3}} + b^5) \\
 & = \frac{a^8}{b^2} - \frac{6a^6}{b^2} \sqrt[3]{b^5} + \frac{15a^5}{ab} \sqrt[3]{b^2} - \frac{20}{a^3} \sqrt[3]{b} + \frac{15b}{a^5} \sqrt[3]{b} - \frac{6b^2}{a^7} \sqrt[3]{b} + \frac{b^3}{a^9}.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & \left(\frac{x}{y} \sqrt{\frac{x}{y}} + \frac{2}{3} \sqrt{\frac{2}{3}} \right)^3 = \left[\left(\frac{x}{y} \right)^{\frac{3}{2}} + \left(\frac{2}{3} \right)^{\frac{3}{2}} \right]^3 \\
 & = \left(\frac{x}{y} \right)^{\frac{9}{2}} + 3 \left(\frac{x}{y} \right)^3 \left(\frac{2}{3} \right)^{\frac{3}{2}} + 3 \cdot \left(\frac{x}{y} \right)^{\frac{3}{2}} \left(\frac{2}{3} \right)^3 + \left(\frac{2}{3} \right)^{\frac{9}{2}} \\
 & = \frac{x^4}{y^4} \sqrt{\frac{x}{y}} + 3 \cdot \frac{x^3}{y^3} \cdot \frac{2}{3} \sqrt{\frac{2}{3}} + 3 \cdot \frac{x}{y} \sqrt{\frac{x}{y}} \cdot \frac{8}{27} + \left(\frac{2}{3} \right)^4 \sqrt{\frac{2}{3}} \\
 & = \frac{x^4 \sqrt{xy}}{y^5} + \frac{2x^3}{3y^3} \sqrt{6} + \frac{8x \sqrt{xy}}{9y^2} + \frac{16 \sqrt{6}}{243}.
 \end{aligned}$$

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3. 4th term = $\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} a^{10-3} 2^3 = 960 a^7.$
4. 4th term = $\frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} x^{12-3} (-3y)^3 = -5940 x^9 y^3.$
5. 8th term = coef. 4th term times $x^{10-7} y^7$
 $= \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} x^3 y^7 = 120 x^3 y^7.$
6. 5th term = $\frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} x^{12-4} (-2y)^4 = 7920 x^8 y^4.$
7. 3d term = $\frac{4 \cdot 3}{1 \cdot 2} (a^2)^{4-2} (-a^{-2})^2 = 6 a^4 a^{-4} = 6 a^0 = 6.$
8. 20th term = coef. 6th term times $1^{24-19} x^{19}$
 $= \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^{19} = 42504 x^{19}.$
9. 16th term = coef. 6th term times $1^{20-15} (-2x)^{15}$
 $= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (-2)^{15} x^{15} = -508035072 x^{15}.$
10. Since there are $6 + 1$ terms, the middle term is the $(3 + 1)$ th.
 4th term = $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} a^{6-3} (3b)^3 = 540 a^3 b^3.$

$$11. \text{ 6th term} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^{10-5} \left(\frac{1}{x}\right)^5 = 252 x^0 = 252.$$

12. Since there are $10 + 1$ terms, the middle term is the $(5 + 1)$ th.

$$\text{6th term} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \left(\frac{x}{y}\right)^{10-5} \left(-\frac{y}{x}\right)^5 = -252 x^0 y^0 = -252.$$

13. Since there are $9 + 1$ terms, the middle terms are the 5th and 6th. They have numerically equal coefficients.

$$\text{5th term} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{a}{b}\right)^{9-4} \left(-\frac{b}{a}\right)^4 = 126 \frac{a}{b}.$$

$$\text{6th term} = 126 \left(\frac{a}{b}\right)^{9-5} \left(-\frac{b}{a}\right)^5 = -126 \frac{b}{a}.$$

$$14. (a^3 + a)^5 = a^5 (a^2 + 1)^5.$$

The term sought corresponds to that term of the expansion of $(a^2 + 1)^5$ which contains a^4 , or $(a^2)^2$. This term is the third from the last, or from the 6th term. Hence, the term sought is the 4th.

$$\text{Coef. 4th term} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10.$$

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$$\begin{aligned} 2. (a+x)^{\frac{1}{2}} &= a^{\frac{1}{2}} + \frac{1}{2} a^{-\frac{1}{2}} x + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2} a^{-\frac{3}{2}} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2 \cdot 3} a^{-\frac{5}{2}} x^3 \\ &\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1 \cdot 2 \cdot 3 \cdot 4} a^{-\frac{7}{2}} x^4 \end{aligned}$$

$$= a^{\frac{1}{2}} + \frac{1}{2} a^{-\frac{1}{2}} x - \frac{1}{8} a^{-\frac{3}{2}} x^2 + \frac{1}{16} a^{-\frac{5}{2}} x^3 - \frac{5}{128} a^{-\frac{7}{2}} x^4.$$

$$\begin{aligned} \text{10th term} &= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})(-\frac{11}{2})(-\frac{13}{2})(-\frac{15}{2})}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} a^{\frac{1}{2}-9} x^9 \\ &= \frac{715}{65536} a^{-\frac{17}{2}} x^9. \end{aligned}$$

$$\begin{aligned} 3. (a+b)^{\frac{2}{3}} &= a^{\frac{2}{3}} + \frac{2}{3} a^{-\frac{1}{3}} b + \frac{\frac{2}{3}(-\frac{1}{3})}{1 \cdot 2} a^{-\frac{4}{3}} b^2 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{1 \cdot 2 \cdot 3} a^{-\frac{5}{3}} b^3 \\ &= a^{\frac{2}{3}} + \frac{2}{3} a^{-\frac{1}{3}} b - \frac{1}{9} a^{-\frac{4}{3}} b^2 + \frac{4}{81} a^{-\frac{5}{3}} b^3. \end{aligned}$$

4.

$$\begin{aligned} (a+b)^{-\frac{1}{2}} &= a^{-\frac{1}{2}} + (-\frac{1}{2}) a^{-\frac{3}{2}} b + \frac{-\frac{1}{2}(-\frac{3}{2})}{1 \cdot 2} a^{-\frac{5}{2}} b^2 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{1 \cdot 2 \cdot 3} a^{-\frac{7}{2}} b^3 \\ &= a^{-\frac{1}{2}} - \frac{1}{2} a^{-\frac{3}{2}} b + \frac{3}{8} a^{-\frac{5}{2}} b^2 - \frac{5}{16} a^{-\frac{7}{2}} b^3. \end{aligned}$$

$$\begin{aligned} 5. (a-b)^{\frac{1}{2}} &= a^{\frac{1}{2}} - \frac{1}{2} a^{-\frac{1}{2}} b + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2} a^{-\frac{3}{2}} b^2 - \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2 \cdot 3} a^{-\frac{5}{2}} b^3 \\ &= a^{\frac{1}{2}} - \frac{1}{2} a^{-\frac{1}{2}} b - \frac{1}{8} a^{-\frac{3}{2}} b^2 - \frac{1}{16} a^{-\frac{5}{2}} b^3. \end{aligned}$$

$$\begin{aligned}
 6. \quad \sqrt[4]{(a-b)^3} &= (a-b)^{\frac{3}{4}} \\
 &= a^{\frac{3}{4}} - \frac{3}{4} a^{-\frac{1}{4}} b + \frac{\frac{3}{4}(-\frac{1}{4})}{1 \cdot 2} a^{-\frac{5}{4}} b^2 - \frac{\frac{3}{4}(-\frac{1}{4})(-\frac{5}{4})}{1 \cdot 2 \cdot 3} a^{-\frac{7}{4}} b^3 \\
 &= a^{\frac{3}{4}} - \frac{3}{4} a^{-\frac{1}{4}} b - \frac{3}{8} a^{-\frac{5}{4}} b^2 + \frac{5}{128} a^{-\frac{7}{4}} b^3.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \frac{1}{\sqrt[4]{(a-b)^3}} &= (a-b)^{-\frac{3}{4}} \\
 &= a^{-\frac{3}{4}} - (-\frac{3}{4}) a^{-\frac{7}{4}} b + \frac{(-\frac{3}{4})(-\frac{7}{4})}{1 \cdot 2} a^{-\frac{11}{4}} b^2 - \frac{(-\frac{3}{4})(-\frac{7}{4})(-\frac{11}{4})}{1 \cdot 2 \cdot 3} a^{-\frac{15}{4}} b^3 \\
 &= a^{-\frac{3}{4}} + \frac{3}{4} a^{-\frac{7}{4}} b + \frac{21}{32} a^{-\frac{11}{4}} b^2 - \frac{77}{128} a^{-\frac{15}{4}} b^3.
 \end{aligned}$$

8.

$$\begin{aligned}
 \sqrt{4+x} &= (4+x)^{\frac{1}{2}} = 4^{\frac{1}{2}} + \frac{1}{2} \cdot 4^{-\frac{1}{2}} x + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2} 4^{-\frac{3}{2}} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2 \cdot 3} 4^{-\frac{5}{2}} x^3 \\
 &= 2 + \frac{1}{4} x - \frac{1}{64} x^2 + \frac{3}{512} x^3.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \sqrt{(9-x)^3} &= (9-x)^{\frac{3}{2}} = 9^{\frac{3}{2}} - \frac{3}{2} \cdot 9^{\frac{1}{2}} x + \frac{\frac{3}{2} \cdot \frac{1}{2}}{1 \cdot 2} 9^{-\frac{1}{2}} x^2 - \frac{\frac{3}{2} \cdot \frac{1}{2}(-\frac{1}{2})}{1 \cdot 2 \cdot 3} 9^{-\frac{3}{2}} x^3 \\
 &= 27 - \frac{9}{2} x + \frac{1}{8} x^2 - \frac{1}{432} x^3.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad (a^{\frac{2}{3}} - x^{-1})^{\frac{3}{2}} &= (a^{\frac{2}{3}})^{\frac{3}{2}} - \frac{3}{2} (a^{\frac{2}{3}})^{\frac{1}{2}} x^{-1} + \frac{\frac{3}{2} \cdot \frac{1}{2}}{1 \cdot 2} (a^{\frac{2}{3}})^{-\frac{1}{2}} (x^{-1})^2 \\
 &\quad - \frac{\frac{3}{2} \cdot \frac{1}{2}(-\frac{1}{2})}{1 \cdot 2 \cdot 3} (a^{\frac{2}{3}})^{-\frac{3}{2}} (x^{-1})^3 \\
 &= a - \frac{3}{2} a^{\frac{1}{3}} x^{-1} + \frac{3}{8} a^{-\frac{1}{3}} x^{-2} + \frac{1}{16} a^{-1} x^{-3}.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad (a^{\frac{1}{2}} - x^{\frac{1}{3}})^{-6} &= (a^{\frac{1}{2}})^{-6} - (-6) (a^{\frac{1}{2}})^{-7} x^{\frac{1}{3}} + \frac{(-6)(-7)}{1 \cdot 2} (a^{\frac{1}{2}})^{-8} (x^{\frac{1}{3}})^2 \\
 &\quad - \frac{(-6)(-7)(-8)}{1 \cdot 2 \cdot 3} (a^{\frac{1}{2}})^{-9} (x^{\frac{1}{3}})^3 \\
 &= a^{-3} + 6 a^{-\frac{7}{2}} x^{\frac{1}{3}} + 21 a^{-4} x^{\frac{2}{3}} + 56 a^{-\frac{9}{2}} x.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \left(\frac{1}{\sqrt{a} - \sqrt[3]{x}} \right)^3 &= (\sqrt{a} - \sqrt[3]{x})^{-3} \\
 &= (\sqrt{a})^{-3} - (-3) (\sqrt{a})^{-4} \sqrt[3]{x} + \frac{(-3)(-4)}{1 \cdot 2} (\sqrt{a})^{-5} (\sqrt[3]{x})^2 \\
 &\quad - \frac{(-3)(-4)(-5)}{1 \cdot 2 \cdot 3} (\sqrt{a})^{-6} (\sqrt[3]{x})^3 \\
 &= a^{-\frac{3}{2}} + 3 a^{-2} x^{\frac{1}{3}} + 6 a^{-\frac{5}{2}} x^{\frac{2}{3}} + 10 a^{-3} x.
 \end{aligned}$$

$$\begin{aligned}
 13. \quad (1+x)^{\frac{2}{3}} &= 1^{\frac{2}{3}} + \frac{2}{3} \cdot 1^{-\frac{1}{3}} x + \frac{\frac{2}{3}(-\frac{1}{3})}{1 \cdot 2} 1^{-\frac{4}{3}} x^2 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{1 \cdot 2 \cdot 3} 1^{-\frac{7}{3}} x^3 \\
 &= 1 + \frac{2}{3} x - \frac{1}{9} x^2 + \frac{8}{27} x^3.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad (1+a)^{-1} &= 1^{-1} + (-1) 1^{-2} a + \frac{(-1)(-2)}{1 \cdot 2} 1^{-3} a^2 \\
 &\quad + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} 1^{-4} a^3 \\
 &= 1 - a + a^2 - a^3.
 \end{aligned}$$

15.
- $$\begin{aligned}(1-a)^{-1} &= 1^{-1} - (-1) 1^{-2} a + \frac{(-1)(-2)}{1 \cdot 2} 1^{-3} a^2 - \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} 1^{-4} a^3 \\ &= 1 + a + a^2 + a^3.\end{aligned}$$
16.
- $$\begin{aligned}(1-x)^{-2} &= 1^{-2} - (-2) 1^{-3} x + \frac{(-2)(-3)}{1 \cdot 2} 1^{-4} x^2 - \frac{(-2)(-3)(-4)}{1 \cdot 2 \cdot 3} 1^{-5} x^3 \\ &= 1 + 2x + 3x^2 + 4x^3.\end{aligned}$$
17.
- $$\begin{aligned}(1-x)^{-3} &= 1^{-3} - (-3) 1^{-4} x + \frac{(-3)(-4)}{1 \cdot 2} 1^{-5} x^2 - \frac{(-3)(-4)(-5)}{1 \cdot 2 \cdot 3} 1^{-6} x^3 \\ &= 1 + 3x + 6x^2 + 10x^3.\end{aligned}$$
18.
- $$\begin{aligned}(r+1)\text{th term} &= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-r+1)}{1 \cdot 2 \cdot 3 \cdots r} a^{\frac{1}{2}-r} x^r \\ &= \frac{1(-1)(-3)\cdots(3-2r)}{2 \cdot 2 \cdot 2 \cdots r \text{ times}} a^{\frac{1}{2}-r} x^r \\ &= \frac{(-1)(-3)\cdots(3-2r)}{2^r \cdot 1 \cdot 2 \cdot 3 \cdots r} a^{\frac{1}{2}-r} x^r.\end{aligned}$$
19.
- $$\begin{aligned}(1-x-x^2)^{-1} &= (\overline{1-x-x^2})^{-1} \\ &= (1-x)^{-1} - (-1)(1-x)^{-2} x^2 + \frac{(-1)(-2)}{1 \cdot 2} (1-x)^{-3} (x^2)^2 \\ &\quad - \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} (1-x)^{-4} (x^2)^3 + \cdots \\ &= (1-x)^{-1} + x^2(1-x)^{-2} + x^4(1-x)^{-3} + x^6(1-x)^{-4} + \cdots \\ \text{Ex. 15,} &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots \\ \text{Ex. 16,} &\quad + x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + \cdots \\ \text{Ex. 17,} &\quad + x^4 + 3x^5 + 6x^6 + 10x^7 + \cdots \\ &\quad + x^6 + 4x^7 + \cdots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots\end{aligned}$$
21.
- $$\begin{aligned}\sqrt{5} &= \sqrt{4+1} = \sqrt{4}\sqrt{1+\frac{1}{4}} = 2(1+\frac{1}{4})^{\frac{1}{2}} \\ &= 2[1^{\frac{1}{2}} + \frac{1}{2} \cdot 1^{-\frac{1}{2}} \cdot \frac{1}{4} + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2} 1^{-\frac{3}{2}} (\frac{1}{4})^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2 \cdot 3} 1^{-\frac{5}{2}} (\frac{1}{4})^3 + \cdots] \\ &= 2[1 + \frac{1}{8} - \frac{1}{128} + \frac{1}{1024} - \cdots] = \frac{1145}{512} = 2.236.\end{aligned}$$
22.
- $$\begin{aligned}\sqrt{17} &= \sqrt{16+1} = \sqrt{16}\sqrt{1+\frac{1}{16}} = 4(1+\frac{1}{16})^{\frac{1}{2}} \\ &= 4[1^{\frac{1}{2}} + \frac{1}{2} \cdot 1^{-\frac{1}{2}} \cdot \frac{1}{16} + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \cdot 2} 1^{-\frac{3}{2}} (\frac{1}{16})^2 \\ &\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2 \cdot 3} 1^{-\frac{5}{2}} (\frac{1}{16})^3 + \cdots] \\ &= 4[1 + \frac{1}{32} - \frac{1}{2048} + \frac{1}{65536} - \cdots] = 4 + \frac{2017}{16384} = 4.123.\end{aligned}$$

$$\begin{aligned}
 23. \quad \sqrt[3]{26} &= \sqrt[3]{25+1} = \sqrt[3]{25} \sqrt[3]{1+\frac{1}{25}} = 5 \left(1 + \frac{1}{25}\right)^{\frac{1}{3}} \\
 &= 5 \left[1^{\frac{1}{3}} + \frac{1}{3} \cdot 1^{-\frac{2}{3}} \cdot \frac{1}{25} + \frac{\frac{1}{3} \left(-\frac{2}{3}\right)}{1 \cdot 2} 1^{-\frac{5}{3}} \left(\frac{1}{25}\right)^2 \right. \\
 &\quad \left. + \frac{\frac{1}{3} \left(-\frac{2}{3}\right) \left(-\frac{2}{3}\right)}{1 \cdot 2 \cdot 3} 1^{-\frac{8}{3}} \left(\frac{1}{25}\right)^3 + \dots \right] \\
 &= 5 + .1 \dots .001 + .00002 = 5.099.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \sqrt[3]{25} &= \sqrt[3]{27-2} = \sqrt[3]{27} \sqrt[3]{1-\frac{2}{27}} = 3 \left(1 - \frac{2}{27}\right)^{\frac{1}{3}} \\
 &= 3 \left[1^{\frac{1}{3}} - \frac{1}{3} \cdot 1^{-\frac{2}{3}} \cdot \frac{2}{27} + \frac{\frac{1}{3} \left(-\frac{2}{3}\right)}{1 \cdot 2} 1^{-\frac{5}{3}} \left(\frac{2}{27}\right)^2 - \frac{\frac{1}{3} \left(-\frac{2}{3}\right) \left(-\frac{5}{3}\right)}{1 \cdot 2 \cdot 3} 1^{-\frac{8}{3}} \left(\frac{2}{27}\right)^3 + \dots \right] \\
 &= 3 - \frac{2}{27} - \frac{4}{3 \cdot 27^2} - \frac{40}{27^4} - \dots \\
 &= 3 - \frac{40378}{581441} = 3 - .0759 = 2.924.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \sqrt[3]{9} &= \sqrt[3]{8+1} = \sqrt[3]{8} \sqrt[3]{1+\frac{1}{8}} = 2 \left(1 + \frac{1}{8}\right)^{\frac{1}{3}} \\
 &= 2 \left[1^{\frac{1}{3}} + \frac{1}{3} \cdot 1^{-\frac{2}{3}} \cdot \frac{1}{8} + \frac{\frac{1}{3} \left(-\frac{2}{3}\right)}{1 \cdot 2} 1^{-\frac{5}{3}} \left(\frac{1}{8}\right)^2 + \frac{\frac{1}{3} \left(-\frac{2}{3}\right) \left(-\frac{5}{3}\right)}{1 \cdot 2 \cdot 3} 1^{-\frac{8}{3}} \left(\frac{1}{8}\right)^3 + \dots \right] \\
 &= 2 \left[1 + \frac{1}{3 \cdot 8} - \frac{1}{9 \cdot 8^2} + \frac{5}{81 \cdot 8^3} - \dots \right] \\
 &= 2 + \frac{3322}{92 \cdot 8^3} = 2.080.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \sqrt[5]{30} &= \sqrt[5]{32-2} = \sqrt[5]{32} \sqrt[5]{1-\frac{1}{16}} = 2 \left(1 - \frac{1}{16}\right)^{\frac{1}{5}} \\
 &= 2 \left[1^{\frac{1}{5}} - \frac{1}{5} \cdot 1^{-\frac{4}{5}} \cdot \frac{1}{16} + \frac{\frac{1}{5} \left(-\frac{4}{5}\right)}{1 \cdot 2} 1^{-\frac{9}{5}} \left(\frac{1}{16}\right)^2 - \dots \right] \\
 &= 2 \left[1 - \frac{1}{80} - \frac{1}{3200} - \dots \right] = 2 - .0256 = 1.974.
 \end{aligned}$$

LOGARITHMS

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| | | | | | |
|-----|------------------------|-----------------|-----|------------------------|----------|
| 16. | log 1050 | = 3.0212 | 19. | log 1900 | = 3.2788 |
| | log 1060 | = 3.0253 | | log 1910 | = 3.2810 |
| | Tab. Diff. | = 41 | | Tab. Diff. | = 22 |
| | | .4 | | | .6 |
| | Tab. Diff. \times .4 | = 164 | | Tab. Diff. \times .6 | = 132 |
| | log 1050 | = 3.0212 | | log 1900 | = 3.2788 |
| | log 1054 | = 3.0228 | | log 1906 | = 3.2801 |
| 17. | log 1270 | = 3.1038 | 20. | log 21.00 | = 1.3222 |
| | log 1280 | = 3.1072 | | log 21.10 | = 1.3243 |
| | Tab. Diff. | = 34 | | Tab. Diff. | = 21 |
| | | .2 | | | .9 |
| | Tab. Diff. \times 2 | = 68 | | Tab. Diff. \times .9 | = 189 |
| | log 1270 | = 3.1038 | | log 21.00 | = 1.3222 |
| | log 1272 | = 3.1045 | | log 21.09 | = 1.3241 |
| 18. | | log .0165 | | = 2.2175 | |
| | | Adding 10 - 10, | | = 8.2175 - 10 | |

| | | | |
|------------------------|------------------------------------|-------------------------|------------------------------------|
| 22. log 441.0 | = 2.6444 | 25. log .10100 | = $\bar{1}.0043$ |
| log 442.0 | = <u>2.6454</u> | log .10200 | = <u>$\bar{1}.0086$</u> |
| Tab. Diff. | = <u>10</u> | Tab. Diff. | = <u>43</u> |
| | <u>.1</u> | | <u>.25</u> |
| Tab. Diff. $\times .1$ | = <u>1</u> | Tab. Diff. $\times .25$ | = <u>1075</u> |
| log 441.0 | = 2.6444 | log .10100 | = <u>$\bar{1}.0043$</u> |
| log 441.1 | = 2.6445 | log .10125 | = <u>$\bar{1}.0054$</u> |
| | | Adding 10 - 10, | = <u>9.0054 - 10</u> |
| 23. log .7850 | = $\bar{1}.8949$ | 26. log 54.600 | = 1.7372 |
| log .7860 | = <u>$\bar{1}.8954$</u> | log 54.700 | = <u>1.7380</u> |
| Tab. Diff. | = <u>5</u> | Tab. Diff. | = <u>8</u> |
| | <u>.4</u> | | <u>.75</u> |
| Tab. Diff. $\times .4$ | = <u>2</u> | Tab. Diff. $\times .75$ | = <u>6</u> |
| log .7850 | = <u>$\bar{1}.8949$</u> | log 54.600 | = <u>1.7372</u> |
| log .7854 | = <u>$\bar{1}.8951$</u> | log 54.675 | = <u>1.7378</u> |
| Adding 10 - 10, | = <u>9.8951 - 10</u> | | |
| 24. log .09090 | = $\bar{2}.9586$ | 27. log .09880 | = $\bar{2}.9948$ |
| log .09100 | = <u>$\bar{2}.9590$</u> | log .09890 | = <u>$\bar{2}.9952$</u> |
| Tab. Diff. | = <u>4</u> | Tab. Diff. | = <u>4</u> |
| | <u>.5</u> | | <u>.5</u> |
| Tab. Diff. $\times .5$ | = <u>2</u> | Tab. Diff. $\times .5$ | = <u>2</u> |
| log .09090 | = <u>$\bar{2}.9586$</u> | log .09880 | = <u>$\bar{2}.9948$</u> |
| log .09095 | = <u>$\bar{2}.9588$</u> | log .09885 | = <u>$\bar{2}.9950$</u> |
| Adding 10 - 10, | = <u>8.9588 - 10</u> | Adding 10 - 10, | = <u>8.9950 - 10</u> |

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6. Given mantissa, .6669.
 Mantissa next less, .6665; figures corresponding, 464.
 Difference, 4
 Tabular difference, 10 | 4 | .4
 Figures corresponding to given mantissa, 4644.
 Hence, the number corresponding to 1.6669 is 46.44.
7. Given mantissa, .7971.
 Mantissa next less, .7966; figures corresponding, 626.
 Difference, 5
 Tabular difference, 7 | 5 | .7
 Figures corresponding to given mantissa, 6267.
 Hence, the number corresponding to 2.7971 is 626.7.
8. Given mantissa, .9545.
 Mantissa next less, .9542; figures corresponding, 900.
 Difference, 3
 Tabular difference, 5 | 3 | .6
 Figures corresponding to given mantissa, 9006.
 Hence, the number corresponding to 3.9545 is 9006.

9. Given mantissa, .8794.
 Mantissa next less, .8791; figures corresponding, 757.
 Difference, 3
 Tabular difference, $\begin{array}{r} 6 \mid 3 \mid .5 \end{array}$
 Figures corresponding to given mantissa, 7575.
 Hence, the number corresponding to 0.8794 is 7.575.

10. Given mantissa, .9371.
 Mantissa next less, .9370; figures corresponding, 865.
 Difference, 1
 Tabular difference, $\begin{array}{r} 5 \mid 1 \mid .2 \end{array}$
 Figures corresponding to given mantissa, 8652.
 Hence, the number corresponding to 2.9371 is 865.2.

11. Given mantissa, .8294.
 Mantissa next less, .8293; figures corresponding, 675.
 Difference, 1
 Tabular difference, $\begin{array}{r} 6 \mid 1 \mid .2 \end{array}$
 Figures corresponding to given mantissa, 6752.
 Hence, the number corresponding to 0.8294 is 6.752.

12. Given mantissa, .9039.
 Mantissa next less, .9036; figures corresponding, 801.
 Difference, 3
 Tabular difference, $\begin{array}{r} 6 \mid 3 \mid .5 \end{array}$
 Figures corresponding to given mantissa, 8015.
 Hence, the number corresponding to 1.9039 is 80.15.

13. Given mantissa, .3685.
 Mantissa next less, .3674; figures corresponding, 233.
 Difference, 11
 Tabular difference, $\begin{array}{r} 18 \mid 11 \mid .6 \end{array}$
 Figures corresponding to given mantissa, 2336.
 Since the characteristic is 9 - 10, or - 1, the number corresponding to the logarithm 9.3685 - 10 is .2336.

14. Given mantissa, .9932.
 Mantissa next less, .9930; figures corresponding, 984.
 Difference, 2
 Tabular difference, $\begin{array}{r} 4 \mid 2 \mid .5 \end{array}$
 Figures corresponding to given mantissa, 9845.
 Since the characteristic is 8 - 10, or - 2, the number corresponding to the logarithm 8.9932 - 10 is .09845.

15. Given mantissa, .9535.
 Mantissa next less, .9533; figures corresponding, 898.
 Difference, 2
 Tabular difference, $\begin{array}{r} 5 \mid 2 \mid .4 \end{array}$
 Figures corresponding to given mantissa, 8984.
 Since the characteristic is 8 - 10, or - 2, the number corresponding to the logarithm 8.9535 - 10 is .08984.

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2. $\log 3.8 = 0.5798$
 $\log 56 = 1.7482$
 $\log \text{ of product} = 2.3280$
 $2.3280 = \log 212.8$
 $\therefore 3.8 \times 56 = 212.8$
4. $\log 8.5 = 0.9294$
 $\log 6.2 = 0.7924$
 $\log \text{ of product} = 1.7218$
 $1.7218 = \log 52.70$
 $\therefore 8.5 \times 6.2 = 52.7$
6. $\log 2.26 = 0.3541$
 $\log 85 = 1.9294$
 $\log \text{ of product} = 2.2835$
 $2.2835 = \log 192.1$
 $\therefore 2.26 \times 85 = 192.1$
8. $\log 3272 = 3.5148$
 $\log 75 = 1.8751$
 $\log \text{ of product} = 5.3899$
 $5.3899 = \log 245400$
 $\therefore 3272 \times 75 = 245400$
10. $\log 1.414 = 0.1504$
 $\log 2.829 = 0.4516$
 $\log \text{ of product} = 0.6020$
 $0.6020 = \log 3.999$
 $\therefore 1.414 \times 2.829 = 3.999$
12. $\log 2912 = 3.4642$
 $\log .7281 = 9.8622-10$
 $\log \text{ of product} = 13.3264-10$
 $= 3.3264$
 $3.3264 = \log 2120$
 $\therefore 2912 \times .7281 = 2120$

3. $\log 72 = 1.8573$
 $\log 39 = 1.5911$
 $\log \text{ of product} = 3.4484$
 $3.4484 = \log 2808$
 $\therefore 72 \times 39 = 2808$
5. $\log 1.64 = 0.2148$
 $\log 35 = 1.5441$
 $\log \text{ of product} = 1.7589$
 $1.7589 = \log 57.40$
 $\therefore 1.64 \times 35 = 57.4$
7. $\log 7.25 = 0.8603$
 $\log 240 = 2.3802$
 $\log \text{ of product} = 3.2405$
 $3.2405 = \log 1740$
 $\therefore 7.25 \times 240 = 1740$
9. $\log .892 = 9.9504-10$
 $\log .805 = 9.9058-10$
 $\log \text{ of product} = 19.8562-20$
 $= 1.8562$
 $1.8562 = \log .7182$
 $\therefore .892 \times .805 = .7182$
11. $\log 42.37 = 1.6271$
 $\log .236 = 9.3729-10$
 $\log \text{ of product} = 11.0000-10$
 $= 1.0000$
 $1.0000 = \log 10.00$
 $\therefore 42.37 \times .236 = 10$
13. $\log 289 = 2.4609$
 $\log .7854 = 9.8951-10$
 $\log \text{ of product} = 12.3560-10$
 $= 2.3560$
 $2.3560 = \log 227.0$
 $\therefore 289 \times .7854 = 227$

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3. $\log 3025 = 3.4807$
 $\log 55 = 1.7404$
 $\log \text{ of quotient} = 1.7403$
 $1.7403 = \log 54.99$
 $\therefore 3025 \div 55 = 54.99$
4. $\log 4096 = 3.6124$
 $\log 32 = 1.5051$
 $\log \text{ of quotient} = 2.1073$
 $2.1073 = \log 128.0$
 $\therefore 4096 \div 32 = 128$

5. $\log 3249 = 3.5118$
 $\log 57 = 1.7559$
 $\log \text{ of quotient} = 1.7559$
 $1.7559 = \log 57.00$
 $\therefore 3249 \div 57 = 57$
6. $\log .2601 = 19.4152-20$
 $\log .68 = 9.8325-10$
 $\log \text{ of quotient} = 9.5827-10$
 $9.5827-10 = \log .3825$
 $\therefore .2601 \div .68 = .3825$

- | | |
|---|---|
| <p>7. $\log 3950 = 13.5966-10$ $\log .250 = 9.3979-10$ <hr/> $\log \text{ of quotient} = 4.1987$ $4.1987 = \log 15800$ $\therefore 3950 \div .250 = 15800.$</p> | <p>12. $\log 26 = 11.4150-10$ $\log .06771 = 8.8307-10$ <hr/> $\log \text{ of quotient} = 2.5843$ $2.5843 = \log 384.0$ $\therefore 26 \div .06771 = 384.$</p> |
| <p>8. $\log 10 = 1.0000$ $\log 3.14 = 0.4969$ <hr/> $\log \text{ of quotient} = 0.5031$ $0.5031 = \log 3.185$ $\therefore 10 \div 3.14 = 3.185.$</p> | <p>13. $\log 1 = 10.0000-10$ $\log 40 = 1.6021$ <hr/> $\log (1 \div 40) = 8.3979-10$ $8.3979-10 = \log .0250$ $\therefore 1 \div 40 = .025.$</p> |
| <p>9. $\log .6911 = 19.8396-20$ $\log .7854 = 9.8951-10$ <hr/> $\log \text{ of quotient} = 9.9445-10$ $9.9445-10 = \log .8800$ $\therefore .6911 \div .7854 = .88.$</p> | <p>14. $\log 1 = 10.0000-10$ $\log 75 = 1.8751$ <hr/> $\log \text{ of quotient} = 8.1249-10$ $8.1249-10 = \log .01333$ $\therefore 1 \div 75 = .01333.$</p> |
| <p>10. $\log 2.816 = 10.4496-10$ $\log 22.5 = 1.3522$ <hr/> $\log \text{ of quotient} = 9.0974-10$ $9.0974-10 = \log .1251$ $\therefore 2.816 \div 22.5 = .1251.$</p> | <p>15. $\log 200 = 12.3010-10$ $\log .5236 = 9.7190-10$ <hr/> $\log \text{ of quotient} = 2.5820$ $2.5820 = \log 381.9$ $\therefore 200 \div .5236 = 381.9.$</p> |
| <p>11. $\log 4 = 10.6021-10$ $\log .00521 = 7.7168-10$ <hr/> $\log \text{ of quotient} = 2.8853$ $2.8853 = \log 767.8$ $\therefore 4 \div .00521 = 767.8.$</p> | <p>16. $\log 300 = 2.4771$ $\log 17.32 = 1.2385$ <hr/> $\log \text{ of quotient} = 1.2386$ $1.2386 = \log 17.32$ $\therefore 300 \div 17.32 = 17.32.$</p> |
| <p>17. $\log .220 = 19.3424-20$ $\log .3183 = 9.5028-10$ <hr/> $\log \text{ of quotient} = 9.8396-10$ $9.8396-10 = \log .6912$ $\therefore .220 \div .3183 = .6912.$</p> | |

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|---|--|
| <p>2. $\log 110 = 2.0414$ $\log 3.1 = 0.4914$ $\log .653 = 9.8149-10$ $\text{colog } 33 = 8.4815-10$ $\text{colog } 7.854 = 9.1049-10$ $\text{colog } 1.7 = 9.7696-10$ <hr/> $\log \text{ of result} = 39.7037-40$ $\therefore \text{result} = .5054.$</p> | <p>3. $\log 6000 = 3.7782$ $\log 5 = 0.6990$ $\log 29 = 1.4624$ $\text{colog } .7854 = 0.1049$ $\text{colog } 25000 = 5.6021-10$ $\text{colog } 81.7 = 8.0878-10$ <hr/> $\log \text{ of result} = 19.7344-20$ $\therefore \text{result} = .5425.$</p> |
|---|--|

$$\begin{aligned}
 4. \log 3.516 &= 0.5460 \\
 \log 485 &= 2.6857 \\
 \log 65 &= 1.8129 \\
 \text{colog } 3.33 &= 9.4776-10 \\
 \text{colog } 17 &= 8.7696-10 \\
 \text{colog } 18 &= 8.7447-10 \\
 \text{colog } 73 &= 8.1367-10 \\
 \hline
 \log \text{ of result} &= 40.1732-40 \\
 &= 0.1732 \\
 \therefore \text{ result} &= 1.49.
 \end{aligned}$$

$$\begin{aligned}
 6. \log 78 &= 1.8921 \\
 \log 52 &= 1.7160 \\
 \log 1605 &= 3.2055 \\
 \text{colog } 338 &= 7.4711-10 \\
 \text{colog } 767 &= 7.1152-10 \\
 \text{colog } 431 &= 7.3655-10 \\
 \hline
 \log \text{ of result} &= 28.7654-30 \\
 \therefore \text{ result} &= .05826.
 \end{aligned}$$

$$\begin{aligned}
 5. \log 15 &= 1.1761 \\
 \log .37 &= 9.5682-10 \\
 \log 26.16 &= 1.4176 \\
 \text{colog } 88 &= 8.0555-10 \\
 \text{colog } .18 &= 0.7447 \\
 \text{colog } 6.67 &= 9.1759-10 \\
 \hline
 \log \text{ of result} &= 30.1380-30 \\
 &= 0.1380 \\
 \therefore \text{ result} &= 1.374.
 \end{aligned}$$

$$\begin{aligned}
 7. \log .5 &= 9.6990-10 \\
 \log .315 &= 9.4983-10 \\
 \log 428 &= 2.6314 \\
 \text{colog } .317 &= 0.4989 \\
 \text{colog } .973 &= 0.0119 \\
 \text{colog } 43.7 &= 8.3595-10 \\
 \hline
 \log \text{ of result} &= 30.6990-30 \\
 &= 0.6990 \\
 \therefore \text{ result} &= 5.
 \end{aligned}$$

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$$\begin{aligned}
 2. \log 7 &= 0.8451 \\
 2 \log 7 &= 1.6902 \\
 1.6902 &= \log 49.00 \\
 \therefore 7^2 &= 49.
 \end{aligned}$$

$$\begin{aligned}
 3. \log 11 &= 1.0414 \\
 2 \log 11 &= 2.0828 \\
 2.0828 &= \log 121.0 \\
 \therefore 11^2 &= 121.
 \end{aligned}$$

$$\begin{aligned}
 4. \log 47 &= 1.6721 \\
 2 \log 47 &= 3.3442 \\
 3.3442 &= \log 2209 \\
 \therefore 47^2 &= 2209.
 \end{aligned}$$

$$\begin{aligned}
 5. \log 4.9 &= 0.6902 \\
 2 \log 4.9 &= 1.3804 \\
 1.3804 &= \log 24.01 \\
 \therefore 4.9^2 &= 24.01.
 \end{aligned}$$

$$\begin{aligned}
 6. \log 5.2 &= 0.7160 \\
 2 \log 5.2 &= 1.4320 \\
 1.4320 &= \log 27.04 \\
 \therefore 5.2^2 &= 27.04.
 \end{aligned}$$

$$\begin{aligned}
 7. \log .78 &= 9.8921-10 \\
 2 \log .78 &= 19.7842-20 \\
 19.7842-20 &= \log .6084 \\
 \therefore .78^2 &= .6084.
 \end{aligned}$$

$$\begin{aligned}
 8. \log 8.05 &= 0.9058 \\
 2 \log 8.05 &= 1.8116 \\
 1.8116 &= \log 64.80 \\
 \therefore 8.05^2 &= 64.8.
 \end{aligned}$$

$$\begin{aligned}
 9. \log 8.33 &= 0.9206 \\
 2 \log 8.33 &= 1.8412 \\
 1.8412 &= \log 69.37 \\
 \therefore 8.33^2 &= 69.37.
 \end{aligned}$$

$$\begin{aligned}
 10. \log 6.61 &= 0.8202 \\
 3 \log 6.61 &= 2.4606 \\
 2.4606 &= \log 288.8 \\
 \therefore 6.61^3 &= 288.8.
 \end{aligned}$$

$$\begin{aligned}
 11. \log .714 &= 9.8537-10 \\
 2 \log .714 &= 19.7074-20 \\
 19.7074-20 &= \log .5098 \\
 \therefore .714^2 &= .5098.
 \end{aligned}$$

$$\begin{aligned}
 12. \log 4.07 &= 0.6096 \\
 3 \log 4.07 &= 1.8288 \\
 1.8288 &= \log 67.42 \\
 \therefore 4.07^3 &= 67.42.
 \end{aligned}$$

$$\begin{aligned}
 13. \log .543 &= 9.7348-10 \\
 3 \log .543 &= 29.2044-30 \\
 29.2044-30 &= \log .1601 \\
 \therefore .543^3 &= .1601.
 \end{aligned}$$

- | | | | |
|--|------------------|---|------------------|
| 14. $\log 7$ | $= 0.8451$ | 15. $\log 1.02$ | $= 0.0086$ |
| $4 \log 7$ | $= 3.3804$ | $5 \log 1.02$ | $= 0.0430$ |
| 3.3804 | $= \log 2401$ | 0.0430 | $= \log 1.104$ |
| $\therefore 7^4$ | $= 2401.$ | $\therefore 1.02^5$ | $= 1.104.$ |
| 16. $\log 1.738$ | $= 0.2400$ | 17. $\log \frac{3}{20} = \log .15$ | $= 9.1761 - 10$ |
| $3 \log 1.738$ | $= 0.7200$ | $2 \log \frac{3}{20}$ | $= 18.3522 - 20$ |
| 0.7200 | $= \log 5.248$ | $18.3522 - 20$ | $= \log .0225$ |
| $\therefore 1.738^3$ | $= 5.248.$ | $\therefore (\frac{3}{20})^2$ | $= .0225.$ |
| 18. $\log \frac{1}{7} = \text{colog } 7$ | $= 9.1549 - 10$ | 19. $\log 64$ | $= 1.8062$ |
| $3 \log \frac{1}{7}$ | $= 27.4647 - 30$ | $\text{colog } 869$ | $= 7.0610 - 10$ |
| $27.4647 - 30$ | $= \log .002915$ | $\log \frac{128}{1738} = \log \frac{64}{869}$ | $= 8.8672 - 10$ |
| $\therefore (\frac{1}{7})^3$ | $= .002915.$ | $2 \log \frac{128}{1738}$ | $= 17.7344 - 20$ |
| | | $17.7344 - 20$ | $= \log .005425$ |
| | | $\therefore (\frac{128}{1738})^2$ | $= .005425.$ |
| 20. $\log 675$ | $= 2.8293$ | 21. $\log \frac{1}{243} = \text{colog } 243$ | $= 7.6144 - 10$ |
| $\text{colog } 4121$ | $= 6.3850 - 10$ | $.4 \log \frac{1}{243}$ | $= 3.0458 - 4$ |
| $\log \frac{675}{4121}$ | $= 9.2143 - 10$ | | $= 9.0458 - 10$ |
| $3 \log \frac{675}{4121}$ | $= 27.6429 - 30$ | $9.0458 - 10$ | $= \log .1111$ |
| $27.6429 - 30$ | $= \log .004394$ | $\therefore (\frac{1}{243})^4$ | $= .1111.$ |
| $\therefore (\frac{675}{4121})^3$ | $= .004394.$ | | |

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- | | | | |
|----------------------------------|-------------------|----------------------------------|------------------|
| 2. $\log 225$ | $= 2.3522$ | 8. $\log 3.375$ | $= 0.5283$ |
| $\frac{1}{2} \log 225$ | $= 1.1761$ | $\frac{1}{3} \log 3.375$ | $= 0.1761$ |
| 1.1761 | $= \log 15.00$ | 0.1761 | $= \log 1.500$ |
| $\therefore 225^{\frac{1}{2}}$ | $= 15.$ | $\therefore 3.375^{\frac{1}{3}}$ | $= 1.5.$ |
| 3. $\log 12.25$ | $= 1.0882$ | 9. $\log 1331$ | $= 3.1242$ |
| $\frac{1}{2} \log 12.25$ | $= 0.5441$ | $\frac{1}{3} \log 1331$ | $= 1.0414$ |
| 0.5441 | $= \log 3.500$ | 1.0414 | $= \log 11.00$ |
| $\therefore 12.25^{\frac{1}{2}}$ | $= 3.5.$ | $\therefore 1331^{\frac{1}{3}}$ | $= 11.$ |
| 4. $\log .2025$ | $= 19.30645 - 20$ | 10. $\log 1024$ | $= 3.0103$ |
| $\frac{1}{2} \log .2025$ | $= 9.6532 - 10$ | $.7 \log 1024$ | $= 2.1072$ |
| $9.6532 - 10$ | $= \log .4500$ | 2.1072 | $= \log 128.0$ |
| $\therefore .2025^{\frac{1}{2}}$ | $= .45.$ | $\therefore 1024^{\frac{7}{10}}$ | $= 128.$ |
| 5. $\log 324$ | $= 2.5105$ | 11. $\log .6724$ | $= 19.8276 - 20$ |
| $\frac{1}{2} \log 324$ | $= 1.2553$ | $\frac{1}{3} \log .6724$ | $= 9.9188 - 10$ |
| 1.2553 | $= \log 18.00$ | $9.9188 - 10$ | $= \log .8200$ |
| $\therefore 324^{\frac{1}{2}}$ | $= 18.$ | $\therefore .6724^{\frac{1}{3}}$ | $= .82.$ |
| 6. $\log .512$ | $= 29.7093 - 30$ | 12. $\log 5.929$ | $= 0.7730$ |
| $\frac{1}{3} \log .512$ | $= 9.9031 - 10$ | $\frac{1}{2} \log 5.929$ | $= 0.3865$ |
| $9.9031 - 10$ | $= \log .8000$ | 0.3865 | $= \log 2.435$ |
| $\therefore .512^{\frac{1}{3}}$ | $= .8.$ | $\therefore 5.929^{\frac{1}{2}}$ | $= 2.435.$ |
| 7. $\log .1181$ | $= 19.07226 - 20$ | 13. $\log .4624$ | $= 19.6650 - 20$ |
| $\frac{1}{2} \log .1181$ | $= 9.5361 - 10$ | $\frac{1}{2} \log .4624$ | $= 9.8325 - 10$ |
| $9.5361 - 10$ | $= \log .3436$ | $9.8325 - 10$ | $= \log .6800$ |
| $\therefore .1181^{\frac{1}{2}}$ | $= .3436.$ | $\therefore .4624^{\frac{1}{2}}$ | $= .68.$ |

| | | | |
|-----------------------------------|----------------|--|-----------------|
| 14. $\log 1.4641$ | $= 0.1656$ | 25. $\log 30.25$ | $= 1.4807$ |
| $\frac{1}{2} \log 1.4641$ | $= 0.0414$ | $\frac{1}{2} \log 30.25$ | $= 0.7404$ |
| 0.0414 | $= \log 1.100$ | 0.7404 | $= \log 5.500.$ |
| $\therefore 1.4641^{\frac{1}{2}}$ | $= 1.1.$ | $\therefore \sqrt{30.25}$ | $= 5.5.$ |
| 15. $\log .00032$ | $= 46.5051-50$ | 26. $\log .90$ | $= 19.9542-20$ |
| $\frac{1}{2} \log .00032$ | $= 9.3010-10$ | $\frac{1}{2} \log .90$ | $= 9.9771-10$ |
| $9.3010-10$ | $= \log .2000$ | $9.9771-10$ | $= \log .9486$ |
| $\therefore .00032^{\frac{1}{2}}$ | $= .2.$ | $\therefore \sqrt{.90}$ | $= .9486.$ |
| 16. $\log 2$ | $= 0.3010$ | 27. $\log .52$ | $= 19.7160-20$ |
| $\frac{1}{2} \log 2$ | $= 0.1505$ | $\frac{1}{2} \log .52$ | $= 9.8580-10$ |
| 0.1505 | $= \log 1.414$ | $9.8580-10$ | $= \log .7212$ |
| $\therefore \sqrt{2}$ | $= 1.414.$ | $\therefore \sqrt{.52}$ | $= .7212.$ |
| 17. $\log 3$ | $= 0.4771$ | 28. $\log .032$ | $= 48.5051-50$ |
| $\frac{1}{2} \log 3$ | $= 0.2386$ | $\frac{1}{2} \log .032$ | $= 9.7010-10$ |
| 0.2386 | $= \log 1.732$ | $9.7010-10$ | $= \log .5023$ |
| $\therefore \sqrt{3}$ | $= 1.732.$ | $\therefore \sqrt[5]{.032}$ | $= .5023.$ |
| 18. $\log 5$ | $= 0.6990$ | 29. See next page. | |
| $\frac{1}{2} \log 5$ | $= 0.3495$ | | |
| 0.3495 | $= \log 2.236$ | 30. $\frac{176}{15 \times 3.1416} = \frac{4}{15 \times .0714}$ | |
| $\therefore \sqrt{5}$ | $= 2.236.$ | $\log 4$ | $= 0.6021$ |
| 19. $\log 6$ | $= 0.7782$ | $\text{colog } 15$ | $= 8.8239-10$ |
| $\frac{1}{2} \log 6$ | $= 0.3891$ | $\text{colog } .0714$ | $= 1.1463$ |
| 0.3891 | $= \log 2.449$ | $\log \text{ result}$ | $= 10.5723-10$ |
| $\therefore \sqrt{6}$ | $= 2.449.$ | | $= 0.5723$ |
| 20. $\log 2$ | $= 0.3010$ | $\therefore \text{result}$ | $= 3.735.$ |
| $\frac{1}{3} \log 2$ | $= 0.1003$ | 31. $\log 100^2$ | $= 4.0000$ |
| 0.1003 | $= \log 1.260$ | $\text{colog } 48$ | $= 8.3188-10$ |
| $\therefore \sqrt[3]{2}$ | $= 1.26.$ | $\text{colog } 64$ | $= 8.1938-10$ |
| 21. $\log 4$ | $= 0.6021$ | $\text{colog } 11$ | $= 8.9586-10$ |
| $\frac{1}{4} \log 4$ | $= 0.1505$ | $\log \text{ result}$ | $= 29.4712-30$ |
| 0.1505 | $= \log 1.414$ | $\therefore \text{result}$ | $= .2959.$ |
| $\therefore \sqrt[4]{4}$ | $= 1.414.$ | 32. $\log 52$ | $= 1.7160$ |
| 22. $\log 3$ | $= 0.4771$ | $\log 52$ | $= 1.7160$ |
| $\frac{1}{3} \log 3$ | $= 0.1590$ | $\log 300$ | $= 2.4771$ |
| 0.1590 | $= \log 1.442$ | $\text{colog } 12$ | $= 8.9208-10$ |
| $\therefore \sqrt[3]{3}$ | $= 1.442.$ | $\text{colog } .31225$ | $= 0.5055$ |
| 23. $\log 2$ | $= 0.3010$ | $\text{colog } 400000$ | $= 4.3979-10$ |
| $\frac{1}{5} \log 2$ | $= 0.0602$ | $\log \text{ result}$ | $= 19.7333-20$ |
| 0.0602 | $= \log 1.149$ | $\therefore \text{result}$ | $= .5411.$ |
| $\therefore \sqrt[5]{2}$ | $= 1.149.$ | 33. $\log 400$ | $= 2.6021$ |
| 24. $\log .027$ | $= 28.4314-30$ | $\text{colog } 55$ | $= 8.2596-10$ |
| $\frac{1}{3} \log .027$ | $= 9.4771-10$ | $\text{colog } 3.1416$ | $= 9.5029-10$ |
| $9.4771-10$ | $= \log .3000$ | $\log \text{ result}^2$ | $= 0.3646$ |
| $\therefore \sqrt[3]{.027}$ | $= .3.$ | $\log \text{ result}$ | $= 0.1823$ |
| | | $\therefore \text{result}$ | $= 1.522.$ |

$$\begin{aligned}
 29. \log .025 &= 18.3979-20 \\
 \frac{1}{2} \log .025 &= 9.1990-10 \\
 9.1990-10 &= \log .1581 \\
 \therefore \sqrt{.025} &= .1581.
 \end{aligned}$$

$$\begin{aligned}
 34. \log 2^{3.5} &= 0.3010 \times 3.5 \\
 &= 1.0535 \quad (1) \\
 \log 8^{1.63} &= 0.9031 \times 1.63 \\
 &= 1.4721 \quad (2)
 \end{aligned}$$

Subtracting (2) from (1),

$$\log \frac{2^{3.5}}{8^{1.63}} = 9.5814-10$$

$$\log 50 = 1.6990$$

$$\log \text{result} = 1.2804$$

$$\therefore \text{result} = 19.07.$$

$$\begin{aligned}
 35. \log 1.6 &= 0.2041 \\
 \frac{1}{3} \log 1.6 &= 0.0680 \\
 \log 14.5 &= 1.1614 \\
 \text{colog } 11 &= 8.9586-10 \\
 \log \text{result} &= 10.1880-10 \\
 &= 0.1880 \\
 \therefore \text{result} &= 1.542.
 \end{aligned}$$

$$\begin{aligned}
 36. \sqrt{\frac{.434 \times 96^4}{64 \times 1500}} &= \frac{96^2}{80} \sqrt{\frac{.434}{15}} \\
 \log .434 &= 9.6375-10 \\
 \text{colog } 15 &= 8.8239-10 \\
 \hline
 2 &18.4614-20
 \end{aligned}$$

$$\log \sqrt{\frac{.434}{15}} = 9.2307-10$$

$$\log 96^2 = 3.9646$$

$$\text{colog } 80 = 8.0969-10$$

$$\log \text{result} = 21.2922-20$$

$$= 1.2922$$

$$\therefore \text{result} = 19.6.$$

$$\begin{aligned}
 37. .32 \times 5000 \times 18 &= 1800 \\
 3.14 \times .1222 \times 8 &= 3.14 \times .0611 \\
 \log 1800 &= 3.2553 \\
 \text{colog } 3.14 &= 9.5031-10 \\
 \text{colog } .0611 &= 1.2140
 \end{aligned}$$

$$\log \text{result} = 13.9724-10$$

$$= 3.9724$$

$$\therefore \text{result} = 9384.$$

$$\begin{aligned}
 38. \log 11 &= 1.0414 \\
 \log 2.63 &= 0.4200 \\
 \log 4.263 &= 0.6297 \\
 \text{colog } 48 &= 8.3188-10 \\
 \text{colog } 3.263 &= 9.4864-10 \\
 \log \text{result} &= 19.8963-20 \\
 \therefore \text{result} &= .7876.
 \end{aligned}$$

$$\begin{aligned}
 39. \log 1.06 &= 0.0253 \\
 5 \log 1.06 &= 0.1265 \\
 \text{colog } 1.06^5 &= 9.8735-10 \\
 \log 3500 &= 3.5441 \\
 \hline
 \log \frac{3500}{1.06^5} &= 13.4176-10
 \end{aligned}$$

$$= 3.4176$$

$$\frac{1}{2} \log \frac{3500}{1.06^5} = 1.7088$$

$$\therefore \text{result} = 51.14.$$

$$\begin{aligned}
 40. 2^{\frac{1}{2}} \times \left(\frac{1}{2}\right)^{\frac{3}{4}} \times \sqrt[3]{\frac{3}{2}} \times \sqrt{.1} \\
 = 2^{\frac{1}{2}} \times \left(\frac{1}{4}\right)^{\frac{3}{4}} \times (1.5)^{\frac{1}{3}} \times (.1)^{\frac{1}{2}} \\
 \frac{1}{2} \log 2 &= 0.1505 \\
 \log .25 &= 9.7993-10 \\
 \log 1.5 &= 0.0587 \\
 \frac{1}{2} \log .1 &= 9.5000-10 \\
 \log \text{result} &= 19.5085-20 \\
 \therefore \text{result} &= .3225.
 \end{aligned}$$

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$$\begin{aligned}
 2. A = P(1+r)^n &= 225 \times 1.08^5. \\
 \log 225 &= 2.3522 \\
 \log 1.08^5 &= 0.1670 \\
 \log A &= 2.5192 \\
 \therefore \text{amount} &= \$330.50.
 \end{aligned}$$

$$\begin{aligned}
 3. A = P(1+r)^n &= 700 \times 1.06^5. \\
 \log 700 &= 2.8451 \\
 \log 1.06^5 &= 0.1265 \\
 \log A &= 2.9716 \\
 \therefore \text{amount} &= \$936.70.
 \end{aligned}$$

$$\begin{aligned}
 4. A = P(1+r)^n &= 400 \times 1.03^{10}. \\
 \log 400 &= 2.6021 \\
 \log 1.03^{10} &= 0.1280 \\
 \log A &= 2.7301 \\
 \therefore \text{amount} &= \$537.10.
 \end{aligned}$$

$$\begin{aligned}
 5. A = P(1+r)^n &= 1200 \times 1.04^{20}. \\
 \log 1200 &= 3.0792 \\
 \log 1.04^{20} &= 0.3400 \\
 \log A &= 3.4192 \\
 \therefore \text{amount} &= \$2625.
 \end{aligned}$$

$$\begin{aligned}
 6. \log P &= \log A - n \log (1 + r) \\
 &= \log 1000 - 10 \log 1.05. \\
 \log 1000 &= 3.0000 \\
 10 \log 1.05 &= 0.2120 \\
 \hline
 \log P &= 2.7880 \\
 \therefore \text{principal} &= \$613.70.
 \end{aligned}$$

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$$\begin{aligned}
 7. \log P &= \log A - n \log (1 + r) \\
 &= \log 743 - 20 \log 1.02. \\
 \log 743 &= 2.8710 \\
 20 \log 1.02 &= 0.1720 \\
 \hline
 \log P &= 2.6990 \\
 \therefore \text{principal} &= \$500.
 \end{aligned}$$

$$\begin{aligned}
 9. \log P &= \log A - n \log (1 + r) \\
 &= \log 1000 - 21 \log 1.04 \\
 \log 1000 &= 3.0000 \\
 21 \log 1.04 &= 0.3570 \\
 \hline
 \log P &= 2.6430 \\
 \therefore \text{principal} &= \$439.50.
 \end{aligned}$$

$$\begin{aligned}
 11. A &= P(1 + r)^n. \\
 \therefore r &= \sqrt[n]{\frac{A}{P}} - 1. \\
 \log 402 &= 2.6042 \\
 \log 300 &= 2.4771 \\
 \hline
 \log (402 \div 300) &= 0.1271 \\
 \frac{1}{n} \log (402 \div 300) &= 0.0212 \\
 0.0212 &= \log 1.05 \\
 \therefore r &= 1.05 - 1 = .05; \\
 \text{i.e., rate} &= 5\%.
 \end{aligned}$$

$$\begin{aligned}
 8. \log P &= \log A - n \log (1 + r) \\
 &= \log 1500 - 10 \log 1.04. \\
 \log 1500 &= 3.1761 \\
 10 \log 1.04 &= 0.1700 \\
 \hline
 \log P &= 3.0061 \\
 \therefore \text{principal} &= \$1014.
 \end{aligned}$$

$$\begin{aligned}
 10. n &= \frac{\log A - \log P}{\log (1 + r)} \\
 &= \frac{\log 1834.5 - \log 800}{\log 1.05} \\
 \log 1834.5 &= 3.2635 \\
 \log 800 &= 2.9031 \\
 \hline
 \text{Diff. of logs} &= 0.3604 \\
 \log 1.05 &= 0.0212 \\
 .3604 \div .0212 &= 17. \\
 \text{Hence, the time is } &17 \text{ years.}
 \end{aligned}$$

$$\begin{aligned}
 12. n &= \frac{\log A - \log P}{\log (1 + r)} \\
 &= \frac{\log 2000 - \log 1000}{\log 1.06} \\
 \log 2000 &= 3.3010 \\
 \log 1000 &= 3.0000 \\
 \hline
 \text{Diff. of logs} &= 0.3010 \\
 \log 1.06 &= 0.0253 \\
 .3010 \div .0253 &= 11.9, \text{ nearly.} \\
 \therefore \text{time} &= 11.9 \text{ years, nearly.}
 \end{aligned}$$

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$$\begin{aligned}
 3. A &= \frac{a}{r} [(1 + r)^n - 1] \\
 &= \frac{25}{.04} [1.04^{20} - 1] \\
 &= 625 [1.04^{20} - 1].
 \end{aligned}$$

By logarithms,

$$\begin{aligned}
 1.04^{20} &= 2.188 \\
 \therefore 1.04^{20} - 1 &= 1.188 \\
 \log 625 &= 2.7959 \\
 \log 1.188 &= 0.0748 \\
 \hline
 \therefore \log A &= 2.8707 \\
 \therefore \text{amount} &= \$742.50.
 \end{aligned}$$

$$\begin{aligned}
 4. P &= \frac{a}{r} \cdot \frac{(1 + r)^n - 1}{(1 + r)^n} \\
 &= \frac{300}{.04} \cdot \frac{1.04^5 - 1}{1.04^5} \\
 &= 7500 \cdot \frac{1.04^5 - 1}{1.04^5}.
 \end{aligned}$$

By logarithms,

$$\begin{aligned}
 1.04^5 &= 1.216 \\
 \therefore 1.04^5 - 1 &= .216 \\
 \log 7500 &= 3.8751 \\
 \log .216 &= 9.3345 - 10 \\
 \hline
 \text{colog } 1.04^5 &= 9.150 - 10 \\
 \therefore \log P &= 3.1246 \\
 \therefore \text{p. v.} &= \$1332.
 \end{aligned}$$

$$5. A = \frac{a}{r} [(1+r)^n - 1] \\ = \frac{17.76}{.035} [1.035^{25} - 1].$$

By logarithms,

$$1.035^{25} = 2.358$$

$$\therefore 1.035^{25} - 1 = 1.358$$

$$\log 17.76 = 1.2494$$

$$\log 1.358 = 0.1329$$

$$\text{colog } .035 = 1.4559$$

$$\therefore \log A = 2.8382$$

$$\therefore \text{amount} = \$689.$$

$$6. P = \frac{a}{r} \cdot \frac{(1+r)^n - 1}{(1+r)^n} \\ = \frac{1000}{.045} \cdot \frac{1.045^{20} - 1}{1.045^{20}}.$$

By logarithms,

$$1.045^{20} = 2.41$$

$$\therefore 1.045^{20} - 1 = 1.41$$

$$\log 1000 = 3.0000$$

$$\log 1.41 = 0.1492$$

$$\text{colog } .045 = 1.3468$$

$$\text{colog } 1.045^{20} = 9.6180 - 10$$

$$\therefore \log P = 4.1140$$

$$\therefore \text{p. v.} = \$13000.$$

$$7. \text{ From } A = \frac{a}{r} [(1+r)^n - 1],$$

$$a = \frac{Ar}{(1+r)^n - 1} \\ = \frac{1000 \times .05}{1.05^{10} - 1} = \frac{50}{1.05^{10} - 1}.$$

By logarithms,

$$1.05^{10} = 1.629$$

$$\therefore 1.05^{10} - 1 = .629$$

$$\log 50 = 1.6990$$

$$\text{colog } .629 = 0.2013$$

$$\therefore \log a = 1.9003$$

$$\therefore \text{annuity} = \$79.48.$$

UNDETERMINED COEFFICIENTS

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$$3. \text{ Assume } \frac{1+x}{1-x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots.$$

Clearing of fractions,

$$1+x = 1 + A|x + B|x^2 + C|x^3 + D|x^4 + \dots \\ - 1|x - A|x - B|x - C|x - \dots$$

Equating the coefficients of like powers of x ,

$$A - 1 = 1; B - A = 0; C - B = 0; D - C = 0, \text{ etc.}$$

$$\therefore D = C = B = A = 2.$$

$$\therefore \text{ to five terms, } \frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + 2x^4.$$

$$4. \text{ Assume } \frac{1+3x}{1+x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots.$$

Clearing of fractions,

$$1+3x = 1 + A|x + B|x^2 + C|x^3 + D|x^4 + \dots \\ + 1|x + A|x + B|x + C|x + D|x$$

Equating the coefficients of like powers of x ,

$$A + 1 = 3; B + A = 0; C + B = 0; D + C = 0; \text{ etc.}$$

$$\therefore A = 2, B = -2, C = 2, D = -2, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{1+3x}{1+x} = 1 + 2x - 2x^2 + 2x^3 - 2x^4.$$

$$5. \text{ Assume } \frac{1}{1-2x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Clearing of fractions,

$$1 = 1 + A|x + B|x^2 + C|x^3 + D|x^4 + \dots$$

$$- 2|-2A|-2B|-2C|- \dots$$

Equating the coefficients of like powers of x ,

$$A - 2 = 0; B - 2A = 0; C - 2B = 0; D - 2C = 0; \text{ etc.}$$

$$\therefore A = 2, B = 4, C = 8, D = 16, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4.$$

$$6. \text{ Assume } \frac{3}{2-x} = \frac{3}{2} + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Multiplying each member by $2-x$,

$$3 = 3 + 2A|x + 2B|x^2 + 2C|x^3 + 2D|x^4 + \dots$$

$$- \frac{3}{2}|-A|-B|-C|- \dots$$

Equating the coefficients of like powers of x ,

$$2A - \frac{3}{2} = 0; 2B - A = 0; 2C - B = 0; 2D - C = 0; \text{ etc.}$$

$$\therefore A = \frac{3}{4}, B = \frac{3}{8}, C = \frac{3}{16}, D = \frac{3}{32}, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{3}{2-x} = \frac{3}{2} + \frac{3}{4}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 + \frac{3}{32}x^4.$$

$$7. \text{ Assume } \frac{1}{1-ax} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Clearing of fractions,

$$1 = 1 + A|x + B|x^2 + C|x^3 + D|x^4 + \dots$$

$$- a|-aA|-aB|-aC|- \dots$$

Equating the coefficients of like powers of x ,

$$A - a = 0; B - aA = 0; C - aB = 0; D - aC = 0; \text{ etc.}$$

$$\therefore A = a, B = a^2, C = a^3, D = a^4, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + a^4x^4.$$

$$8. \text{ Assume } \frac{2+3x^2}{1-2x^2} = 2 + Ax^2 + Bx^4 + Cx^6 + Dx^8 + \dots$$

Clearing of fractions,

$$2 + 3x^2 = 2 + A|x^2 + B|x^4 + C|x^6 + D|x^8 + \dots$$

$$- 4|-2A|-2B|-2C|- \dots$$

Equating the coefficients of like powers of x ,

$$A - 4 = 3; B - 2A = 0; C - 2B = 0; D - 2C = 0; \text{ etc.}$$

$$\therefore A = 7, B = 14, C = 28, D = 56, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{2+3x^2}{1-2x^2} = 2 + 7x^2 + 14x^4 + 28x^6 + 56x^8.$$

9. Assume $\frac{4x - 3x^2}{1 + 2x} = 4x + Ax^2 + Bx^3 + Cx^4 + Dx^5 + \dots$

Clearing of fractions,

$$4x - 3x^2 = 4x + A|x^2 + B|x^3 + C|x^4 + D|x^5 + \dots \\ + 8| + 2A| + 2B| + 2C| + \dots$$

Equating the coefficients of like powers of x ,

$$A + 8 = -3; B + 2A = 0; C + 2B = 0; D + 2C = 0; \text{ etc.} \\ \therefore A = -11, B = 22, C = -44, D = 88, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{4x + 3x^2}{1 + 2x} = 4x - 11x^2 + 22x^3 - 44x^4 + 88x^5.$$

10. Assume $\frac{1 - x - 2x^2}{1 - 2x - x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

Clearing of fractions,

$$1 - x - 2x^2 = 1 + A|x + B|x^2 + C|x^3 + D|x^4 + \dots \\ - 2|-2A|-2B|-2C| - \dots \\ - 1| - A| - B| - \dots$$

Equating the coefficients of like powers of x ,

$$A - 2 = -1; B - 2A - 1 = -2; C - 2B - A = 0; \\ D - 2C - B = 0; \text{ etc.} \\ \therefore A = 1, B = 1, C = 3, D = 7, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{1 - x - 2x^2}{1 - 2x - x^2} = 1 + x + x^2 + 3x^3 + 7x^4.$$

11. Assume $\frac{x - x^2}{1 + 2x - x^2} = x + Ax^2 + Bx^3 + Cx^4 + Dx^5 + \dots$

Clearing of fractions,

$$x - x^2 = x + A|x^2 + B|x^3 + C|x^4 + D|x^5 + \dots \\ + 2| + 2A| + 2B| + 2C| + \dots \\ - 1| - A| - B|$$

Equating the coefficients of like powers of x ,

$$A + 2 = -1; B + 2A - 1 = 0; C + 2B - A = 0; D + 2C - B = 0. \\ \therefore A = -3, B = 7, C = -17, D = 41.$$

$$\therefore \text{ to five terms, } \frac{x - x^2}{1 + 2x - x^2} = x - 3x^2 + 7x^3 - 17x^4 + 41x^5.$$

12. $\frac{1 - x}{1 - x + x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

Clearing of fractions,

$$1 - x = 1 + A|x + B|x^2 + C|x^3 + D|x^4 + \dots \\ - 1| - A| - B| - C| - \dots \\ 1 + A| + B| + \dots$$

Equating the coefficients of like powers of x ,

$$A - 1 = -1; B - A + 1 = 0; C - B + A = 0; D - C + B = 0; \text{ etc.} \\ \therefore A = 0, B = -1, C = -1, D = 0, \text{ etc.}$$

Dropping the terms having zero coefficients, we have less than the required number of terms. The series may be extended, however, by

means of the relation between any coefficient and the two preceding it, shown in the equations involving three coefficients. Thus, since $D - C + B = 0$, $D = C - B$; similarly, $C = B - A$, etc.; that is, *any coefficient is equal to the preceding coefficient minus that which in turn precedes it.*

Hence,

$$\frac{1-x}{1-x+x^2} = 1 + 0x - x^2 - x^3 + 0x^4 + [0 - (-1)]x^5 + (1-0)x^6 + \dots$$

to five terms $= 1 - x^2 - x^3 + x^5 + x^6$.

$$13. \text{ Assume } \frac{1}{1-x-x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Clearing of fractions,

$$\begin{array}{rccccccc} 1 & = & 1 & + & A|x & + & B|x^2 & + & C|x^3 & + & D|x^4 & + & \dots \\ & & - & 1| & - & A| & - & B| & - & C| & - & D| & - & \dots \\ & & & & - & 1| & - & A| & - & B| & - & C| & - & \dots \end{array}$$

Equating the coefficients of like powers of x ,

$$A - 1 = 0; B - A - 1 = 0; C - B - A = 0; D - C - B = 0; \text{ etc.}$$

$$\therefore A = 1, B = 2, C = 3, D = 5, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4.$$

$$14. \text{ Assume } \frac{2+x-2x^2}{1-x+2x^2} = 2 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Clearing of fractions,

$$\begin{array}{rccccccc} 2 & + & x & - & 2x^2 & = & 2 & + & A|x & + & B|x^2 & + & C|x^3 & + & D|x^4 & + & \dots \\ & & & & & & - & 2| & - & A| & - & B| & - & C| & - & D| & - & \dots \\ & & & & & & & & + & 4| & + & 2A| & + & 2B| & + & \dots \end{array}$$

Equating the coefficients of like powers of x ,

$$A - 2 = 1; B - A + 4 = -2; C - B + 2A = 0; D - C + 2B = 0; \text{ etc.}$$

$$\therefore A = 3, B = -3, C = -9, D = -3, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{2+x-2x^2}{1-x+2x^2} = 2 + 3x - 3x^2 - 9x^3 - 3x^4.$$

$$15. \text{ Assume } \frac{x^2+x^3}{1-2x+x^2} = x^2 + Ax^3 + Bx^4 + Cx^5 + Dx^6 + \dots$$

Clearing of fractions,

$$\begin{array}{rccccccc} x^2 & + & x^3 & = & x^2 & + & A|x^3 & + & B|x^4 & + & C|x^5 & + & D|x^6 & + & \dots \\ & & & & - & 2| & - & 2A| & - & 2B| & - & 2C| & - & 2D| & - & \dots \\ & & & & & & + & 1| & + & A| & + & B| & + & C| & + & \dots \end{array}$$

Equating the coefficients of like powers of x ,

$$A - 2 = 1; B - 2A + 1 = 0; C - 2B + A = 0; D - 2C + B = 0; \text{ etc.}$$

$$\therefore A = 3, B = 5, C = 7, D = 9, \text{ etc.}$$

$$\therefore \text{ to five terms, } \frac{x^2+x^3}{1-2x+x^2} = x^2 + 3x^3 + 5x^4 + 7x^5 + 9x^6.$$

16. Assume $\frac{1-2x}{x^2+x^3+x^4} = x^{-2} + Ax^{-1} + B + Cx + Dx^2 + \dots$

Clearing of fractions,

$$1 - 2x = 1 + \begin{array}{c} Ax + Bx^2 + Cx^3 + Dx^4 + \dots \\ + 1 \mid + A \mid + B \mid + C \mid + \dots \\ + 1 \mid + A \mid + B \mid + \dots \end{array}$$

Equating the coefficients of like powers of x ,

$A + 1 = -2$; $B + A + 1 = 0$; $C + B + A = 0$; $D + C + B = 0$; etc.
 $\therefore A = -3$, $B = 2$, $C = 1$, $D = -3$, etc.

\therefore to five terms, $\frac{1-2x}{x^2+x^3+x^4} = x^{-2} - 3x^{-1} + 2 + x - 3x^2$.

17. Assume $\frac{2-5x}{2x-x^2} = x^{-1} + A + Bx + Cx^2 + Dx^3 + \dots$

Clearing of fractions,

$$2 - 5x = 2 + \begin{array}{c} 2A \mid x + 2B \mid x^2 + 2C \mid x^3 + 2D \mid x^4 + \dots \\ - 1 \mid - A \mid - B \mid - C \mid - \dots \end{array}$$

Equating the coefficients of like powers of x ,

$2A - 1 = -5$; $2B - A = 0$; $2C - B = 0$; $2D - C = 0$; etc.
 $\therefore A = -2$, $B = -1$, $C = -\frac{1}{2}$, $D = -\frac{1}{4}$, etc.

\therefore to five terms, $\frac{2-5x}{2x-x^2} = x^{-1} - 2 - x - \frac{1}{2}x^2 - \frac{1}{4}x^3$.

18. Assume $\frac{1+x+x^2}{x+x^3+x^4} = x^{-1} + A + Bx + Cx^2 + Dx^3 + \dots$

Clearing of fractions, $1 + x + x^2 = 1 + \begin{array}{c} Ax + Bx^2 + Cx^3 + Dx^4 + \dots \\ + 1 \mid + A \mid + B \mid + C \mid + \dots \\ + 1 \mid + A \mid + \dots \end{array}$

Equating the coefficients of like powers of x ,

$A = 1$; $B + 1 = 1$; $C + A + 1 = 0$; $D + B + A = 0$; etc.
 $\therefore A = 1$, $B = 0$, $C = -2$, $D = -1$, next coef. $= -C - B = 2$, etc.

$\therefore \frac{1+x+x^2}{x+x^3+x^4} = x^{-1} + 1 + 0x - 2x^2 - x^3 + 2x^4 - \dots$

to five terms, $= x^{-1} + 1 - 2x^2 - x^3 + 2x^4$.

19. Assume $\frac{1}{a-x} = a^{-1} + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

Clearing of fractions, $1 = 1 + \begin{array}{c} aA \mid x + aB \mid x^2 + aC \mid x^3 + aD \mid x^4 + \dots \\ - a^{-1} \mid - A \mid - B \mid - C \mid - \dots \end{array}$

Equating the coefficients of like powers of x ,

$aA - a^{-1} = 0$; $aB - A = 0$; $aC - B = 0$; $aD - C = 0$; etc.

$\therefore A = a^{-2}$, $B = a^{-3}$, $C = a^{-4}$, $D = a^{-5}$, etc.

\therefore to five terms, $\frac{1}{a-x} = a^{-1} + a^{-2}x + a^{-3}x^2 + a^{-4}x^3 + a^{-5}x^4$.

20. Since, by Ex. 19, $\frac{1}{a-x} = a^{-1} + a^{-2}x + a^{-3}x^2 + a^{-4}x^3 + a^{-5}x^4 + \dots$,
 substituting $-x$ for x , we have
 to five terms, $\frac{1}{a+x} = a^{-1} - a^{-2}x + a^{-3}x^2 - a^{-4}x^3 + a^{-5}x^4$.

21. Since, by Ex. 19, $\frac{1}{a-x} = a^{-1} + a^{-2}x + a^{-3}x^2 + a^{-4}x^3 + a^{-5}x^4 + \dots$,
 substituting 1 for a , $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$.
 \therefore to five terms, $\frac{a}{1-x} = a + ax + ax^2 + ax^3 + ax^4$.

22. Since, by Ex. 19, $\frac{1}{a-x} = a^{-1} + a^{-2}x + a^{-3}x^2 + a^{-4}x^3 + a^{-5}x^4 + \dots$,
 substituting b for a , $\frac{1}{b-x} = b^{-1} + b^{-2}x + b^{-3}x^2 + b^{-4}x^3 + b^{-5}x^4 + \dots$.
 \therefore to five terms, $\frac{a}{b-x} = ab^{-1} + ab^{-2}x + ab^{-3}x^2 + ab^{-4}x^3 + ab^{-5}x^4$.

23. Assume $\frac{c}{b-ax} = b^{-1}c + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$.

Clearing of fractions, $c = c + \frac{bA}{ab^{-1}c}|x + \frac{bB}{-aA}|x^2 + \frac{bC}{-aB}|x^3 + \frac{bD}{-aC}|x^4 + \dots$

Equating the coefficients of like powers of x ,

$$bA - ab^{-1}c = 0; bB - aA = 0; bC - aB = 0; bD - aC = 0; \text{etc.}$$

$$\therefore A = ab^{-2}c, B = a^2b^{-3}c, C = a^3b^{-4}c, D = a^4b^{-5}c, \text{etc.}$$

\therefore to five terms, $\frac{c}{b-ax} = b^{-1}c + ab^{-2}cx + a^2b^{-3}cx^2 + a^3b^{-4}cx^3 + a^4b^{-5}cx^4$.

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2. Assume $\sqrt{1-x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$.

$$\text{Squaring, } 1-x = 1 + 2Ax + \frac{2B}{A^2}|x^2 + \frac{2C}{2AB}|x^3 + \frac{2D}{B^2}|x^4 + \dots$$

Equating the coefficients of like powers of x ,

$$2A = -1; 2B + A^2 = 0; 2C + 2AB = 0; 2D + 2AC + B^2 = 0; \text{etc.}$$

$$\therefore A = -\frac{1}{2}, B = -\frac{1}{8}, C = -\frac{1}{16}, D = -\frac{5}{128}, \text{etc.}$$

Hence, to five terms, $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4$.

3. Assume $\sqrt{1+2x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$.

$$\text{Squaring, } 1+2x = 1 + 2Ax + \frac{2B}{A^2}|x^2 + \frac{2C}{2AB}|x^3 + \frac{2D}{B^2}|x^4 + \dots$$

Equating the coefficients of like powers of x ,

$$2A = 2; 2B + A^2 = 0; 2C + 2AB = 0; 2D + 2AC + B^2 = 0; \text{etc.}$$

$$\therefore A = 1, B = -\frac{1}{2}, C = -\frac{1}{2}, D = -\frac{5}{8}, \text{etc.}$$

Hence, to five terms, $\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4$.

4. Assume $\sqrt{1+4x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

$$\text{Squaring, } \begin{array}{r} 1 + 4x = 1 + 2Ax + 2Bx^2 + 2Cx^3 + 2Dx^4 + \dots \\ \quad \quad \quad + A^2x^2 + 2ABx^3 + 2ACx^4 + \dots \\ \quad \quad \quad \quad \quad \quad + B^2x^4 \end{array}$$

Equating the coefficients of like powers of x ,

$$2A = 4; 2B + A^2 = 0; 2C + 2AB = 0; 2D + 2AC + B^2 = 0; \text{ etc.}$$

$$\therefore A = 2, B = -2, C = 4, D = -10.$$

$$\text{Hence, to five terms, } \sqrt{1+4x} = 1 + 2x - 2x^2 + 4x^3 - 10x^4.$$

5. Assume $\sqrt{4-x} = 2 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

$$\text{Squaring, } \begin{array}{r} 4 + x = 4 + 4Ax + 4Bx^2 + 4Cx^3 + 4Dx^4 + \dots \\ \quad \quad \quad + A^2x^2 + 2ABx^3 + 2ACx^4 + \dots \\ \quad \quad \quad \quad \quad \quad + B^2x^4 \end{array}$$

Equating the coefficients of like powers of x ,

$$4A = 1; 4B + A^2 = 0; 4C + 2AB = 0; 4D + 2AC + B^2 = 0; \text{ etc.}$$

$$\therefore A = \frac{1}{4}, B = -\frac{1}{64}, C = \frac{1}{512}, D = -\frac{5}{16384}, \text{ etc.}$$

$$\text{Hence, to five terms, } \sqrt{4-x} = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 - \frac{5}{16384}x^4.$$

6. See next page.

7. Assume $\sqrt{a^2 - x^2} = a + Ax^2 + Bx^4 + Cx^6 + Dx^8 + \dots$

$$\text{Squaring, } \begin{array}{r} a^2 - x^2 = a^2 + 2aAx^2 + 2aBx^4 + 2aCx^6 + 2aDx^8 + \dots \\ \quad \quad \quad + A^2x^4 + 2ABx^6 + 2ACx^8 + \dots \\ \quad \quad \quad \quad \quad \quad + B^2x^8 \end{array}$$

Equating the coefficients of like powers of x ,

$$2aA = -1; 2aB + A^2 = 0; 2aC + 2AB = 0; 2aD + 2AC + B^2 = 0.$$

$$\therefore A = -\frac{1}{2a}, B = -\frac{1}{8a^3}, C = -\frac{1}{16a^5}, D = -\frac{5}{128a^7}, \text{ etc.}$$

$$\text{Hence, to five terms, } \sqrt{a^2 - x^2} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}.$$

8. Assume $\sqrt{1+x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

$$\text{Squaring, } \begin{array}{r} 1 + x = 1 + 2Ax + 2Bx^2 + 2Cx^3 + 2Dx^4 + \dots \\ \quad \quad \quad + A^2x^2 + 2ABx^3 + 2ACx^4 + \dots \\ \quad \quad \quad \quad \quad \quad + B^2x^4 \end{array}$$

Equating the coefficients of like powers of x ,

$$2A = 1; 2B + A^2 = 0; 2C + 2AB = 0; 2D + 2AC + B^2 = 0; \text{ etc.}$$

$$\therefore A = \frac{1}{2}, B = -\frac{1}{8}, C = \frac{1}{16}, D = -\frac{5}{128}, \text{ etc.}$$

$$\text{Hence, to five terms, } \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4.$$

This result may be obtained by substituting $-x$ for x in Ex. 2.

9. Assume $\sqrt{1+x+x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$

$$\text{Squaring, } \begin{array}{r} 1 + x + x^2 = 1 + 2Ax + 2Bx^2 + 2Cx^3 + 2Dx^4 + \dots \\ \quad \quad \quad + A^2x^2 + 2ABx^3 + 2ACx^4 + \dots \\ \quad \quad \quad \quad \quad \quad + B^2x^4 \end{array}$$

Equating the coefficients of like powers of x ,

$$2A = 1; 2B + A^2 = 1; 2C + 2AB = 0; 2D + 2AC + B^2 = 0; \text{ etc.}$$

$$\therefore A = \frac{1}{2}, B = \frac{3}{8}, C = -\frac{3}{16}, D = \frac{3}{128}, \text{ etc.}$$

$$\text{Hence, to five terms, } \sqrt{1+x+x^2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{3}{16}x^3 + \frac{3}{128}x^4.$$

6. Assume $\sqrt{a-x} = \sqrt{a} + Ax + Bx^2 + Cx^3 + Dx^4 \dots$

$$\text{Squaring, } a - x = a + 2\sqrt{a}Ax + 2\sqrt{a}B|x^2 + 2\sqrt{a}C|x^3 + 2\sqrt{a}D|x^4 + \dots \\ + \begin{matrix} A^2 \\ + 2AB \\ + B^2 \end{matrix}$$

Equating the coefficients of like powers of x ,

$$2\sqrt{a}A = -1; 2\sqrt{a}B + A^2 = 0; 2\sqrt{a}C + 2AB = 0; 2\sqrt{a}D + 2AC + B^2 = 0.$$

$$\therefore A = -\frac{\sqrt{a}}{2a}, B = -\frac{\sqrt{a}}{8a^2}, C = -\frac{\sqrt{a}}{16a^3}, D = -\frac{5\sqrt{a}}{128a^4}, \text{ etc.}$$

$$\text{Hence, to five terms, } \sqrt{a-x} = \sqrt{a} \left(1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4} \right).$$

This result may be obtained by substituting $-x$ for x in Ex. 1.

10. Assume

$$(1 + 4x + 6x^2 + 4x^3 + x^4)^{\frac{1}{2}} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Squaring,

$$1 + 4x + 6x^2 + 4x^3 + x^4 = 1 + 2Ax + 2B|x^2 + 2C|x^3 + 2D|x^4 + \dots \\ + \begin{matrix} A^2 \\ + 2AB \\ + B^2 \end{matrix}$$

Equating the coefficients of like powers of x ,

$$2A = 4; 2B + A^2 = 6; 2C + 2AB = 4; 2D + 2AC + B^2 = 1; \text{ etc.}$$

$$\therefore A = 2, B = 1, C = 0, D = 0.$$

It will be observed that the equation $2D + 2AC + B^2 = 1$, from which D is found when the preceding coefficients have been obtained, contains C and B in each term of the first member except that containing D ; that $2C + 2AB = 4$ contains A and B ; and in general that the equation from which any coefficient is found has one or the other of the two preceding coefficients in every term not containing the coefficient itself. Thus, the next coefficient E , found from the equation $2E + 2AD + 2BC = 0$, contains either C or D as a factor in each of the last three terms. Since $C = 0$ and $D = 0$, $E = 0$. Likewise, since $D = 0$ and $E = 0$, $F = 0$; and so on. Hence, the series is finite.

$$\text{Hence, } (1 + 4x + 6x^2 + 4x^3 + x^4)^{\frac{1}{2}} = 1 + 2x + x^2.$$

11. See next page.

$$12. (1+x)^{\frac{3}{2}} = \sqrt{(1+x)^3} = \sqrt{1+3x+3x^2+x^3}.$$

$$\text{Assume } \sqrt{1+3x+3x^2+x^3} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Squaring,

$$1 + 3x + 3x^2 + x^3 = 1 + 2Ax + 2B|x^2 + 2C|x^3 + 2D|x^4 + \dots \\ + \begin{matrix} A^2 \\ + 2AB \\ + B^2 \end{matrix}$$

Equating the coefficients of like powers of x ,

$$2A = 3; 2B + A^2 = 3; 2C + 2AB = 1; 2D + 2AC + B^2 = 0; \text{ etc.}$$

$$\therefore A = \frac{3}{2}, B = \frac{3}{8}, C = -\frac{1}{16}, D = \frac{3}{128}, \text{ etc.}$$

$$\text{Hence, to five terms, } (1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4.$$

5.
$$\frac{x^2 + x - 4}{x^2 - 1} = 1 + \frac{x - 3}{x^2 - 1}.$$

Assume that $\frac{x - 3}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1}$ is an identity.

Clearing of fractions, $x - 3 = (A + B)x - A + B.$

Equating the coefficients of like powers of x ,

$$A + B = 1 \text{ and } B - A = -3.$$

$$\therefore A = 2, \text{ and } B = -1.$$

Hence,
$$\frac{x^2 + x - 4}{x^2 - 1} = 1 + \frac{2}{x + 1} - \frac{1}{x - 1}.$$

6. Assume that $\frac{2x}{8 - 6x + x^2} = \frac{A}{2 - x} + \frac{B}{4 - x}$ is an identity.

Clearing of fractions, $2x = 4A + 2B - (A + B)x.$

Equating the coefficients of like powers of x ,

$$4A + 2B = 0 \text{ and } -(A + B) = 2.$$

$$\therefore A = 2 \text{ and } B = -4.$$

Hence,
$$\frac{2x}{8 - 6x + x^2} = \frac{2}{2 - x} - \frac{4}{4 - x}.$$

7. Assume that $\frac{3 + 4x}{1 + 8x + 16x^2} = \frac{A}{1 + 4x} + \frac{B}{(1 + 4x)^2}$ is an identity.

Clearing of fractions, $3 + 4x = A + 4Ax + B.$

Equating the coefficients of like powers of x ,

$$A + B = 3 \text{ and } 4A = 4.$$

$$\therefore A = 1 \text{ and } B = 2.$$

Hence,
$$\frac{3 + 4x}{1 + 8x + 16x^2} = \frac{1}{1 + 4x} + \frac{2}{(1 + 4x)^2}.$$

8. Assume that $\frac{3x}{1 + x^3} = \frac{A}{1 + x} + \frac{B + Cx}{1 - x + x^2}$ is an identity.

Clearing of fractions, $3x = \frac{A - A|x + A|x^2}{+ B + C} + \frac{B + C}{+ B + C}$

Equating the coefficients of like powers of x ,

$$A + B = 0; -A + C + B = 3; A + C = 0.$$

$$\therefore A = -1, B = 1, C = 1.$$

Hence,
$$\frac{3x}{1 + x^3} = -\frac{1}{1 + x} + \frac{1 + x}{1 - x + x^2}, \text{ or } \frac{1 + x}{1 - x + x^2} - \frac{1}{1 + x}.$$

9. Assume that $\frac{1 - x - 6x^2}{x - x^3} = \frac{A}{x} + \frac{B}{1 + x} + \frac{C}{1 - x}$ is an identity.

Clearing of fractions, $1 - x - 6x^2 = A - Ax^2 + Bx - Bx^2 + Cx + Cx^2.$

Equating the coefficients of like powers of x ,

$$A = 1; B + C = -1; -A - B + C = -6.$$

$$\therefore A = 1, B = 2, C = -3.$$

Hence,
$$\frac{1 - x - 6x^2}{x - x^3} = \frac{1}{x} + \frac{2}{1 + x} - \frac{3}{1 - x}.$$

10. Assume that

$$\frac{1-2x+2x^2}{(1-x)(1-2x)^2} = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2} \text{ is an identity.}$$

Clearing of fractions, $1-2x+2x^2 = \frac{A-4A|x+4A|x^2}{+B-3B|+2B|} + \frac{C}{C|}$

Equating the coefficients of like powers of x ,

$$A+B+C=1; -4A-3B-C=-2; 4A+2B=2.$$

$$\therefore A=1, B=-1, C=1.$$

Hence, $\frac{1-2x+2x^2}{(1-x)(1-2x)^2} = \frac{1}{1-x} - \frac{1}{1-2x} + \frac{1}{(1-2x)^2}.$

11. See next page.

12. Assume that $\frac{x^2-6x}{(x-5)^3} = \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{(x-5)^3}$ is an identity.

Clearing of fractions, $x^2-6x = Ax^2-10A|x+25A$
 $+ \frac{B}{B|} - \frac{5B}{5B} + \frac{C}{C|}$

Equating the coefficients of like powers of x ,

$$A=1; -10A+B=-6; 25A-5B+C=0.$$

$$\therefore A=1, B=4, C=-5.$$

Hence, $\frac{x^2-6x}{(x-5)^3} = \frac{1}{x-5} + \frac{4}{(x-5)^2} - \frac{5}{(x-5)^3}.$

13. $\frac{x^2-5}{x^2-1} = 1 - \frac{4}{x^2-1}.$

Assume that $\frac{-4}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$ is an identity.

Clearing of fractions, $-4 = Ax - A + Bx + B.$

Equating the coefficients of like powers of x ,

$$A+B=0 \text{ and } -A+B=-4.$$

$$\therefore A=2 \text{ and } B=-2.$$

Hence, $\frac{x^2-5}{x^2-1} = 1 + \frac{2}{x+1} - \frac{2}{x-1}.$

14. $\frac{2x^2+9x+11}{x^2+4x+4} = 2 + \frac{x+3}{(x+2)^2}.$

Assume that $\frac{x+3}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$ is an identity.

Clearing of fractions, $x+3 = Ax+2A+B.$

Equating the coefficients of like powers of x ,

$$A=1 \text{ and } 2A+B=3, \text{ whence } B=1.$$

Hence, $\frac{2x^2+9x+11}{x^2+4x+4} = 2 + \frac{1}{x+2} + \frac{1}{(x+2)^2}.$

11. Assume that $\frac{3x-2}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$ is an identity.

Clearing of fractions, $3x-2 = Ax-3A+B$.

Equating the coefficients of like powers of x ,

$$A = 3 \text{ and } -3A + B = -2, \text{ whence } B = 7.$$

Hence,
$$\frac{3x-2}{(x-3)^2} = \frac{3}{x-3} + \frac{7}{(x-3)^2}.$$

15. Assume that $\frac{2-6x+6x^2}{1-6x+11x^2-6x^3} = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x}$ is an identity.

Clearing of fractions,

$$2-6x+6x^2 = A(1-2x)(1-3x) + B(1-x)(1-3x) + C(1-x)(1-2x).$$

Let $x = 1$; then, $2 = A(-1)(-2)$, whence $A = 1$.

Let $x = \frac{1}{2}$; then, $\frac{1}{2} = B(\frac{1}{2})(-\frac{1}{2})$, whence $B = -2$.

Let $x = \frac{1}{3}$; then, $\frac{2}{3} = C(\frac{2}{3})(\frac{1}{3})$, whence $C = 3$.

Hence,
$$\frac{2-6x+6x^2}{1-6x+11x^2-6x^3} = \frac{1}{1-x} - \frac{2}{1-2x} + \frac{3}{1-3x}.$$

16. Assume that $\frac{49}{(2-3x)^2(3-x)} = \frac{A}{3-x} + \frac{B}{2-3x} + \frac{C}{(2-3x)^2}$ is an identity.

Clearing of fractions, $49 = A(2-3x)^2 + B(3-x)(2-3x) + C(3-x)$.

Let $x = 3$; then, $49 = A(2-9)^2$, whence $A = 1$.

Let $x = \frac{2}{3}$; then, $49 = C(\frac{7}{3})$, whence $C = 3 \times 7 = 21$.

Let $x = 0$; then, $49 = 4A + 6B + 3C$.

Substituting 1 for A and 21 for C ,

$$49 = 4 + 6B + 63, \text{ whence } B = -3.$$

Hence,
$$\frac{49}{(2-3x)^2(3-x)} = \frac{1}{3-x} - \frac{3}{2-3x} + \frac{21}{(2-3x)^2}.$$

17. Assume that $\frac{1+2x+3x^2+2x^3}{x-x^5} = \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D+Ex}{1+x^2}$ (1) is an identity.

Clearing of fractions,

$$1+2x+3x^2+2x^3 = A(1-x^4) + Bx(1-x)(1+x^2) + Cx(1+x)(1+x^2) + Dx(1-x^2) + Ex^2(1-x^2). \quad (2)$$

Let $x = 0$; then, $1 = A \cdot 1$, whence $A = 1$.

Let $x = 1$; then, $8 = C(1 \cdot 2 \cdot 2)$, whence $C = 2$.

Let $x = -1$; then, $0 = B(-1 \cdot 2 \cdot 2)$, whence $B = 0$.

Substituting these values in (2),

$$1+2x+3x^2+2x^3 = 1-x^4+2x+2x^2+2x^3+2x^4+(Dx+Ex^2)(1-x^2).$$

Canceling, etc., $x^2-x^4 = (Dx+Ex^2)(1-x^2)$.

Dividing by $x(1-x^2)$, $x = D + Ex$.

Equating the coefficients of like powers of x ,

$$D = 0 \text{ and } E = 1.$$

Hence,
$$\frac{1+2x+3x^2+2x^3}{x-x^5} = \frac{1}{x} + \frac{2}{1-x} + \frac{x}{1+x^2}.$$

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$$10. \quad \frac{1}{2} = x + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{60} + \dots$$

$$\begin{aligned} \text{Reverting,} \quad x &= 1\left(\frac{1}{2}\right) - \frac{1}{6}\left(\frac{1}{2}\right)^2 + \frac{1}{24}\left(\frac{1}{2}\right)^3 - \frac{1}{60}\left(\frac{1}{2}\right)^4 + \dots \\ &= \frac{1}{2} - \frac{1}{24} + \frac{1}{8 \times 72} - \frac{11}{2160 \times 16} + \dots \\ &= \frac{15889}{34560} = .45975. \end{aligned}$$

$$11. \quad \frac{1}{5} = x - \frac{x^2}{3} + \frac{3x^3}{10} - \frac{2x^4}{7} + \dots$$

$$\begin{aligned} \text{Reverting,} \quad x &= 1\left(\frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{5}\right)^2 - \frac{7}{90}\left(\frac{1}{5}\right)^3 + \frac{11}{378}\left(\frac{1}{5}\right)^4 + \dots \\ &= .2 + \frac{1}{75} - \frac{7}{90 \times 125} - \frac{11}{378 \times 625} + \dots \\ &= .2 + .01266 + = .21266. \end{aligned}$$

PERMUTATIONS AND COMBINATIONS

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4. The number of ways is equal to the number of permutations of 4 things taken all together.

$$P_4^4 = \underline{4} = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

$$5. \quad P_5^5 = \underline{5} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

6. The first prize may be awarded to any one of the 10 athletes, and after that the second prize may be awarded to any one of the 9 athletes remaining. Hence, the number of ways is 10×9 , or 90.

$$7. \quad P_7^7 = \underline{7} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

8. At the bottom of the mountain the traveler has the choice of 5 routes, and having reached the top by any one of them, he may return by any one of the four remaining routes, since he may return by any of the routes except the one he has taken to go up. Hence, the number of ways is 5×4 , or 20.

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$$3. \quad C_3^{12} = C_3^{12} = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220.$$

$$4. \quad C_2^{10} = \frac{10 \cdot 9}{1 \cdot 2} = 45; \quad C_8^{12} = C_4^{12} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495;$$

$$C_{22}^{25} = C_3^{25} = \frac{25 \cdot 24 \cdot 23}{1 \cdot 2 \cdot 3} = 2300.$$

$$5. \quad C_5^{52} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 2598960.$$

6. $C_8^{10} = C_2^{10}$, which is less than C_3^{10} , since $C_2^{10} = \frac{10 \cdot 9}{1 \cdot 2}$ while $C_3^{10} = \frac{10 \cdot 9}{1 \cdot 2} \times \frac{8}{3}$; $C_4^{10} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4}$ and $C_5^{10} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{6}{5}$.

Hence, $C_3^{10} > C_8^{10}$, and $C_5^{10} > C_4^{10}$.

7. From 11 Republicans 6 Republicans may be selected in C_6^{11} ways; from 10 Democrats 5 Democrats can be selected in C_5^{10} ways.

Since each combination of 6 Republicans may be associated with each combination of 5 Democrats to form a committee, the number of committees that can be selected is equal to

$$C_6^{11} \times C_5^{10} = C_5^{11} \times C_6^{10} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 116424.$$

8. He may have to make 10 trials to hit upon the right mark on the first wheel, 13 trials, the second wheel, and 13 trials, the third wheel. In order to get the right combination, therefore, he may have to make $10 \times 13 \times 13$, or 1690, trials.

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11. From 20 consonants 3 consonants may be selected in C_3^{20} ways; from 5 vowels 3 vowels can be selected in C_3^5 ways.

Since each combination of 3 consonants may be associated with each combination of 3 vowels to form a word of six letters, the number of words of six letters differing in the letters composing them is equal to

$$C_3^{20} \times C_3^5 = C_3^{20} \times C_2^5 = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} \times \frac{5 \cdot 4}{1 \cdot 2} = 11400.$$

Finally, since from each of these 11400 words of six letters [6 words may be formed by permuting the six letters in their places, the whole number of words that may be formed under the conditions of the problem is equal to

$$11400 \times [6] = 11400 \times 720 = 8208000.$$

12. Since there are 5 different coins and these may be taken 1, 2, 3, 4, or 5 at a time, the total number of different sums that may be paid is equal to

$$C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = \frac{5}{1} + \frac{5 \cdot 4}{1 \cdot 2} + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} + 1 \\ = 5 + 10 + 10 + 5 + 1 = 31.$$

13. The number of boys selected may be 2 or 3 or 4 or 5, and the corresponding number of girls is therefore 4 or 3 or 2 or 1.

From 5 boys 2 boys may be selected in C_2^5 ways and from 5 girls 4 girls may be selected in C_4^5 ways.

Since those committees of 6 which are composed of 2 boys and 4 girls may be formed by associating each group of 2 boys with each group of 4 girls, the number of these committees is $C_2^5 \times C_4^5$.

Similarly, the number of committees of 6 composed of 3 boys and 3 girls is $C_3^5 \times C_3^5$; of 4 boys and 2 girls is $C_4^5 \times C_2^5$; of 5 boys and 1 girl is $C_5^5 \times C_1^5$.

Hence, the whole number of committees of 6 is equal to

$$C_2^5 \times C_4^5 + C_3^5 \times C_3^5 + C_4^5 \times C_2^5 + C_5^5 \times C_1^5 = 10 \cdot 5 + 10 \cdot 10 + 5 \cdot 10 + 1 \cdot 5 \\ = 205.$$

14. $(a + b)$ soldiers are divided into groups of c , m , and n soldiers, respectively. From $(m + n + c)$ soldiers c soldiers may be detailed in C_c^{m+n+c} ways and from $(m + n)$ soldiers m soldiers may be selected in C_m^{m+n} , or C_n^{m+n} , ways, leaving in each case a group of n soldiers.

Since for every one of the C_c^{m+n+c} ways in which c soldiers may be detailed to garrison the fort the remaining $m + n$ soldiers may be divided into two parties in C_m^{m+n} ways, the number of ways of grouping the soldiers under the conditions of the problem is equal to

$$C_c^{m+n+c} \times C_m^{m+n} = \frac{m+n+c}{c} \times \frac{m+n}{m} = \frac{m+n+c}{m}.$$

16. Let

$$3 C_3^n = 2 C_4^{n+1}.$$

Then,

$$3 \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = 2 \frac{(n+1)(n)(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

Dividing both members by $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$,

$$3 = \frac{2(n+1)}{4} = \frac{n+1}{2}.$$

$$\therefore n = 5.$$

Since $n = 5$,

$$C_3^n = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10,$$

and

$$C_4^{n+1} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} = 15.$$

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1. The number of orders is equal to the number of permutations of 6 things in a circle taken 6 at a time.

$$P_6^6 \text{ (circular)} = \underline{5} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

2.

$$P_8^8 \text{ (circular)} = \underline{7} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

3. If each gentleman sits opposite his wife, when the wives have been seated, which can be done in $P_4^4 \text{ (circular)} = \underline{3} = 3 \cdot 2 \cdot 1 = 6$ ways, each gentleman has but one place to sit. Hence, the number of orders is 6.

4. If each gentleman sits opposite a lady, when the ladies have been seated in any one of the 6 ways possible, the gentlemen may be seated in the 4 vacant places also in 6 ways. Hence, the number of orders is 6×6 , or 36.

5.

$$P_7^7 \text{ (circular)} = \underline{6} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

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2. Since three of the seven letters of the word *zoölogy* are alike, the number of permutations of the letters all at a time is $\frac{1}{3}$ of $\underline{7}$, or 840.

In the word *coefficient* there are two *c*'s, two *e*'s, two *f*'s, and two *i*'s, and in all eleven letters. Hence, the number of permutations is

$$\frac{\underline{11}}{\underline{2} \underline{2} \underline{2} \underline{2}} = 2404800.$$

In the word *ecclesiastical* there are two *e*'s, three *c*'s, two *l*'s, two *s*'s, two *i*'s, and two *a*'s, and in all fourteen letters. Hence, the number of permutations is

$$\frac{14}{\begin{array}{|c|c|c|c|c|c|} \hline 2 & 3 & 2 & 2 & 2 & 2 \\ \hline \end{array}} = 454053600.$$

In the word *divisibility* there are five *i*'s and in all twelve letters. Hence, the number of permutations is

$$\frac{12}{\begin{array}{|c|} \hline 5 \\ \hline \end{array}} = 3991680.$$

3. The number is $\frac{11}{\begin{array}{|c|c|c|} \hline 4 & 5 & 2 \\ \hline \end{array}} = 6930.$

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2. Total $C^{10} = 2^{10} - 1 = 1023.$

3. Total $C^5 = 2^5 - 1 = 31.$

4. Total $C^7 = 2^7 - 1 = 127.$

5. (a) As many as the number of permutations of the letters in *count*, *er* being attached to each permutation.

$$P_5^5 = \begin{array}{|c|} \hline 5 \\ \hline \end{array} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

(b) With *n* as the middle letter the six other letters may be permuted in $P_6^6 = \begin{array}{|c|} \hline 6 \\ \hline \end{array} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ ways.

(c) If each vowel keeps its position the four consonants may be permuted in $P_4^4 = \begin{array}{|c|} \hline 4 \\ \hline \end{array} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways.

(d) The first letter may be taken in 4 ways, since any of the four consonants may be taken. The six letters remaining may be permuted in P_6^6 , or $\begin{array}{|c|} \hline 6 \\ \hline \end{array}$, ways, and each of these permutations may be attached to any one of the 4 initial letters.

Hence, the number of permutations beginning with a consonant is $4 \begin{array}{|c|} \hline 6 \\ \hline \end{array} = 4 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 2880.$

6. There are 4 odd places and 3 even places. To fill the odd places two different digits, 1 and 3, are to be selected, and this can be done in C_2^4 ways, since there are two 1's and two 3's. To fill the even places two different digits, 2 and 4, are to be selected, and this can be done in C_2^3 ways since there are two 2's and one 4.

Hence, the whole number of ways is equal to

$$C_2^4 \times C_2^3 = \frac{4 \cdot 3}{1 \cdot 2} \times \frac{3 \cdot 2}{1 \cdot 2} = 18.$$

7.
$$\begin{aligned} P_5^n &= 24 P_2^n = P_2^n \times 4 \times 3 \times 2. \\ n(n-1)(n-2)(n-3)(n-4) &= n(n-1) \times 4 \times 3 \times 2. \\ (n-2)(n-3)(n-4) &= 4 \times 3 \times 2 = (6-2)(6-3)(6-4). \\ \therefore n &= 6. \end{aligned}$$

DETERMINANTS

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3-9. See pp. 418-419 Key.

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4-10. See pp. 420-421 Key.

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1-4. See p. 422 Key.

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11-23. See pp. 419-420 Key.

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2-4. See p. 422 Key.

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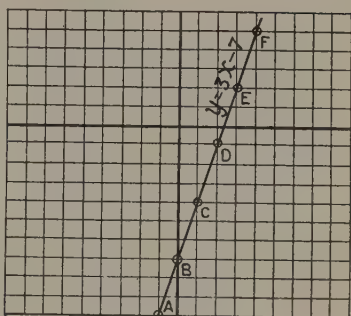
5-17. See pp. 423-428 Key.

GRAPHIC ALGEBRA

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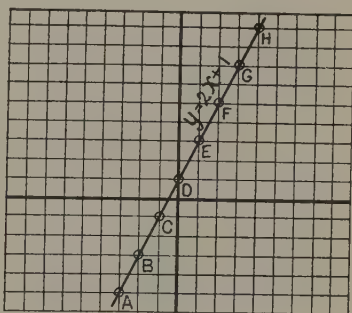
16. $y = 3x - 7$.

| x | y | POINT |
|-----|-----|-------|
| -1 | -10 | A |
| 0 | -7 | B |
| 1 | -4 | C |
| 2 | -1 | D |
| 3 | 2 | E |
| 4 | 5 | F |

A line drawn through A, B, C, D, etc., is the graph of $y = 3x - 7$.

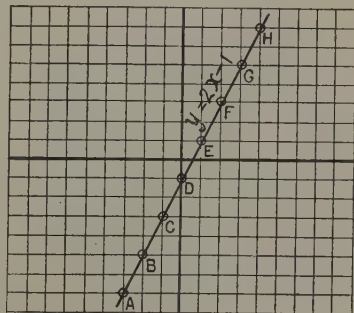
17. $y = 2x + 1$.

| x | y | POINT |
|-----|-----|-------|
| -3 | -5 | A |
| -2 | -3 | B |
| -1 | -1 | C |
| 0 | 1 | D |
| 1 | 3 | E |
| 2 | 5 | F |
| 3 | 7 | G |
| 4 | 9 | H |

A line drawn through A, B, C, D, etc., is the graph of $y = 2x + 1$.

18. $y = 2x - 1$.

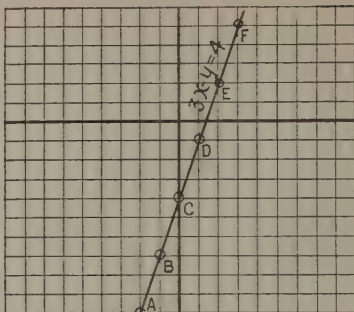
| x | y | POINT |
|-----|-----|-------|
| -3 | -7 | A |
| -2 | -5 | B |
| -1 | -3 | C |
| 0 | -1 | D |
| 1 | 1 | E |
| 2 | 3 | F |
| 3 | 5 | G |
| 4 | 7 | H |

A line drawn through A, B, C, D, etc., is the graph of $y = 2x - 1$.

19. Solving for y ,

$$y = 3x - 4.$$

| x | y | POINT |
|-----|-----|----------|
| -2 | -10 | <i>A</i> |
| -1 | -7 | <i>B</i> |
| 0 | -4 | <i>C</i> |
| 1 | -1 | <i>D</i> |
| 2 | 2 | <i>E</i> |
| 3 | 5 | <i>F</i> |

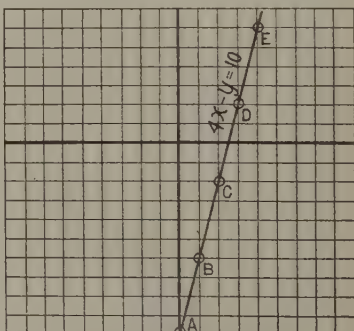


A line drawn through A , B , C , D , etc., is the graph of $3x - y = 4$.

20. Solving for y ,

$$y = 4x - 10.$$

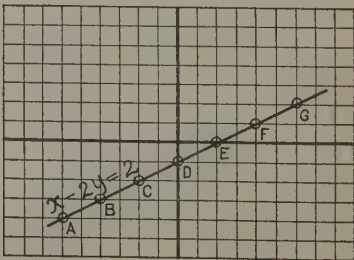
| x | y | POINT |
|-----|-----|----------|
| 0 | -10 | <i>A</i> |
| 1 | -6 | <i>B</i> |
| 2 | -2 | <i>C</i> |
| 3 | 2 | <i>D</i> |
| 4 | 6 | <i>E</i> |



A line drawn through A , B , C , D , etc., is the graph of $4x - y = 10$.

21. Solving for y , $y = \frac{1}{2}(x - 2)$.

| x | y | POINT |
|-----|-----|----------|
| -6 | -4 | <i>A</i> |
| -4 | -3 | <i>B</i> |
| -2 | -2 | <i>C</i> |
| 0 | -1 | <i>D</i> |
| 2 | 0 | <i>E</i> |
| 4 | 1 | <i>F</i> |
| 6 | 2 | <i>G</i> |

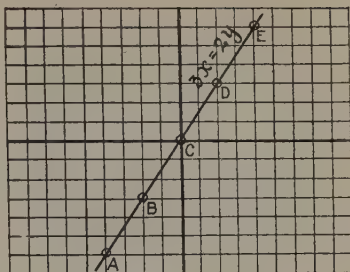


A line drawn through A , B , C , D , etc., is the graph of $x - 2y = 2$.

22. Solving for y ,

$$y = \frac{3}{2}x.$$

| x | y | POINT |
|-----|-----|----------|
| -4 | -6 | <i>A</i> |
| -2 | -3 | <i>B</i> |
| 0 | 0 | <i>C</i> |
| 2 | 3 | <i>D</i> |
| 4 | 6 | <i>E</i> |

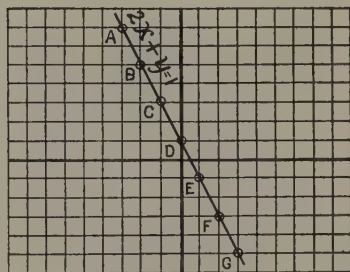


A line drawn through A , B , C , D , etc., is the graph of $3x = 2y$.

23. Solving for y ,

$$y = 1 - 2x.$$

| x | y | POINT |
|-----|-----|----------|
| -3 | 7 | <i>A</i> |
| -2 | 5 | <i>B</i> |
| -1 | 3 | <i>C</i> |
| 0 | 1 | <i>D</i> |
| 1 | -1 | <i>E</i> |
| 2 | -3 | <i>F</i> |
| 3 | -5 | <i>G</i> |

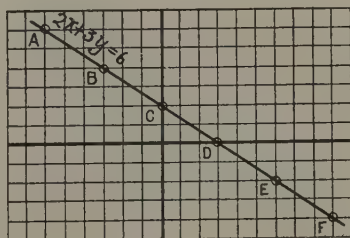


A line drawn through A , B , C , D , etc., is the graph of $2x + y = 1$.

24. Solving for y ,

$$y = 2 - \frac{2}{3}x.$$

| x | y | POINT |
|-----|-----|----------|
| -6 | 6 | <i>A</i> |
| -3 | 4 | <i>B</i> |
| 0 | 2 | <i>C</i> |
| 3 | 0 | <i>D</i> |
| 6 | -2 | <i>E</i> |
| 9 | -4 | <i>F</i> |



A line drawn through A , B , C , D , etc., is the graph of $2x + 3y = 6$.

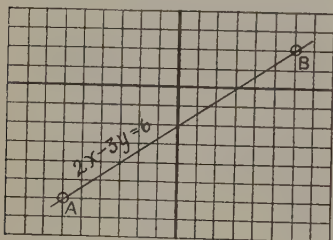
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1. Solving for y , $y = \frac{2}{3}x - 2$.

When $x = -6$, $y = -6$;
when $x = 6$, $y = 2$.

Locate $A = (-6, -6)$, $B = (6, 2)$.

A straight line drawn through A and B is the graph of $2x - 3y = 6$

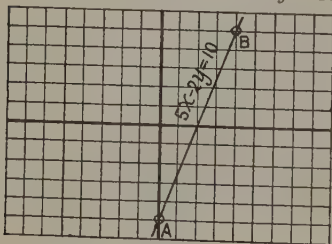


3. Solving for y , $y = \frac{5}{2}x - 5$.

When $x = 0$, $y = -5$;
when $x = 4$, $y = 5$.

Locate $A = (0, -5)$, $B = (4, 5)$.

A straight line drawn through A and B is the graph of $5x - 2y = 10$.

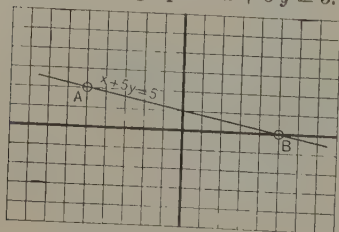


5. Solving for y , $y = 1 - \frac{1}{5}x$.

When $x = -5$, $y = 2$;
when $x = 5$, $y = 0$.

Locate $A = (-5, 2)$, $B = (5, 0)$.

A straight line drawn through A and B is the graph of $x + 5y = 5$.

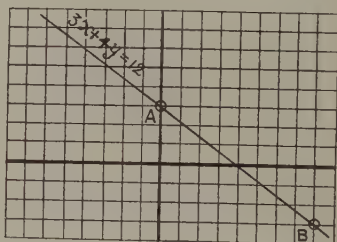


2. Solving for y , $y = 3 - \frac{3}{4}x$.

When $x = 0$, $y = 3$;
when $x = 8$, $y = -3$.

Locate $A = (0, 3)$, $B = (8, -3)$.

A straight line drawn through A and B is the graph of $3x + 4y = 12$.

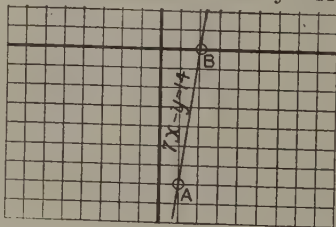


4. Solving for y , $y = 7x - 14$.

When $x = 1$, $y = -7$;
when $x = 2$, $y = 0$.

Locate $A = (1, -7)$, $B = (2, 0)$.

A straight line drawn through A and B is the graph of $7x - y = 14$.

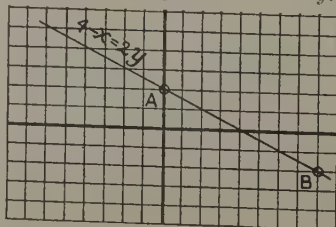


6. Solving for y , $y = 2 - \frac{1}{2}x$.

When $x = 0$, $y = 2$;
when $x = 8$, $y = -2$.

Locate $A = (0, 2)$, $B = (8, -2)$.

A straight line drawn through A and B is the graph of $4 - x = 2y$.



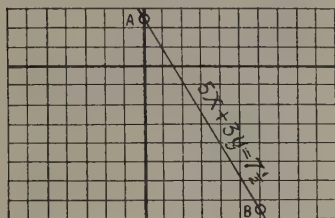
7. Solving for y , $y = \frac{5}{2} - \frac{5}{3}x$.

When $x = 0$, $y = 2\frac{1}{2}$;

when $x = 6$, $y = -7\frac{1}{2}$.

Locate $A = (0, 2\frac{1}{2})$, $B = (6, -7\frac{1}{2})$.

A straight line drawn through A and B is the graph of $5x + 3y = 7\frac{1}{2}$.



9. Solving for y , $y = 6 - \frac{3}{2}x$.

When $x = 0$, $y = 6$;

when $x = 6$, $y = -3$.

Locate $A = (0, 6)$, $B = (6, -3)$.

A straight line drawn through A and B is the graph of $\frac{1}{2}x + \frac{1}{3}y = 2$.

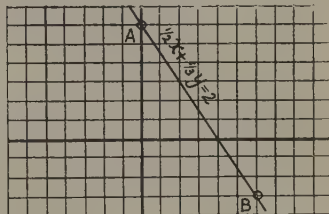
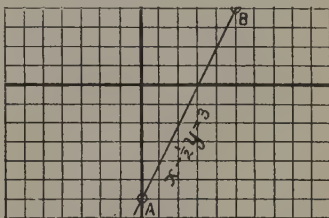
8. Solving for y , $y = 2x - 6$.

When $x = 0$, $y = -6$;

when $x = 5$, $y = 4$.

Locate $A = (0, -6)$, $B = (5, 4)$.

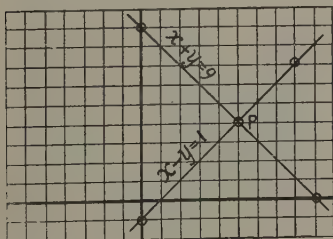
A straight line drawn through A and B is the graph of $x - \frac{1}{2}y = 3$.



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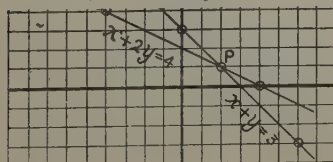
2. The graphs of the equations intersect at $P = (5, 4)$.

Hence, $x = 5$ and $y = 4$.



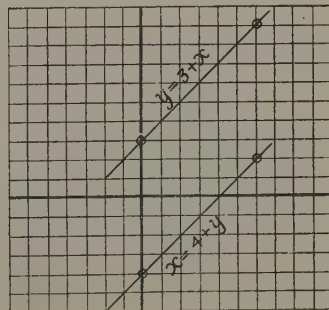
3. The graphs of the equations intersect at $P = (2, 1)$.

Hence, $x = 2$ and $y = 1$.



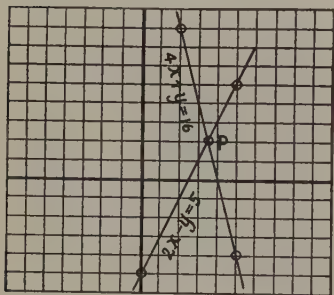
4. Since the graphs of the equations are everywhere seven units apart vertically, they are parallel straight lines and have no point in common.

Hence, the equations are *inconsistent*.

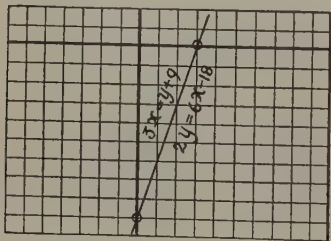


5. The graphs of the equations intersect at $P = (3.5, 2)$.

Hence, $x = 3.5$ and $y = 2$.

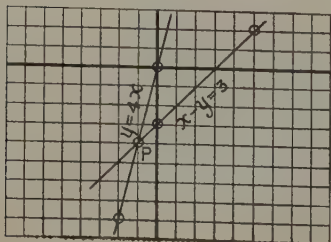


6. The graphs of the equations coincide. Hence, the equations are *indeterminate*.

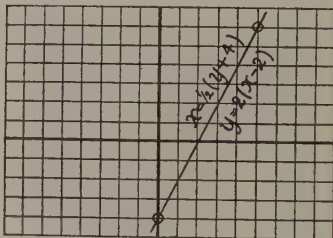


7. The graphs of the equations intersect at $P = (-1, -4)$.

Hence, $x = -1$ and $y = -4$.

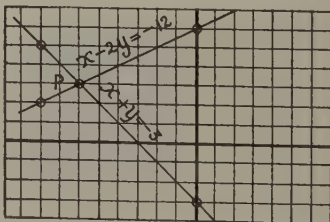


8. The graphs of the equations coincide. Hence, the equations are *indeterminate*.

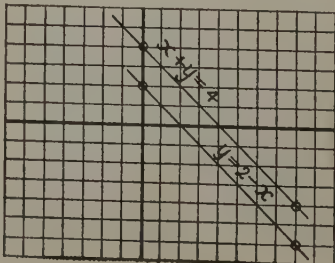


9. The graphs of the equations intersect at $P = (-6, 3)$.

Hence, $x = -6$ and $y = 3$.



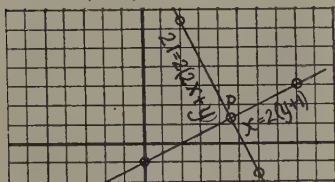
10. Since the graphs of the equations are everywhere two units apart vertically, they are parallel straight lines and have no point in common. Hence, the equations are *inconsistent*.



11. The graphs of the equations intersect approximately at

$$P = (4.6, 1.3).$$

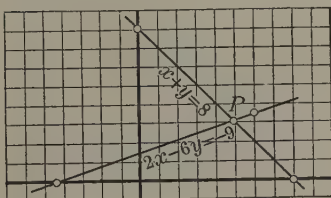
Hence, $x = 4.6$ and $y = 1.3$.



12. The graphs of the equations intersect approximately at

$$P = (4.9, 3.1).$$

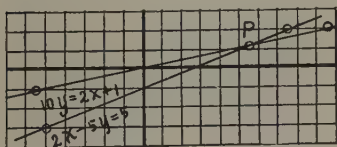
Hence, $x = 4.9$ and $y = 3.1$.



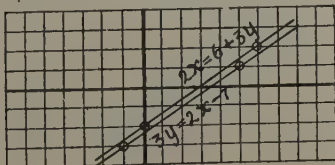
13. The graphs of the equations intersect approximately at

$$P = (5.5, 1.2).$$

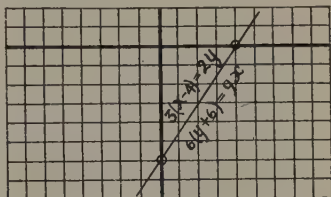
Hence, $x = 5.5$ and $y = 1.2$.



14. The graphs of the equations are parallel straight lines and have no point in common. Hence, the equations are *inconsistent*.



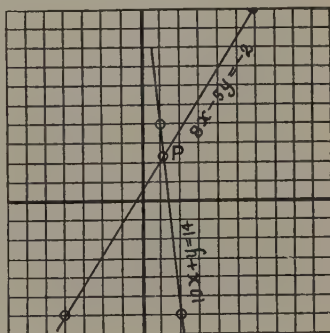
15. The graphs of the equations coincide. Hence, the equations are *indeterminate*.



16. The graphs of the equations intersect approximately at

$$P = (1.2, 2.3).$$

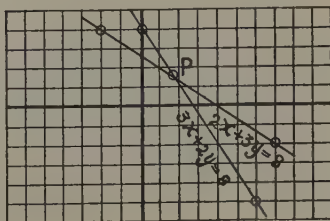
Hence, $x = 1.2$ and $y = 2.3$.



17. The graphs of the equations intersect approximately at

$$P = (1.6, 1.6).$$

Hence, $x = 1.6$ and $y = 1.6$.

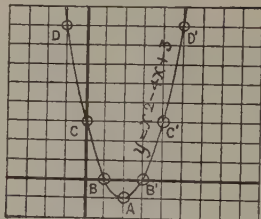


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1. Since the coefficient of x is -4 , § 544, first substitute 2 for x .

$$y = x^2 - 4x + 3.$$

| x | y | POINTS |
|-------|-----|---------|
| 2 | -1 | A |
| 1, 3 | 0 | B, B' |
| 0, 4 | 3 | C, C' |
| -1, 5 | 8 | D, D' |



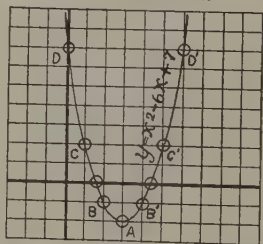
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 4x + 3$, which crosses the x -axis at 1 and 3.

Hence, the roots of $x^2 - 4x + 3 = 0$ are 1 and 3.

2. Since the coefficient of x is -6 , § 544, first substitute 3 for x .

$$y = x^2 - 6x + 7.$$

| x | y | POINTS |
|------|-----|---------|
| 3 | -2 | A |
| 2, 4 | -1 | B, B' |
| 1, 5 | 2 | C, C' |
| 0, 6 | 7 | D, D' |

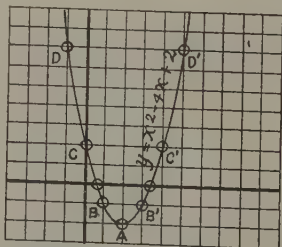


Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 6x + 7$, which crosses the x -axis approximately at 1.6 and 4.4. Hence, the roots of $x^2 - 6x + 7 = 0$ are 1.6 and 4.4.

3. Putting $x^2 - 4x = -2$ in the form $x^2 - 4x + 2 = 0$, since the coefficient of x is -4 , § 544, first substitute 2 for x .

$$y = x^2 - 4x + 2.$$

| x | y | POINTS |
|-------|-----|---------|
| 2 | -2 | A |
| 1, 3 | -1 | B, B' |
| 0, 4 | 2 | C, C' |
| -1, 5 | 7 | D, D' |



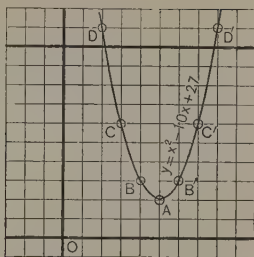
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 4x + 2$, which crosses the x -axis approximately at .6 and 3.4.

Hence, the roots of $x^2 - 4x + 2 = 0$, or of $x^2 - 4x = -2$, are .6 and 3.4.

4. Since the coefficient of x is -10 , § 544, first substitute 5 for x .

$$y = x^2 - 10x + 27.$$

| x | y | POINTS |
|------|-----|---------|
| 5 | 2 | A |
| 4, 6 | 3 | B, B' |
| 3, 7 | 6 | C, C' |
| 2, 8 | 11 | D, D' |



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 10x + 27$, whose minimum point lies above the x -axis.

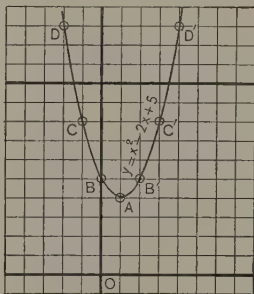
Hence, § 546, 4, the roots of $x^2 - 10x + 27 = 0$ are imaginary.

5. Since the coefficient of x is -2 ,

§ 544, first substitute 1 for x .

$$y = x^2 - 2x + 5.$$

| x | y | POINTS |
|---------|-----|---------|
| 1 | 4 | A |
| 0, 2 | 5 | B, B' |
| $-1, 3$ | 8 | C, C' |
| $-2, 4$ | 13 | D, D' |



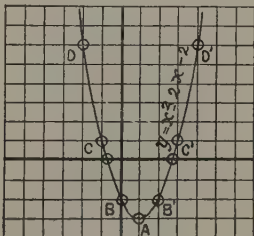
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 2x + 5$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 - 2x + 5 = 0$ are imaginary.

6. Putting $x^2 = 2(x + 1)$ in the form $x^2 - 2x - 2 = 0$, since the coefficient of x is -2 , § 544, first substitute 1 for x .

$$y = x^2 - 2x - 2.$$

| x | y | POINTS |
|---------|------|---------|
| 1 | -3 | A |
| 0, 2 | -2 | B, B' |
| $-1, 3$ | 1 | C, C' |
| $-2, 4$ | 6 | D, D' |



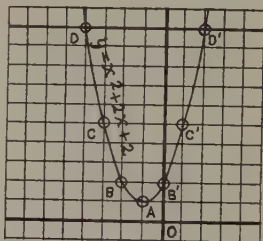
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 2x - 2$, which crosses the x -axis approximately at $-.7$ and 2.7 .

Hence, the roots of $x^2 - 2x - 2 = 0$, or of $x^2 = 2(x + 1)$, are $-.7$ and 2.7 .

7. Putting $x^2 + 2(x+1) = 0$ in the form $x^2 + 2x + 2 = 0$, since the coefficient of x is 2, § 544, first substitute -1 for x .

$$y = x^2 + 2x + 2.$$

| x | y | POINTS |
|---------|------|---------|
| -1 | 1 | A |
| $-2, 0$ | 2 | B, B' |
| $-3, 1$ | 5 | C, C' |
| $-4, 2$ | 10 | D, D' |



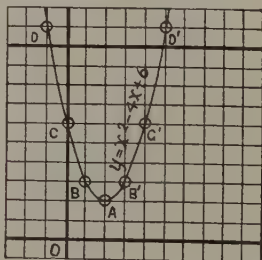
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 2x + 2$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 + 2x + 2 = 0$, or of $x^2 + 2(x+1) = 0$, are imaginary.

8. Since the coefficient of x is -4 , § 544, first substitute 2 for x .

$$y = x^2 - 4x + 6.$$

| x | y | POINTS |
|---------|------|---------|
| 2 | 2 | A |
| $1, 3$ | 3 | B, B' |
| $0, 4$ | 6 | C, C' |
| $-1, 5$ | 11 | D, D' |



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 4x + 6$, whose minimum point lies above the x -axis.

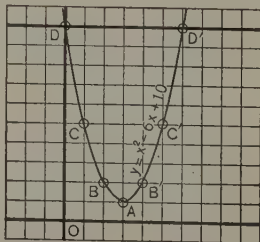
Hence, § 546, 4, the roots of $x^2 - 4x + 6 = 0$ are imaginary.

9. Since this equation is the same as the one in Ex. 6, though in different form, its graph and roots are the same. (For solution see Ex. 6.)

10. Putting $x^2 = 6x - 10$ in the form $x^2 - 6x + 10 = 0$, since the coefficient of x is -6 , § 544, first substitute 3 for x .

$$y = x^2 - 6x + 10.$$

| x | y | POINTS |
|--------|------|---------|
| 3 | 1 | A |
| $2, 4$ | 2 | B, B' |
| $1, 5$ | 5 | C, C' |
| $0, 6$ | 10 | D, D' |



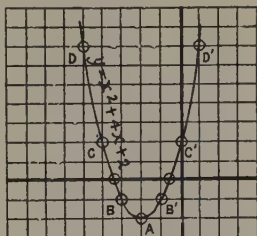
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 6x + 10$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 - 6x + 10 = 0$, or of $x^2 = 6x - 10$, are imaginary.

11. Since the coefficient of x is 4, § 544, first substitute -2 for x .

$$y = x^2 + 4x + 2.$$

| x | y | POINTS |
|----------|------|---------|
| -2 | -2 | A |
| $-3, -1$ | -1 | B, B' |
| $-4, 0$ | 2 | C, C' |
| $-5, 1$ | 7 | D, D' |



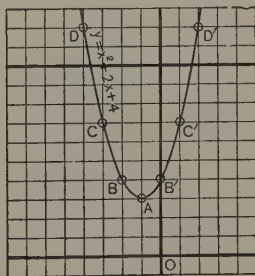
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 4x + 2$, which crosses the x -axis approximately at -3.4 and $-.6$.

Hence, the roots of $x^2 + 4x + 2 = 0$ are -3.4 and $-.6$.

12. Since the coefficient of x is 2, § 544, first substitute -1 for x .

$$y = x^2 + 2x + 4.$$

| x | y | POINTS |
|---------|------|---------|
| -1 | 3 | A |
| $-2, 0$ | 4 | B, B' |
| $-3, 1$ | 7 | C, C' |
| $-4, 2$ | 12 | D, D' |



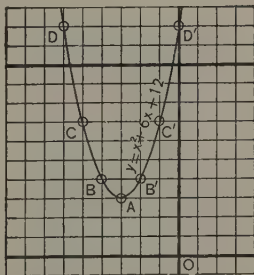
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 2x + 4$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 + 2x + 4 = 0$ are imaginary.

13. Since the coefficient of x is 6, § 544, first substitute -3 for x .

$$y = x^2 + 6x + 12.$$

| x | y | POINTS |
|----------|------|---------|
| -3 | 3 | A |
| $-4, -2$ | 4 | B, B' |
| $-5, -1$ | 7 | C, C' |
| $-6, 0$ | 12 | D, D' |



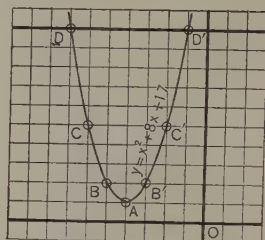
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 6x + 12$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 + 6x + 12 = 0$ are imaginary.

14. Since the coefficient of x is 8, § 544, first substitute -4 for x .

$$y = x^2 + 8x + 17.$$

| x | y | POINTS |
|----------|------|---------|
| -4 | 1 | A |
| $-5, -3$ | 2 | B, B' |
| $-6, -2$ | 5 | C, C' |
| $-7, -1$ | 10 | D, D' |



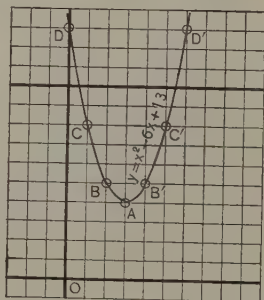
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 8x + 17$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 + 8x + 17 = 0$ are imaginary.

15. Since the coefficient of x is -6 , § 544, first substitute 3 for x .

$$y = x^2 - 6x + 13.$$

| x | y | POINTS |
|--------|------|---------|
| 3 | 4 | A |
| $2, 4$ | 5 | B, B' |
| $1, 5$ | 8 | C, C' |
| $0, 6$ | 13 | D, D' |



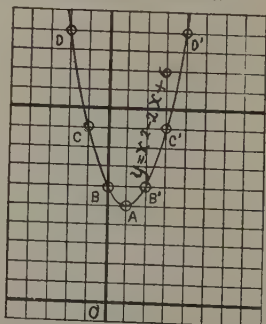
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 6x + 13$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 - 6x + 13 = 0$ are imaginary.

16. Since the coefficient of x is -2 , § 544, first substitute 1 for x .

$$y = x^2 - 2x + 6.$$

| x | y | POINTS |
|---------|------|---------|
| 1 | 5 | A |
| $0, 2$ | 6 | B, B' |
| $-1, 3$ | 9 | C, C' |
| $-2, 4$ | 14 | D, D' |



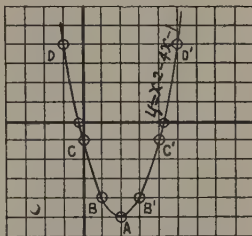
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 2x + 6$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 - 2x + 6 = 0$ are imaginary.

17. Since the coefficient of x is -4 , § 544, first substitute 2 for x .

$$y = x^2 - 4x - 1.$$

| x | y | POINTS |
|-------|-----|---------|
| 2 | -5 | A |
| 1, 3 | -4 | B, B' |
| 0, 4 | -1 | C, C' |
| -1, 5 | 4 | D, D' |



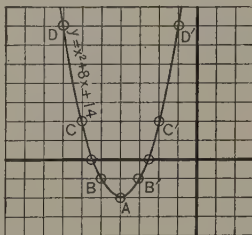
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 4x - 1$, which crosses the x -axis approximately at $-.2$ and 4.2 .

Hence, the roots of $x^2 - 4x - 1 = 0$ are $-.2$ and 4.2 .

18. Since the coefficient of x is 8, § 544, first substitute -4 for x .

$$y = x^2 + 8x + 14.$$

| x | y | POINTS |
|--------|-----|---------|
| -4 | -2 | A |
| -5, -3 | -1 | B, B' |
| -6, -2 | 2 | C, C' |
| -7, -1 | 7 | D, D' |



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 8x + 14$, which crosses the x -axis approximately at -2.6 and -5.4 .

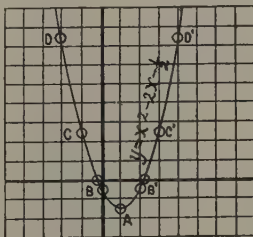
Hence, the roots of $x^2 + 8x + 14 = 0$ are -2.6 and -5.4 .

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19. Since in $y = x^2 - 2x - \frac{1}{2}$, the coefficient of x is -2 , § 544, first substitute 1 for x .

$$y = x^2 - 2x - \frac{1}{2}.$$

| x | y | POINTS |
|-------|-----------------|---------|
| 1 | $-1\frac{1}{2}$ | A |
| 0, 2 | $-\frac{1}{2}$ | B, B' |
| -1, 3 | $2\frac{1}{2}$ | C, C' |
| -2, 4 | $7\frac{1}{2}$ | D, D' |



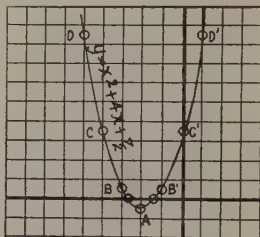
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 2x - \frac{1}{2}$, which crosses the x -axis approximately at $-.2$ and 2.2 .

Hence, the roots of $x^2 - 2x - \frac{1}{2} = 0$, or of $4x - 2x^2 + 1 = 0$, are $-.2$ and 2.2 .

20. Putting $2x^2 + 8x + 7 = 0$ in the form $x^2 + 4x + \frac{7}{2} = 0$, since the coefficient of x is 4, § 544, first substitute -2 for x .

$$y = x^2 + 4x + \frac{7}{2}.$$

| x | y | POINTS |
|----------|----------------|---------|
| -2 | $-\frac{1}{2}$ | A |
| $-3, -1$ | $\frac{1}{2}$ | B, B' |
| $-4, 0$ | $3\frac{1}{2}$ | C, C' |
| $-5, 1$ | $8\frac{1}{2}$ | D, D' |



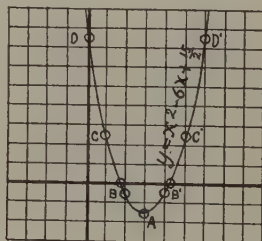
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 4x + \frac{7}{2}$, which crosses the x -axis approximately at -2.7 and -1.3 .

Hence, the roots of $x^2 + 4x + \frac{7}{2} = 0$, or of $2x^2 + 8x + 7 = 0$, are -2.7 and -1.3 .

21. Putting $2x^2 - 12x + 15 = 0$ in the form $x^2 - 6x + \frac{5}{2} = 0$, since the coefficient of x is -6 , § 544, first substitute 3 for x .

$$y = x^2 - 6x + \frac{5}{2}.$$

| x | y | POINTS |
|--------|-----------------|---------|
| 3 | $-1\frac{1}{2}$ | A |
| $2, 4$ | $-\frac{1}{2}$ | B, B' |
| $1, 5$ | $2\frac{1}{2}$ | C, C' |
| $0, 6$ | $7\frac{1}{2}$ | D, D' |



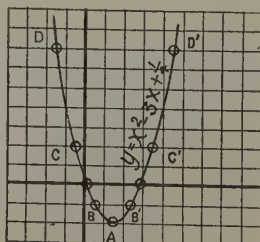
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 6x + \frac{5}{2}$, which crosses the x -axis approximately at 1.8 and 4.2 .

Hence, the roots of $x^2 - 6x + \frac{5}{2} = 0$, or of $2x^2 - 12x + 15 = 0$, are 1.8 and 4.2 .

22. Putting $12x - 4x^2 - 1 = 0$ in the form $x^2 - 3x + \frac{1}{4} = 0$, since the coefficient of x is -3 , § 544, first substitute $1\frac{1}{2}$ for x .

$$y = x^2 - 3x + \frac{1}{4}.$$

| x | y | POINTS |
|-------------------------------|------|---------|
| $1\frac{1}{2}$ | -2 | A |
| $\frac{1}{2}, 2\frac{1}{2}$ | -1 | B, B' |
| $-\frac{1}{2}, 3\frac{1}{2}$ | 2 | C, C' |
| $-1\frac{1}{2}, 4\frac{1}{2}$ | 7 | D, D' |



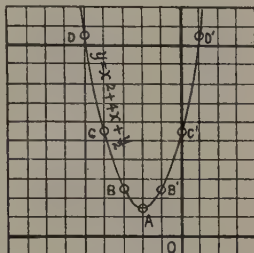
Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 3x + \frac{1}{4}$, which crosses the x -axis approximately at .1 and 2.9.

Hence, the roots of $x^2 - 3x + \frac{1}{4} = 0$, or of $12x - 4x^2 - 1 = 0$, are .1 and 2.9.

23. Putting $11 + 8x + 2x^2 = 0$ in the form $x^2 + 4x + \frac{11}{2} = 0$, since the coefficient of x is 4, § 544, first substitute -2 for x .

$$y = x^2 + 4x + \frac{11}{2}.$$

| x | y | POINTS |
|----------|-----------------|---------|
| -2 | $1\frac{1}{2}$ | A |
| $-3, -1$ | $2\frac{1}{2}$ | B, B' |
| $-4, 0$ | $5\frac{1}{2}$ | C, C' |
| $-5, 1$ | $10\frac{1}{2}$ | D, D' |



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 + 4x + \frac{11}{2}$, whose minimum point lies above the x -axis.

Hence, § 546, 4, the roots of $x^2 + 4x + \frac{11}{2} = 0$, or of $11 + 8x + 2x^2 = 0$, are imaginary.

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6. Imagine the circle $x^2 + y^2 = 25$ in example 5 to become smaller and smaller until it coincides with the circle $x^2 + y^2 = 9$ (see dotted circle in the cut). The four real unequal roots represented by the coordinates of the points of intersection of the graphs come together in pairs at the points $(3, 0)$ and $(-3, 0)$ where the circle $x^2 + y^2 = 9$ is tangent to the hyperbola $4x^2 - 9y^2 = 36$; consequently, the equations $4x^2 - 9y^2 = 36$ and $x^2 + y^2 = 9$ have two pairs of equal real roots, namely :

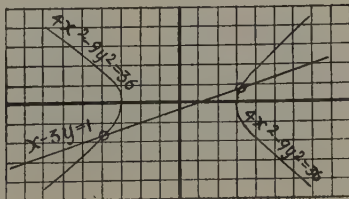
$$\begin{cases} x = 3, 3, -3, -3; \\ y = 0, 0, 0, 0. \end{cases}$$

7. Solving $4x^2 - 9y^2 = 36$ for y , $y = \pm \frac{2}{3}\sqrt{x^2 - 9}$.

Since any value numerically less than 3 substituted for x will make the value of y imaginary, no point of the graph lies between $x = +3$ and $x = -3$; consequently, we substitute for x only ± 3 and values numerically greater than 3.

Corresponding values of x and y are given in the table :

| x | y |
|---------|-----------|
| ± 3 | 0 |
| ± 4 | ± 1.8 |
| ± 5 | ± 2.7 |
| ± 6 | ± 3.5 |



Plotting these fourteen points, it is found that half of them are on one side of the y -axis and half on the other side, and since there are no points of the curve between $x = +3$ and $x = -3$, the graph has two separate branches.

Drawing a smooth curve through each group of points, the two branches thus constructed constitute the graph of the equation $4x^2 - 9y^2 = 36$, which is an hyperbola.

The graph of the equation $x - 3y = 1$ (§ 538) is a straight line.

The straight line intersects the hyperbola approximately at the points (3.2, .7) and (-3.8, -1.6). Hence, there are two real roots, namely :

$$\begin{cases} x = 3.2, & -3.8; \\ y = .7, & -1.6. \end{cases}$$

8. The graph of $4x^2 - 9y^2 = 36$, the same as that constructed in example 7, is an hyperbola.

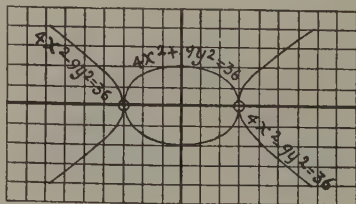
By a similar method we may find the graph of $4x^2 + 9y^2 = 36$.

Solving $4x^2 + 9y^2 = 36$ for y , $y = \pm \frac{2}{3}\sqrt{9 - x^2}$.

Since any value numerically greater than 3 substituted for x will make the value of y imaginary, no point of the graph lies farther to the right or to the left of the origin than 3 units ; consequently, we substitute for x only values between and including $+3$ and -3 .

Corresponding values of x and y are given in the table :

| x | y |
|---------|-----------|
| 0 | ± 2 |
| ± 1 | ± 1.6 |
| ± 2 | ± 1.5 |
| ± 3 | 0 |



Plotting these twelve points and drawing a smooth curve through them, we have the graph of $4x^2 + 9y^2 = 36$, which is an ellipse.

The hyperbola and the ellipse are tangent at the points (3, 0) and (-3, 0); consequently, the equations have two pairs of equal real roots, namely :

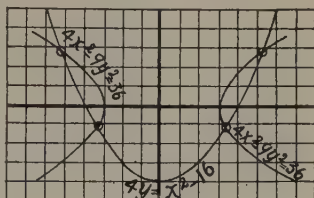
$$\begin{cases} x = 3, & 3, & -3, & -3; \\ y = 0, & 0, & 0, & 0. \end{cases}$$

9. The graph of $4x^2 - 9y^2 = 36$, the same as that constructed in example 7, is an hyperbola; and that of $4y = x^2 - 16$, constructed by solving for y and employing the method of § 543, is a parabola.

These graphs intersect approximately at the points (5.2, 2.8), (-5.2, 2.8), (3.4, -1.1), and (-3.4, -1.1).

Hence, the equations have four real roots, namely :

$$\begin{cases} x = 5.2, & -5.2, & 3.4, & -3.4; \\ y = 2.8, & 2.8, & -1.1, & -1.1. \end{cases}$$

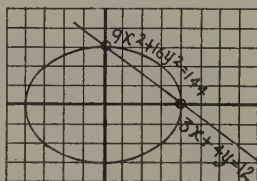


10. The graph of $9x^2 + 16y^2 = 144$, constructed by the method employed in example 8, is an ellipse; and that for $3x + 4y = 12$ (§ 538) is a straight line.

The straight line intersects the ellipse at the points (4, 0) and (0, 3).

Hence, the equations have two real roots,

$$\begin{cases} x = 4, & 0; \\ y = 0, & 3. \end{cases}$$



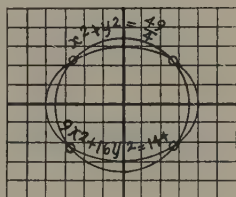
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11. The graph of $9x^2 + 16y^2 = 144$, constructed by the method employed in example 8, is an ellipse; and that for $x^2 + y^2 = \frac{49}{4}$, constructed by the method of example 1, is a circle.

These graphs intersect approximately at the points (2.7, 2.2), (2.7, -2.2), (-2.7, 2.2), and (-2.7, -2.2).

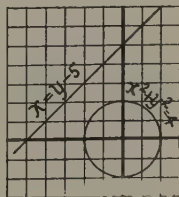
Hence, the equations have four real roots,

$$\begin{cases} x = 2.7, & 2.7, & -2.7, & -2.7; \\ y = 2.2, & -2.2, & 2.2, & -2.2. \end{cases}$$



12. The graph of $x^2 + y^2 = 4$, constructed by the method of example 1, is a circle; and that of $x = y - 5$ (§ 538) is a straight line.

Since one equation is simple and the other quadratic, they have two roots, and since their graphs have no points in common, the roots are imaginary.

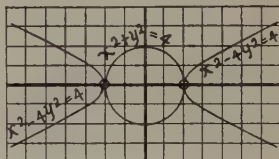


13. The graph of $x^2 - 4y^2 = 4$, constructed by the method of example 7, is an hyperbola; and that of $x^2 + y^2 = 4$, same as the first equation of example 12, is a circle.

The hyperbola and circle are tangent to each other at the points $(2, 0)$ and $(-2, 0)$.

Hence, the equations have two pairs of equal real roots, namely:

$$\begin{cases} x = 2, 2, -2, -2; \\ y = 0, 0, 0, 0. \end{cases}$$

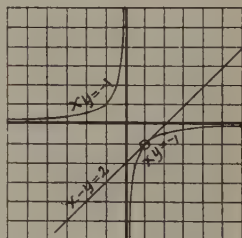


14. The graph of $x - y = 2$ (§ 538) is a straight line; and that of $xy = -1$, constructed by solving for y and substituting values of x just as in the case of a linear equation, is an hyperbola.

The straight line is tangent to the hyperbola at the point $(1, -1)$.

Hence, the equations have a pair of equal real roots, namely:

$$\begin{cases} x = 1, 1; \\ y = -1, -1. \end{cases}$$

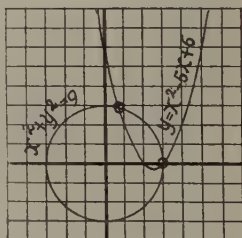


15. The graph of $x^2 + y^2 = 9$, constructed by the method of example 1, is a circle; and that of $y = x^2 - 5x + 6$, constructed by the method of § 543, is a parabola.

These two graphs intersect approximately at the points $(3, 0)$ and $(.7, 2.9)$.

Hence, the equations have two real roots, namely:

$$\begin{cases} x = 3, .7; \\ y = 0, 2.9. \end{cases}$$



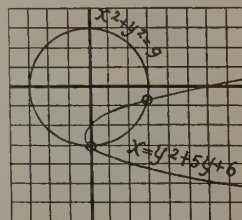
Since both equations are quadratic, they have four roots; hence, the other two roots are imaginary.

16. The graph of $x^2 + y^2 = 9$, constructed by the method of example 1, is a circle; and that of $x = y^2 + 5y + 6$, constructed by the method of § 543 by substituting values of y and solving for x , is a parabola.

These graphs intersect approximately at the points $(0, -3)$ and $(2.9, -.7)$.

Hence, the equations have two real roots, namely:

$$\begin{cases} x = 0, 2.9; \\ y = -3, -.7. \end{cases}$$

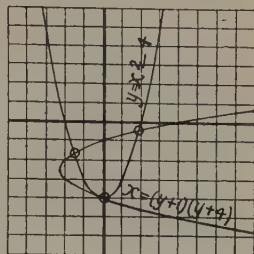


Since both equations are quadratic they have four roots; hence, the other two roots are imaginary.

17. The graph of $y = x^2 - 4$, constructed by the method of § 543, is a parabola; and that of $x = (y + 1)(y + 4)$, constructed by the same method by substituting values of y and solving for x , is also a parabola.

These graphs are tangent to each other at the point $(0, -4)$ and intersect approximately at the points $(1.8, -.5)$ and $(-1.6, -1.7)$.

Hence, the equations have a pair of equal real roots and two other real roots, namely:



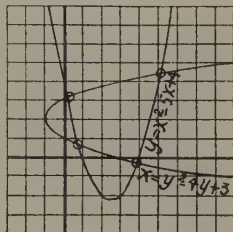
$$\begin{cases} x = 0, & 0, & 1.8, & -1.6; \\ y = -4, & -4, & -.5, & -1.7. \end{cases}$$

18. The graph of $y = x^2 - 5x + 4$, constructed by the method of § 543, is a parabola; and that of $x = y^2 - 4y + 3$, constructed by the same method by substituting values of y and solving for x , is also a parabola.

These graphs intersect approximately at the points $(.2, 3.1)$, $(.7, .7)$, $(3.9, -.2)$, and $(5.1, 4.5)$.

Hence, the equations have four real roots,

namely:
$$\begin{cases} x = .2, .7, & 3.9, & 5.1; \\ y = 3.1, .7, & -.2, & 4.5. \end{cases}$$



KEY TO ADVANCED ALGEBRA

Pages 7-371

Pages 7-371 of the Advanced Algebra are the same as pages 7-371 of the Academic Algebra. See the preceding part of this Key, pages 3-350.

IMAGINARY AND COMPLEX NUMBERS

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3. Let $\sqrt{x} + i\sqrt{y} = \sqrt{\frac{1}{3} + \frac{4}{3}\sqrt{-1}}.$ (1)

Then, $\sqrt{x} - i\sqrt{y} = \sqrt{\frac{1}{3} - \frac{4}{3}\sqrt{-1}}.$ (2)

Multiplying, $x + y = \sqrt{\frac{1}{9} + \frac{16}{9}} = \frac{5}{3}.$ (3)

Squaring (1), $x + 2i\sqrt{xy} - y = \frac{1}{3} + \frac{4}{3}\sqrt{-1}.$ (4)

Therefore, $x - y = \frac{1}{3}.$ (4)

Solving (3) and (4), $x = \frac{2}{3}, y = \frac{1}{3}.$
 $\therefore \sqrt{x} = \frac{2}{3}, i\sqrt{y} = \frac{1}{3}\sqrt{-1}.$

Hence, by (1), $\sqrt{\frac{1}{3} + \frac{4}{3}\sqrt{-1}} = \frac{2}{3} + \frac{1}{3}\sqrt{-1}.$

4. Let $\sqrt{x} - i\sqrt{y} = \sqrt{\frac{7}{4} - \sqrt{-15}}.$ (1)

Then, $\sqrt{x} + i\sqrt{y} = \sqrt{\frac{7}{4} + \sqrt{-15}}.$ (2)

Multiplying, $x + y = \sqrt{\frac{49}{16} + 15} = \frac{17}{4}.$ (3)

Squaring (1), $x - 2i\sqrt{xy} - y = \frac{7}{4} - \sqrt{-15}.$ (4)

Therefore, $x - y = \frac{7}{4}.$ (4)

Solving (3) and (4), $x = 3, y = \frac{1}{4}.$
 $\therefore \sqrt{x} = \sqrt{3}, i\sqrt{y} = \frac{1}{2}\sqrt{-1}.$

Hence, by (1), $\sqrt{\frac{7}{4} - \sqrt{-15}} = \sqrt{3} - \frac{1}{2}\sqrt{-1}.$

5. Let $\sqrt{x} - i\sqrt{y} = \sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}.$ (1)

Then, $\sqrt{x} + i\sqrt{y} = \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}}.$ (2)

Multiplying, $x + y = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$ (3)

Squaring (1), $x - 2i\sqrt{xy} - y = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$ (4)

Therefore, $x - y = -\frac{1}{2}.$ (4)

Solving (3) and (4), $x = \frac{1}{4}, y = \frac{3}{4}.$
 $\therefore \sqrt{x} = \frac{1}{2}, i\sqrt{y} = \frac{1}{2}\sqrt{-3}.$

Hence, by (1), $\sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}} = \frac{1}{2} - \frac{1}{2}\sqrt{-3}.$

6. Let $\sqrt{x} - i\sqrt{y} = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$. (1)

Then, $\sqrt{x} + i\sqrt{y} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{-3}}$. (2)

Multiplying, $x + y = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$. (3)

Squaring (1), $x - 2i\sqrt{xy} - y = \frac{1}{2} - \frac{1}{2}\sqrt{-3}$. (4)

Therefore, $x - y = \frac{1}{2}$. (4)

Solving (3) and (4), $x = \frac{3}{4}, y = \frac{1}{4}$.

$$\therefore \sqrt{x} = \frac{1}{2}\sqrt{3}, i\sqrt{y} = \frac{1}{2}\sqrt{-1}.$$

Hence, by (1), $\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{-3}} = \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{-1}.$

INEQUALITIES

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4-10. See pp. 350-351 Key.

12-34. See pp. 351-355 Key.

VARIABLES AND LIMITS

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Page 387

1-10. See pp. 355-356 Key.

1-15. See pp. 356-358 Key.

INTERPRETATION OF RESULTS

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1-4. See p. 358 Key.

5. See p. 359 Key.

1-5. See p. 359 Key.

INDETERMINATE EQUATIONS

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4-24. See pp. 360-363 Key.

25-35. See pp. 363-365 Key.

MATHEMATICAL INDUCTION

Page 407

1. By trial, $1^2 = \frac{1}{3}(2-1)(2+1)$, and $1^2 + 3^2 = \frac{2}{3}(4-1)(4+1)$. Assume that this law holds for the first n terms.

Then, $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n}{3}(2n-1)(2n+1)$. (1)

Adding the $(n+1)$ th term $(2n+1)^2$ to each member,

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 + (2n+1)^2 &= \frac{n}{3}(2n-1)(2n+1) + (2n+1)^2 \\ &= \frac{2n+1}{3}[n(2n-1) + 3(2n+1)] \\ &= \frac{2n+1}{3}(2n^2 + 5n + 3) \\ &= \frac{2n+1}{3}(n+1)(2n+3) \\ &= \frac{n+1}{3}(2n+1)(2n+3). \end{aligned} \quad (2)$$

Since (2) has the same form as (1), $n + 1$ simply taking the place of n , if the law is true for n terms, it holds for $n + 1$ terms.

Therefore, since the law is true for two terms, it holds for three; being true for three terms, it holds for four; etc.

Hence, (1) is true for all integral values of n .

2. It is evident that $2n^2 + 9n + 7 = (n + 1)(2n + 7)$.

By trial, $1 \cdot 3 = \frac{1}{6}(1 + 1)(2 + 7)$, and $1 \cdot 3 + 2 \cdot 4 = \frac{2}{6}(2 + 1)(4 + 7)$.

Assume that this law holds for the first n terms.

Then, $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots + n(n + 2) = \frac{n}{6}(n + 1)(2n + 7)$. (1)

Adding the $(n + 1)$ th term $(n + 1)(n + 3)$ to each member,

$$\begin{aligned} 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots + n(n + 2) + (n + 1)(n + 3) \\ &= \frac{n}{6}(n + 1)(2n + 7) + (n + 1)(n + 3) \\ &= \frac{n + 1}{6}[n(2n + 7) + 6(n + 3)] \\ &= \frac{n + 1}{6}(2n^2 + 13n + 18) \\ &= \frac{n + 1}{6}(n + 2)(2n + 9). \end{aligned} \quad (2)$$

Since (2) has the same form as (1), $n + 1$ simply taking the place of n , if the law is true for n terms, it holds for $n + 1$ terms.

Therefore, since the law is true for two terms, it holds for three; being true for three terms, it holds for four; etc.

Hence, (1) is true for all integral values of n .

3. By trial, $1 \cdot 3 = \frac{1}{3}(4 \cdot 1^2 + 6 \cdot 1 - 1)$, $1 \cdot 3 + 3 \cdot 5 = \frac{2}{3}(4 \cdot 2^2 + 6 \cdot 2 - 1)$.

Assume that this law holds for the first n terms.

Then, $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \dots + (2n - 1)(2n + 1) = \frac{n}{3}(4n^2 + 6n - 1)$. (1)

Adding the $(n + 1)$ th term $(2n + 1)(2n + 3)$ to each member,

$$\begin{aligned} 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \dots + (2n - 1)(2n + 1) + (2n + 1)(2n + 3) \\ &= \frac{n}{3}(4n^2 + 6n - 1) + (2n + 1)(2n + 3) \\ &= \frac{1}{3}(4n^3 + 6n^2 - n + 12n^2 + 24n + 9) \\ &= \frac{1}{3}(4n^3 + 18n^2 + 23n + 9) \\ &= \frac{n + 1}{3}(4n^2 + 14n + 9). \end{aligned} \quad (2)$$

Since $4n^2 + 14n + 9 = 4n^2 + 8n + 4 + 6n + 6 - 1$

$$= 4(n + 1)^2 + 6(n + 1) - 1,$$

(2) has the same form as (1), $n + 1$ simply taking the place of n .

Hence, if the law is true for n terms, it holds for $n + 1$ terms.

Therefore, since the law is true for two terms, it holds for three; being true for three terms, it holds for four; etc.

Hence, (1) is true for all integral values of n .

4. It is evident that

$$a = \frac{a(1 - r^1)}{1 - r}, \quad a + ar = \frac{a(1 - r^2)}{1 - r}, \quad a + ar + ar^2 = \frac{a(1 - r^3)}{1 - r}.$$

Assume that this law holds for the first n terms.

Then, $a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}$. (1)

Adding the $(n+1)$ th term ar^n to each member,

$$\begin{aligned} a + ar + ar^2 + \cdots + ar^{n-1} + ar^n &= \frac{a(1-r^n)}{1-r} + ar^n \\ &= \frac{a - ar^n + ar^n - ar^{n+1}}{1-r} \\ &= \frac{a(1-r^{n+1})}{1-r}. \end{aligned} \quad (2)$$

Since (2) has the same form as (1), $n+1$ simply taking the place of n , if the law is true for n terms, it holds for $n+1$ terms.

Therefore, since the law is true for three terms, it holds for four terms; being true for four terms, it holds for five terms; etc.

Hence, (1) is true to any number of terms.

5. By trial, $x^2 - y^2$, $x^4 - y^4$, and $x^6 - y^6$, say, are found to be divisible by $x + y$.

Assume that $x^{2n} - y^{2n}$ is divisible by $x + y$, n being a positive integer.

$$\begin{aligned} \text{Now } x^{2n+2} - y^{2n+2} &= x^{2n+2} - x^{2n}y^2 + x^{2n}y^2 - y^{2n+2} \\ &= x^{2n}(x^2 - y^2) - y^2(x^{2n} - y^{2n}). \end{aligned}$$

Since $x^2 - y^2$ is divisible by $x + y$ and by supposition $x^{2n} - y^{2n}$ is divisible by $x + y$, by § 104, 3, $x^{2n+2} - y^{2n+2}$ is divisible by $x + y$; that is, if $x^{2n} - y^{2n}$ is divisible by $x + y$ for any integral value of n , it is divisible by $x + y$ for the next integral value of n .

Therefore, since $x^6 - y^6$ is divisible by $x + y$, $x^8 - y^8$ is divisible by $x + y$, since $x^8 - y^8$ is divisible by $x + y$, $x^{10} - y^{10}$ is divisible by $x + y$; etc.

Hence, $x^{2n} - y^{2n}$ is divisible by $x + y$ when n is any positive integer.

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$$6. \quad 3^x = 81 = 3^4. \\ \therefore x = 4.$$

$$7. \quad 4^x = 10. \\ x \log 4 = \log 10 = 1. \\ \therefore x = \frac{1}{\log 4} \\ = \frac{1}{0.6021}.$$

$$\therefore \log x = \text{colog } .6021 \\ = 0.2203. \\ \therefore x = 1.660.$$

$$8. \quad 2^x = 80. \\ x \log 2 = \log 80. \\ \therefore x = \frac{\log 80}{\log 2} \\ = \frac{1.9031}{0.3010} \\ \log 1.9031 = 0.2795 \\ \log .3010 = \bar{1}.4786 \\ \therefore \log x = 0.8009 \\ \therefore x = 6.323.$$

$$9. \quad 3^{2x} = 9^{2x} = (3^2)^{2x}. \\ \therefore 3^{2x} = 3^{4x}. \\ \therefore x^2 = 4x. \\ \therefore x = 4 \text{ or } 0.$$

$$10. \quad 2^{x^2} = 512 = 2^9. \\ \therefore x^2 = 9. \\ \therefore x = \pm 3.$$

$$17. \quad \begin{cases} 2^{x+y} = 6, & (1) \\ 2^{x+1} = 3y, & (2) \\ 2^{x+y} = 2 \cdot 3, & (3) \end{cases}$$

By (1),
Dividing (3) by (2),
or

$$\begin{aligned} \therefore (y-2) \log 2 &= (1-y) \log 3. \\ (\log 2 + \log 3)y &= \log 3 + 2 \log 2 = \log (3 \cdot 2^2). \\ (\log 6)y &= \log 12. \end{aligned}$$

$$\therefore y = \frac{\log 12}{\log 6} = \frac{1.0792}{0.7782}.$$

$$\log y = \log 1.0792 - \log .7782 = 0.1420.$$

$$\therefore y = 1.387.$$

$$\text{By (1),} \quad (x+y) \log 2 = \log 6. \\ x = \frac{\log 6 - y \log 2}{\log 2} = \frac{\log 6}{\log 2} - y$$

$$= \frac{0.7782}{0.3010} - 1.387$$

$$\text{by logarithms,} \quad = 2.585 - 1.387 = 1.198.$$

$$11. \quad (2^x)^2 = 256 = 16^2. \\ \therefore 2^x = 16 = 2^4. \\ \therefore x = 4.$$

$$12. \quad 3^{2x} - 36 \cdot 3^x + 243 = 0. \\ (3^x - 9)(3^x - 27) = 0. \\ \therefore 3^x = 9 \text{ or } 27; \text{ that is,} \\ 3^x = 3^2 \text{ or } 3^3. \\ \therefore x = 2 \text{ or } 3.$$

$$13. \quad \log \log x = \log 2. \\ \therefore \log x = 2. \\ \therefore x = 10^2 = 100.$$

$$14. \quad \begin{cases} 3^x = 2y, & (1) \\ 4^x = 20y, & (2) \end{cases} \\ 4^x = 10 \cdot 2y = 10 \cdot 3^x. \\ \therefore x \log 4 = \log 10 + x \log 3. \\ \therefore x = \frac{1}{\log 4 - \log 3} \\ = \frac{1}{0.1250} = 8.$$

$$\text{By (1), } \log y = x \log 3 - \log 2 \\ = 8 \log 3 - \log 2 \\ = 3.5158. \\ \therefore y = 3279.$$

$$15. \quad 2^{3y} = 512 = 2^9. \\ \therefore 3y = 9 = 3^2. \\ \therefore y = 2.$$

$$16. \quad 5^{x^6} = 625 = 5^4. \\ \therefore x^6 = 4. \\ \therefore x = \sqrt[6]{4} = \sqrt[3]{2}.$$

18.
$$\begin{cases} 4x+y = 32, \\ 2^{2x-y} = 4, \\ 2^{2x+2y} = 2^5, \\ 2^{2x-y} = 2^2, \\ 2x+2y = 5, \\ 2x-y = 2, \end{cases} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{matrix}$$
- By (1),
By (2),
By (3),
By (4),
Solving (5) and (6),
$$x = \frac{3}{2}, y = 1.$$
19.
$$\begin{cases} 2^x = y, \\ x = 1 + \log y. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$
- By (1) and (2),
$$x = 1 + \log 2^x = 1 + x \log 2.$$

$$\therefore x = \frac{1}{1 - \log 2} = \frac{1}{.6990}.$$

$$\log x = \text{colog } .6990 = 0.1555.$$

$$\therefore x = 1.4306.$$

By (2),
$$\log y = x - 1 = 0.4306.$$

$$\therefore y = 2.695.$$

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PERMUTATIONS AND COMBINATIONS

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8. In the word *international* there are two *i*'s, three *n*'s, two *t*'s, two *a*'s and in all thirteen letters. Hence, the number of permutations is $\frac{13!}{2!3!2!2!}$, or 129729600.

9. Since with any of the eight pairs of trousers any of the six vests and any of the five coats may be taken, the number of different suits is $8 \cdot 6 \cdot 5$, or 240.

10. Five men may be selected from ten men in C_{10}^5 ways, and three women from five women in C_3^5 , or C_2^5 , ways. Hence, the number of committees that may be formed by associating any selection of five men with any selection of three women is $C_{10}^5 \cdot C_3^5$, or 2520.

11. After selecting any one of the five right-hand gloves he may select any one of the five left-hand gloves except the mate of the right-hand glove selected. Hence, there are $5 \cdot 4$, or 20, ways of selecting a glove for each hand without selecting a pair.

12. The number of ways is $\underline{12 - 1} = \underline{11} = 39916800$.

13. The five ladies may be seated in $\underline{4}$ ways, and for each way the five gentlemen may be seated between them in $\underline{4}$ ways. Hence, the number of ways is $\underline{4} \cdot \underline{4}$, or 576.

14. From the remaining 15 members he may select 3 members in C_3^{15} , or 455 ways.

15. Six persons may be selected for one table and six for the other in C_6^{12} ways, and at each table the six persons may be permuted in P_6^6 (circular) ways for each permutation of the other table. Hence, the number of ways is $C_6^{12} \cdot P_6^6$ (circular) $\cdot P_6^6$ (circular), or 13305600.

16. Eight books may be arranged in $\underline{8}$ ways. If two particular books, say A and B, are together, the other six books and the combination AB may be arranged in $\underline{7}$ ways. Also the other six books and the combination BA may be arranged in $\underline{7}$ ways. Hence, out of the $\underline{8}$ arrangements, $\underline{8} - 2\underline{7}$ arrangements are left in which A and B do not stand side by side.

$$\underline{8} - 2\underline{7} = 8\underline{7} - 2\underline{7} = 6\underline{7} = 30240.$$

17. There may be 4 boys and 1 girl or 3 boys and 2 girls. From 12 boys and 10 girls, 4 boys and 1 girl may be selected in $C_4^{12} \cdot C_1^{10}$ ways, and 3 boys and 2 girls may be selected in $C_3^{12} \cdot C_2^{10}$ ways. Hence, the number of committees possible is $C_4^{12} \cdot C_1^{10} + C_3^{12} \cdot C_2^{10}$, or 14850.

18. From 3 men who may sit on either side 2 men may be selected to fill any 2 vacant places on the starboard side in C_2^3 ways, the third man taking the vacant place on the port side in each case. Now when the places have been filled, the four men on each side may be permuted in $\underline{4}$ ways, and to obtain all the different ways of seating the crew each permutation on the port side may be associated with each permutation on the starboard side. Hence, the crew may be seated in $C_2^3 \cdot P_4^4 \cdot P_4^4$, or 1728 ways.

19. There are ten digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0. Of numbers expressed by one digit, only one, 9, contains the digit 9.

Numbers expressed by two digits one of which is 9 have the form $9*$ or else $*9$. When 9 is the first digit, any of the ten digits may be taken for the second digit; when 9 is the second digit, any of the ten digits except 0, and except 9 previously accounted for, may be taken for the first digit. Hence, $C_1^{10} + C_1^8$, or 18, numbers of two figures have 9 for one of them.

Numbers expressed by three digits one of which is 9 have one of the forms $9**$, $*9*$, $**9$. When 9 is the first digit, any one of the ten digits may be taken for the second digit and any one of the ten may be taken also for the third digit. Hence, $C_1^{10} \cdot C_1^{10}$ numbers of three figures begin with 9.

When 9 is the second digit, any one of the ten digits except 0 and 9 may fill the first place and any one of the ten may fill the third place. Hence, $C_1^8 \cdot C_1^{10}$ numbers of three figures, besides those previously accounted for, have 9 for the middle digit.

When 9 is the third digit, any one of the ten digits except 0 and 9 may fill the first place and any one of the ten except 9 may fill the second place. Hence, $C_1^8 \cdot C_1^9$ numbers of three figures, besides those previously accounted for, have 9 for the third digit.

Hence, $1 + (C_1^{10} + C_1^8) + (C_1^{10} \cdot C_1^{10} + C_1^8 \cdot C_1^{10} + C_1^8 \cdot C_1^9)$ numbers, or $1 + 10 + 8 + 100 + 80 + 72$ numbers, or 271 numbers, less than 1000 have the digit 9.

20. From 20 men C_4^{20} , or 4845, different pickets of 4 soldiers can be detailed. The number of these that contain any particular soldier is the same as the number of selections of 3 other soldiers from 19 other soldiers, or C_3^{19} , or 969.

PROBABILITY

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10. From 9 men a committee of 4 can be chosen in C_4^9 ways. Since 2 Democrats can be chosen from 5 in C_2^5 ways and 2 Republicans from 4 in C_2^4 ways, there are $C_2^5 \cdot C_2^4$ ways in which 2 Democrats and 2 Republicans can be chosen.

Hence, the probability that of 4 men chosen by lot 2 will be Democrats and 2 Republicans is

$$\frac{C_2^5 \cdot C_2^4}{C_4^9} = \frac{(5 \cdot 4)(4 \cdot 3)(1 \cdot 2 \cdot 3 \cdot 4)}{(1 \cdot 2)(1 \cdot 2)(9 \cdot 8 \cdot 7 \cdot 6)} = \frac{10}{21}.$$

11. From 18 balls 4 may be drawn in C_4^{18} ways. From 5 black balls 4 may be drawn in C_4^5 ways. Hence, the probability of drawing 4 black balls is

$$\frac{C_4^5}{C_4^{18}} = \frac{C_1^5}{C_4^{18}} = \frac{5(4 \cdot 3 \cdot 2 \cdot 1)}{18 \cdot 17 \cdot 16 \cdot 15} = \frac{1}{612}.$$

Again, the joint probability of drawing 2 of the 5 black balls and 2 of the 7 white balls when there are C_4^{18} ways in which 4 balls may be drawn, is

$$\frac{C_2^5 \cdot C_2^7}{C_4^{18}} = \frac{(5 \cdot 4)(7 \cdot 6)(4 \cdot 3 \cdot 2 \cdot 1)}{(1 \cdot 2)(1 \cdot 2)(18 \cdot 17 \cdot 16 \cdot 15)} = \frac{7}{102}.$$

12. Five different ways of throwing 6 are possible, namely,

$$1 + 5, 5 + 1, 2 + 4, 4 + 2, \text{ and } 3 + 3.$$

The probability of throwing 1 with the first die and 5 with the second is $\frac{1}{6} \cdot \frac{1}{6}$, or $\frac{1}{36}$, and similarly for each of the other ways.

Hence, the total probability is $\frac{5}{36}$.

13. The probability of throwing say 6 with the first die, 6 with the second, 5 with the third, and 4 with the fourth, is $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$, or $\frac{1}{1296}$. But the numbers 6, 6, 5, 4 can be thrown with the dice in as many different orders, such as $6 + 6 + 5 + 4$, $6 + 5 + 4 + 6$, etc., as there are permutations of four things two of which are alike. Hence, the probability of throwing 6, 6, 5, and 4 is

$$\frac{1}{1296} \times \frac{4}{2}, \text{ or } \frac{12}{1296}.$$

Then, taking all the different combinations that aggregate more than 20, the probabilities are as follows :

$$6 + 6 + 6 + 3, \text{ in } \underline{4} \div \underline{3} \text{ orders.} \quad \therefore p_1 = \frac{4}{1296}.$$

$$6 + 6 + 6 + 4, \text{ in } \underline{4} \div \underline{3} \text{ orders.} \quad \therefore p_2 = \frac{4}{1296}.$$

$$6 + 6 + 6 + 5, \text{ in } \underline{4} \div \underline{3} \text{ orders.} \quad \therefore p_3 = \frac{4}{1296}.$$

$$6 + 6 + 6 + 6, \text{ in } \underline{4} \div \underline{4} \text{ orders.} \quad \therefore p_4 = \frac{1}{1296}.$$

$$6 + 6 + 5 + 4, \text{ in } \underline{4} \div \underline{2} \text{ orders.} \quad \therefore p_5 = \frac{12}{1296}.$$

$$6 + 6 + 5 + 5, \text{ in } \underline{4} \div (\underline{2} \times \underline{2}) \text{ orders.} \quad \therefore p_6 = \frac{6}{1296}.$$

$$6 + 5 + 5 + 5, \text{ in } \underline{4} \div \underline{3} \text{ orders.} \quad \therefore p_7 = \frac{4}{1296}.$$

$$\text{Adding, total probability} = P = \frac{35}{1296}.$$

14. From 40 tickets 4 may be drawn in C_{40}^4 ways, and C_3^{38} of these ways will include either designated ticket, but not the other. Hence, A's chance

$$\text{of drawing a prize with either ticket is } \frac{C_3^{38}}{C_{40}^4} = \frac{38 \cdot 37 \cdot 36 \cdot 4}{40 \cdot 39 \cdot 38 \cdot 37} = \frac{6}{65}.$$

Similarly, his chance of drawing prizes with both tickets is

$$\frac{C_2^{38}}{C_{40}^4} = \frac{38 \cdot 37 \cdot 3 \cdot 4}{40 \cdot 39 \cdot 38 \cdot 37} = \frac{1}{130}.$$

$$\text{Hence, his total chance is } 2\left(\frac{6}{65}\right) + \frac{1}{130} = \frac{5}{26}.$$

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15. As shown in the suggestion, A's chance of drawing at least one prize is $\frac{27}{35}$. Similarly, B's chance of drawing at least one prize is

$$1 - \frac{C_2^{25}}{C_2^{30}} = 1 - \frac{25 \cdot 24}{30 \cdot 29} = \frac{9}{29} = \frac{27}{87}.$$

$$\therefore \text{A's chance : B's chance} = \frac{27}{35} : \frac{27}{87}, \text{ or } 87 : 95.$$

16. The respective probabilities of A and B drawing 3 blanks are

$$\frac{C_3^{90}}{C_3^{100}} = \frac{90 \cdot 89 \cdot 88}{100 \cdot 99 \cdot 98} = \frac{178}{245}, \text{ and } \frac{C_3^{45}}{C_3^{50}} = \frac{45 \cdot 44 \cdot 43}{50 \cdot 49 \cdot 48} = \frac{1419}{1960}.$$

Therefore, A's chance of winning at least one prize is $1 - \frac{178}{245}$, or $\frac{67}{245}$, or $\frac{536}{1960}$, and B's chance is $1 - \frac{1419}{1960}$, or $\frac{541}{1960}$. Hence, B's chance is slightly better than A's.

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2. The probability that A will meet a friend in any particular city is $\frac{1}{4}$. Hence, the probability that he will meet a friend in each of the five cities is $\left(\frac{1}{4}\right)^5$, or $\frac{1}{1024}$.

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2. The probability that A will be lost at sea is the product of the probability that the ship will encounter a storm, the probability that she will spring a leak, the probability that her engines will fail to pump her out, the probability that her compartments will fail to keep her afloat, and the probability that any one passenger will not be saved by passing boats.

$$\text{Hence, } P = \frac{1}{2} \cdot \frac{1}{10} \cdot \frac{1}{10+1} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{1760}.$$

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4. The possible throws that give more than 10 are $5 + 6$, $6 + 5$, and $6 + 6$. Since the probability of throwing an assigned number with each die is $\frac{1}{6} \cdot \frac{1}{6}$, or $\frac{1}{36}$, and since more than 10 can be thrown with three such throws of two dice, the probability of throwing more than 10 is $\frac{3}{36}$, or $\frac{1}{12}$.

5. The probability of throwing 3 with one die is $\frac{1}{6}$.

By Ex. 1, the probability of throwing 6 with two dice is $\frac{5}{36}$, the five different ways being $1 + 5$, $5 + 1$, $2 + 4$, $4 + 2$, $3 + 3$.

Hence, the probabilities are as $\frac{6}{36}$ to $\frac{5}{36}$, or as 6 to 5.

6. Since the occurrence of each event is associated with the failure of the others to occur, if p_1 denotes the probability that only the first occurs, p_2 the probability that only the second occurs, and p_3 the probability that only the third occurs,

$$\begin{aligned} p_1 &= \frac{2}{3}(1 - \frac{3}{4})(1 - \frac{1}{2}) = \frac{1}{12}; \\ p_2 &= (1 - \frac{2}{3}) \cdot \frac{3}{4} \cdot (1 - \frac{1}{2}) = \frac{1}{8}; \\ p_3 &= (1 - \frac{2}{3})(1 - \frac{3}{4}) \cdot \frac{1}{2} = \frac{1}{24}. \end{aligned}$$

Hence, the probability that one and only one of the events will happen is $p_1 + p_2 + p_3 = \frac{1}{12} + \frac{1}{8} + \frac{1}{24} = \frac{1}{4}$.

7.

Probability that A will fail = $\frac{1}{3}$;

probability that B will fail = $\frac{1}{6}$;

\therefore probability that both will fail = $\frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$;

\therefore probability that the problem will be solved = $1 - \frac{1}{18} = \frac{17}{18}$.

Again,

probability that A will succeed = $\frac{2}{3}$;

and

probability that B will succeed = $\frac{5}{6}$;

\therefore probability that both will succeed = $\frac{2}{3} \cdot \frac{5}{6} = \frac{5}{9}$.

8. The probability that all will fall tails is $(\frac{1}{2})^4$, or $\frac{1}{16}$. Therefore, the probability that at least one will fall head is $1 - \frac{1}{16}$, or $\frac{15}{16}$. Again, the probability that a particular coin will fall head and the other three tails is $\frac{1}{2} (\frac{1}{2})^3$, or $\frac{1}{16}$; and since the particular coin that falls head may be any of the four, the probability that one and only one will fall head is $\frac{1}{16} \times 4$, or $\frac{1}{4}$.

9. By Ex. 2 (b), the probability of throwing at least one ace in six trials is a little less than $\frac{2}{3}$.

The probability of throwing tails twice in succession is $\frac{1}{2} \cdot \frac{1}{2}$, or $\frac{1}{4}$. Hence, the probability of throwing head at least once in two trials is $1 - \frac{1}{4}$, or $\frac{3}{4}$. Hence, this event is the more probable.

10. The probability of success on the first trial is $\frac{1}{6}$. The probability that a second trial is needed is $1 - \frac{1}{6}$, or $\frac{5}{6}$, and therefore the probability of success on the second trial is $\frac{5}{6} \cdot \frac{1}{6}$, or $\frac{5}{36}$.

The probability that a third trial is needed is $(1 - \frac{1}{6})(1 - \frac{5}{6})$, or $\frac{25}{36}$; and therefore the probability of success on the third trial is $\frac{25}{36} \cdot \frac{1}{6}$, or $\frac{25}{216}$.

Hence, the probability of throwing an ace in two trials is $\frac{1}{6} + \frac{5}{36}$, or $\frac{11}{36}$; and in three trials, $\frac{1}{6} + \frac{5}{36} + \frac{25}{216}$, or $\frac{91}{216}$.

Or, the probability of failure is $\frac{5}{6}$ for one trial, $(\frac{5}{6})^2$ for two successive trials, and $(\frac{5}{6})^3$ for three successive trials. Hence, the probability that an ace will be thrown in two trials is $1 - (\frac{5}{6})^2$, or $\frac{11}{36}$; and in three trials, $1 - (\frac{5}{6})^3$, or $\frac{91}{216}$.

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11. In two throws, 10 can be made in 7 ways, namely,

$2 + 8$, $8 + 2$, $3 + 7$, $7 + 3$, $4 + 6$, $6 + 4$, $5 + 5$.

The first number in each case represents the first throw with two dice, and the second number the second throw. Now since, Ex. 1, the probabilities

of throwing 2, 3, 4, 5, 6, 7, 8 with two dice are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}$, respectively, the total probability for the 7 ways mentioned above is

$$P = \frac{1}{36} \cdot \frac{5}{36} + \frac{2}{36} \cdot \frac{1}{36} + \frac{3}{36} \cdot \frac{2}{36} + \frac{4}{36} \cdot \frac{3}{36} + \frac{5}{36} \cdot \frac{4}{36} + \frac{6}{36} \cdot \frac{5}{36} + \frac{5}{36} \cdot \frac{4}{36} \\ = \frac{2}{36^2} (5 + 12 + 15 + 8) = \frac{80}{36^2} = \frac{5}{81}.$$

12. The probability that the door is not locked and that A will get in without a key is $\frac{2}{5}$, while the probability that it is locked is $\frac{3}{5}$.

1st key, probability of success, $\frac{1}{6} \cdot \frac{3}{5} = \frac{1}{10}$; of failure, $\frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}$.

2d key, probability of success, $\frac{1}{6} \cdot \frac{1}{5} = \frac{1}{30}$; of failure, $\frac{4}{6} \cdot \frac{1}{5} = \frac{2}{15}$.

3d key, probability of success, $\frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10}$.

Hence, the probability that he gets in by trying only one key is $\frac{2}{5} + \frac{1}{10} = \frac{1}{2}$; by trying two keys, $\frac{2}{5} + \frac{1}{10} + \frac{1}{10} = \frac{3}{5}$; by trying three keys, $\frac{2}{5} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{7}{10}$.

13. By Ex. 1, the probabilities of throwing 12, 11, 10, 9, 8, 7, 6 with two dice are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}$, respectively. The probability of three of these events concurring so that the sum of the numbers thrown is 30 is found as follows:

$$30 = 12 + 12 + 6, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_1 = 3 \left(\frac{1}{36} \cdot \frac{1}{36} \cdot \frac{5}{36} \right) = \frac{15}{36^3}.$$

$$30 = 12 + 11 + 7, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_2 = 6 \left(\frac{1}{36} \cdot \frac{2}{36} \cdot \frac{6}{36} \right) = \frac{72}{36^3}.$$

$$30 = 12 + 10 + 8, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_3 = 6 \left(\frac{1}{36} \cdot \frac{3}{36} \cdot \frac{5}{36} \right) = \frac{90}{36^3}.$$

$$30 = 12 + 9 + 9, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_4 = 3 \left(\frac{1}{36} \cdot \frac{4}{36} \cdot \frac{4}{36} \right) = \frac{48}{36^3}.$$

$$30 = 11 + 11 + 8, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_5 = 3 \left(\frac{2}{36} \cdot \frac{2}{36} \cdot \frac{5}{36} \right) = \frac{60}{36^3}.$$

$$30 = 11 + 10 + 9, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_6 = 6 \left(\frac{2}{36} \cdot \frac{3}{36} \cdot \frac{4}{36} \right) = \frac{144}{36^3}.$$

$$30 = 10 + 10 + 10, \text{ in } \frac{3}{2} \text{ orders. } \therefore p_7 = 1 \left(\frac{3}{36} \cdot \frac{3}{36} \cdot \frac{3}{36} \right) = \frac{27}{36^3}.$$

$$\text{Adding, the total probability is } P = \frac{456}{36^3} = \frac{19}{1944}.$$

14. In any trial the probability of throwing double fives is $(\frac{1}{6})^2$, or $\frac{1}{36}$, and the probability of failing is $\frac{35}{36}$. Therefore, the probability of throwing double fives three times in succession and failing the fourth time is

$\left(\frac{1}{36} \right)^3 \cdot \frac{35}{36}$, or $\frac{35}{36^4}$. But under the conditions of the problem the failure may occur on any trial, provided the other three trials are successful. Hence, the probability of throwing double fives exactly three times out of four is

$$\frac{35}{36^4} \cdot 4 = \frac{140}{36^4} = \frac{35}{419904}.$$

Or, using formula (A), the probability is

$$\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot \frac{1}{36^3} \left(1 - \frac{1}{36} \right)^{4-3} = \frac{140}{36^4} = \frac{35}{419904}.$$

15. *1st trial.* The probability that A draws black is $\frac{1}{3}$; the probability that he fails, leaving 2 black balls and 3 others, is $\frac{2}{3}$.

2d trial. The probability that B draws black is $\frac{2}{3} \cdot \frac{2}{3}$, or $\frac{4}{9}$; the probability that he fails, leaving 2 black balls and 2 others, is $\frac{2}{3} \cdot \frac{1}{3}$, or $\frac{2}{9}$.

3d trial. The probability that C draws black is $\frac{2}{3} \cdot \frac{1}{3}$, or $\frac{2}{9}$; the probability that he fails, leaving 2 black balls and 1 other, is $\frac{2}{3} \cdot \frac{1}{3}$, or $\frac{2}{9}$.

4th trial. The probability that A draws black is $\frac{1}{3} \cdot \frac{2}{3}$, or $\frac{2}{9}$; the probability that he fails, leaving 2 black balls, is $\frac{1}{3} \cdot \frac{1}{3}$, or $\frac{1}{9}$.

5th trial. The probability that B draws black is $\frac{1}{3} \cdot \frac{2}{3}$, or $\frac{2}{9}$.

A's chance = $\frac{1}{3} + \frac{2}{9} = \frac{5}{9}$; B's chance = $\frac{4}{9} + \frac{2}{9} = \frac{2}{3}$; C's chance = $\frac{2}{9}$. Hence, A's expectation is $\frac{5}{9}$ of \$150, or \$70; B's expectation is $\frac{2}{3}$ of \$150, or \$50; and C's expectation is $\frac{2}{9}$ of \$150, or \$30.

16. If one bill is drawn at random, the probability that it is a 5-dollar bill is $\frac{1}{3}$, or $\frac{1}{3}$, and the probability that it is a 2-dollar bill is $\frac{2}{3}$, or $\frac{2}{3}$. Hence, the value of the privilege of drawing one bill at random is ($\frac{1}{3}$ of \$5) + ($\frac{2}{3}$ of \$2), or \$3; and consequently the value of the privilege of drawing two bills at random is \$6.

Or, consider the expectations of drawing \$5 + \$5, \$5 + \$2, \$2 + \$5, and \$2 + \$2, respectively. The probability of the first event is $\frac{1}{3} \cdot \frac{2}{3}$, or $\frac{2}{9}$, and the expectation from this possibility is $\frac{2}{9}$ of \$10, or $\frac{20}{9}$; the probability of the second event, should the first not occur, is $\frac{1}{3} \cdot \frac{2}{3}$, or $\frac{2}{9}$, and the corresponding expectation is $\frac{2}{9}$ of \$7, or $\frac{14}{9}$; the probability of the third event, should the first two not occur, is $\frac{2}{3} \cdot \frac{1}{3}$, or $\frac{2}{9}$, and the corresponding expectation is $\frac{2}{9}$ of \$7, or $\frac{14}{9}$; the probability of the fourth event, should the others not occur, is $\frac{2}{3} \cdot \frac{1}{3}$, or $\frac{2}{9}$, and the corresponding expectation is $\frac{2}{9}$ of \$4, or $\frac{8}{9}$. Hence, the total expectation is $\frac{20}{9} + \frac{14}{9} + \frac{14}{9} + \frac{8}{9}$, or \$6.

17. *1st trial.* A's chance of success, $\frac{1}{2}$; of failure, $\frac{1}{2}$.

2d trial. B's chance of success, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$; of failure, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

3d trial. C's chance of success, $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$; of failure, $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$.

4th trial. A's chance of success, $\frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$; of failure, $\frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$.

5th trial. B's chance of success, $\frac{1}{16} \cdot \frac{1}{2} = \frac{1}{32}$; of failure, $\frac{1}{16} \cdot \frac{1}{2} = \frac{1}{32}$.

6th trial. C's chance of success, $\frac{1}{32} \cdot \frac{1}{2} = \frac{1}{64}$; of failure, $\frac{1}{32} \cdot \frac{1}{2} = \frac{1}{64}$.

And so on indefinitely. It is evident that

$$\text{A's chance of success} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3};$$

$$\text{B's chance of success} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots = \frac{1}{2} \text{ of } \frac{4}{3} = \frac{2}{3};$$

$$\text{and C's chance of success} = \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \dots = \frac{1}{2} \text{ of } \frac{2}{3} = \frac{1}{3}.$$

18. *1st trial.* A's chance of success, $\frac{1}{3}$; of failure, $\frac{2}{3}$.

2d trial. B's chance of success, $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$; of failure, $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$.

3d trial. A's chance of success, $\frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27}$; of failure, $\frac{4}{9} \cdot \frac{2}{3} = \frac{8}{27}$.

4th trial. B's chance of success, $\frac{8}{27} \cdot \frac{1}{3} = \frac{8}{81}$; of failure, $\frac{8}{27} \cdot \frac{2}{3} = \frac{16}{81}$.

And so on indefinitely. It is evident that

$$\text{A's chance of success} = \frac{1}{3} + \frac{4}{9} \left(\frac{2}{3}\right)^2 + \frac{16}{27} \left(\frac{2}{3}\right)^4 + \dots = \frac{1}{1 - \frac{4}{9}} = \frac{9}{5};$$

$$\text{and B's chance of success} = \frac{2}{9} + \frac{8}{81} \left(\frac{2}{3}\right)^2 + \frac{32}{729} \left(\frac{2}{3}\right)^4 + \dots = \frac{2}{9} \text{ of } \frac{9}{5} = \frac{2}{5}.$$

19. *1st trial.* A's chance of success, $\frac{1}{2}$; of failure, $\frac{1}{2}$.

2d trial. B's chance of success, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$; of failure, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

3d trial. A's chance of success, $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$; of failure, $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$.

4th trial. B's chance of success, $\frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$; of failure, $\frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$.

And so on indefinitely. It is evident that

$$A's \text{ chance of success} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{2}{3};$$

and $B's \text{ chance of success} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots = \frac{1}{2} \text{ of } \frac{2}{3} = \frac{1}{3}.$

$\therefore A's \text{ expectation} = \frac{2}{3} \text{ of } \$1 = 66\frac{2}{3} \text{ cents,}$

and $B's \text{ expectation} = \frac{1}{3} \text{ of } \$1 = 33\frac{1}{3} \text{ cents.}$

Since A and B own equal shares of the dollar, A's expectation of gain is $66\frac{2}{3}$ cents less 50 cents, or $16\frac{2}{3}$ cents, and B's expectation of loss is 50 cents less $33\frac{1}{3}$ cents, or $16\frac{2}{3}$ cents.

20. By formula (A),

$$P = C^5 \left(\frac{1}{10} \right)^3 \left(\frac{9}{10} \right)^2 = \frac{5 \cdot 4 \cdot 3 \cdot 81}{1 \cdot 2 \cdot 3 \cdot 100000} = .0081, \text{ or } \frac{81}{10000}.$$

Again, by formula (B),

$$\begin{aligned} P &= \left(\frac{1}{10} \right)^5 + 5 \left(\frac{1}{10} \right)^4 \cdot \frac{9}{10} + \frac{5 \cdot 4}{2} \left(\frac{1}{10} \right)^3 \left(\frac{9}{10} \right)^2 + \frac{5 \cdot 4 \cdot 3}{3} \left(\frac{1}{10} \right)^2 \left(\frac{9}{10} \right)^3 \\ &= .00001 + .00045 + .00810 + .07290 \\ &= .08146, \text{ or } \frac{4073}{50000}. \end{aligned}$$

21. The probability of escaping injury for 1 day is $\frac{1000}{1001}$; for 2 days, $\left(\frac{1000}{1001} \right)^2$; for 3 days, $\left(\frac{1000}{1001} \right)^3$; etc. Hence, the probability of escaping injury for 1500 days is $\left(\frac{1000}{1001} \right)^{1500}$.

$$\begin{aligned} \log \left(\frac{1000}{1001} \right)^{1500} &= 1500(\log 1000 - \log 1001) \\ &= 1500(3 - 3.0004) = 1500(-.0004) \\ &= -.6000 = \bar{1}.4000 = \log .2512. \end{aligned}$$

Hence, the chance of escaping injury for 5 years is about $\frac{1}{4}$; or, the odds against escaping injury are about 3 to 1.

SIMPLE CONTINUED FRACTIONS

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$$2. \quad \pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \dots}}}}}$$

$$\text{Convergents,} \quad \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{355}{33102}, \dots$$

Since the reciprocal of the product of the denominators of the fourth and fifth convergents is $\frac{1}{113 \times 33102}$, which is less than $\frac{1}{2}$ of .000001, the fourth

convergent $\frac{355}{113}$ gives the value of π to the nearest sixth decimal place.

By division, $\frac{355}{113} = 3.141593$ to the nearest sixth decimal place.

3. The ratio of 2.20462 to 1 is the same as the ratio of 220462 to 100000. Reducing this ratio to a continued fraction,

$$\frac{220462}{100000} = 2 + \frac{1}{4 + \frac{1}{1 + \frac{1}{7 + \frac{1}{1 + \frac{1}{6 + \frac{1}{23 + \frac{1}{2 + \frac{1}{3}}}}}}}}$$

$$\text{Convergents,} \quad \frac{2}{1}, \frac{9}{4}, \frac{11}{5}, \frac{86}{39}, \frac{97}{44}, \frac{668}{303}, \frac{(\quad)}{7013}, \text{ etc.}$$

Errors are less than $\frac{1}{4}$, $\frac{1}{20}$, $\frac{1}{105}$, $\frac{1}{1716}$, $\frac{1}{3332}$, $\frac{1}{2124633}$, etc.; that is, 2 is too small by less than $\frac{1}{4}$, $\frac{2}{3}$ is too great by less than $\frac{1}{20}$, $\frac{11}{5}$ is too small by less than $\frac{1}{105}$, etc. The sixth convergent is a very near approximation.

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$$1. \quad \sqrt{15} = 3 + \frac{\sqrt{15} - 3}{1} = 3 + \frac{6}{\sqrt{15} + 3} = 3 + \frac{1}{\frac{\sqrt{15} + 3}{6}} \quad (1)$$

$$\frac{\sqrt{15} + 3}{6} = 1 + \frac{\sqrt{15} - 3}{6} = 1 + \frac{1}{\sqrt{15} + 3} \quad (2)$$

$$\text{By (1), } \sqrt{15} + 3 = 6 + \frac{1}{\frac{\sqrt{15} + 3}{6}} \quad (3)$$

Substituting (2) and (3) in (1),

$$\sqrt{15} = 3 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \frac{1}{6 + \dots}}}}$$

Convergents, $\frac{3}{1}, \frac{4}{1}, \frac{27}{7}, \frac{31}{8}, \dots$

The 4th convergent is a good approximation (§ 525).

$$2. \quad \sqrt{126} = 11 + \frac{\sqrt{126} - 11}{1} = 11 + \frac{5}{\sqrt{126} + 11} = 11 + \frac{1}{\frac{\sqrt{126} + 11}{5}} \quad (1)$$

$$\frac{\sqrt{126} + 11}{5} = 4 + \frac{\sqrt{126} - 9}{5} = 4 + \frac{9}{\sqrt{126} + 9} = 4 + \frac{1}{\frac{\sqrt{126} + 9}{9}} \quad (2)$$

$$\frac{\sqrt{126} + 9}{9} = 2 + \frac{\sqrt{126} - 9}{9} = 2 + \frac{5}{\sqrt{126} + 9} = 2 + \frac{1}{\frac{\sqrt{126} + 9}{5}} \quad (3)$$

$$\frac{\sqrt{126} + 9}{5} = 4 + \frac{\sqrt{126} - 11}{5} = 4 + \frac{1}{\sqrt{126} + 11} \quad (4)$$

By (1),

$$\sqrt{126} + 11 = 22 + \frac{1}{\frac{\sqrt{126} + 11}{5}} \quad (5)$$

Substituting (2), (3), (4), and (5) in (1),

$$\sqrt{126} = 11 + \frac{1}{4 + \frac{1}{2 + \frac{1}{4 + \frac{1}{22 + \frac{1}{4 + \dots}}}}}$$

Convergents, $\frac{11}{1}, \frac{43}{4}, \frac{191}{9}, \frac{449}{10}, \dots$

The 4th convergent is a good approximation (§ 525).

$$3. \quad \sqrt{7} = 2 + \frac{\sqrt{7} - 2}{1} = 2 + \frac{3}{\sqrt{7} + 2} = 2 + \frac{1}{\frac{\sqrt{7} + 2}{3}} \quad (1)$$

$$\frac{\sqrt{7} + 2}{3} = 1 + \frac{\sqrt{7} - 1}{3} = 1 + \frac{2}{\sqrt{7} + 1} = 1 + \frac{1}{\frac{\sqrt{7} + 1}{2}} \quad (2)$$

$$\frac{\sqrt{7} + 1}{2} = 1 + \frac{\sqrt{7} - 1}{2} = 1 + \frac{3}{\sqrt{7} + 1} = 1 + \frac{1}{\frac{\sqrt{7} + 1}{3}} \quad (3)$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} = 1 + \frac{1}{\sqrt{7}+2} = 1 + \frac{1}{\frac{\sqrt{7}+2}{1}} \quad (4)$$

Since $\frac{\sqrt{7}+2}{1} = \sqrt{7}+2$, by (1), $\frac{\sqrt{7}+2}{1} = 4 + \frac{1}{\frac{\sqrt{7}+2}{3}}$ (5)

Substituting (2), (3), (4), and (5) in (1),

$$\sqrt{7} = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \dots}}}}$$

Convergents, $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \dots$

The 4th convergent is a good approximation (§ 525).

4. By Ex. 3, $2 + \sqrt{7} = 4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \dots}}}}$

Convergents, $\frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{14}{3}, \frac{(\quad)}{14}, \dots$

The 4th convergent is a good approximation (§ 525).

5. $\sqrt{21} = 4 + \frac{\sqrt{21}-4}{1} = 4 + \frac{5}{\sqrt{21}+4} = 4 + \frac{1}{\frac{\sqrt{21}+4}{5}} \quad (1)$

$$\frac{\sqrt{21}+4}{5} = 1 + \frac{\sqrt{21}-1}{5} = 1 + \frac{4}{\sqrt{21}+1} = 1 + \frac{1}{\frac{\sqrt{21}+1}{4}} \quad (2)$$

$$\frac{\sqrt{21}+1}{4} = 1 + \frac{\sqrt{21}-3}{4} = 1 + \frac{3}{\sqrt{21}+3} = 1 + \frac{1}{\frac{\sqrt{21}+3}{3}} \quad (3)$$

$$\frac{\sqrt{21}+3}{3} = 2 + \frac{\sqrt{21}-3}{3} = 2 + \frac{4}{\sqrt{21}+3} = 2 + \frac{1}{\frac{\sqrt{21}+3}{4}} \quad (4)$$

$$\frac{\sqrt{21}+3}{4} = 1 + \frac{\sqrt{21}-1}{4} = 1 + \frac{5}{\sqrt{21}+1} = 1 + \frac{1}{\frac{\sqrt{21}+1}{5}} \quad (5)$$

$$\frac{\sqrt{21}+1}{5} = 1 + \frac{\sqrt{21}-4}{5} = 1 + \frac{1}{\sqrt{21}+4} \quad (6)$$

By (1), $\sqrt{21} + 4 = 8 + \frac{1}{\frac{\sqrt{21}+4}{5}} \quad (7)$

Substituting (2), (3), (4), (5), (6), and (7) in (1),

$$\sqrt{21} = 4 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{8 + \frac{1}{1 + \dots}}}}}}$$

Convergents, $\frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{23}{5}, \frac{32}{7}, \frac{55}{12}, \frac{(\quad)}{103}, \dots$

The 6th convergent is a good approximation (§ 525).

$$6. \quad \sqrt{19} = 4 + \frac{\sqrt{19} - 4}{1} = 4 + \frac{3}{\sqrt{19} + 4} = 4 + \frac{1}{\frac{3}{\sqrt{19} + 4}}. \quad (1)$$

$$\frac{\sqrt{19} + 4}{3} = 2 + \frac{\sqrt{19} - 2}{3} = 2 + \frac{5}{\sqrt{19} + 2} = 2 + \frac{1}{\frac{5}{\sqrt{19} + 2}}. \quad (2)$$

$$\frac{\sqrt{19} + 2}{5} = 1 + \frac{\sqrt{19} - 3}{5} = 1 + \frac{2}{\sqrt{19} + 3} = 1 + \frac{1}{\frac{2}{\sqrt{19} + 3}}. \quad (3)$$

$$\frac{\sqrt{19} + 3}{2} = 3 + \frac{\sqrt{19} - 3}{2} = 3 + \frac{5}{\sqrt{19} + 3} = 3 + \frac{1}{\frac{5}{\sqrt{19} + 3}}. \quad (4)$$

$$\frac{\sqrt{19} + 3}{5} = 1 + \frac{\sqrt{19} - 2}{5} = 1 + \frac{3}{\sqrt{19} + 2} = 1 + \frac{1}{\frac{3}{\sqrt{19} + 2}}. \quad (5)$$

$$\frac{\sqrt{19} + 2}{3} = 2 + \frac{\sqrt{19} - 4}{3} = 2 + \frac{1}{\sqrt{19} + 4}. \quad (6)$$

$$\text{By (1), } \sqrt{19} + 4 = 8 + \frac{1}{\frac{3}{\sqrt{19} + 4}}. \quad (7)$$

Substituting (2), (3), (4), (5), (6), and (7) in (1),

$$\sqrt{19} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{8 + \frac{1}{2 + \dots}}}}}}}$$

Convergents, $\frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}, \frac{(\quad)}{326}, \dots$

The 6th convergent is a good approximation (§ 525).

$$7. \quad \sqrt{10} = 3 + \frac{\sqrt{10} - 3}{1} = 3 + \frac{1}{\sqrt{10} + 3}. \quad (1)$$

$$\text{By (1), } \sqrt{10} + 3 = 6 + \frac{1}{\sqrt{10} + 3}.$$

$$\therefore \sqrt{10} = 3 + \frac{1}{6 + \frac{1}{6 + \dots}}.$$

Convergents, $\frac{3}{1}, \frac{19}{6}, \frac{117}{37}, \frac{(\quad)}{228}, \dots$

The second convergent is a good approximation (§ 525).

$$8. \quad 1 \div \sqrt{10} = \frac{1}{\sqrt{10}}$$

$$\text{by Ex. 7, } = \frac{1}{3 + \frac{1}{6 + \frac{1}{6 + \dots}}} = 0 + \frac{1}{3 + \frac{1}{6 + \frac{1}{6 + \dots}}}.$$

Convergents, $\frac{0}{1}, \frac{1}{3}, \frac{6}{19}, \frac{37}{117}, \frac{(\quad)}{721}, \dots$

The third convergent is a good approximation (§ 525).

9. Let
$$x = 2 + \frac{1}{*} \frac{1}{8 + \frac{1}{2 + \dots}}.$$

Then,
$$x = 2 + \frac{1}{*} \frac{1}{8 + \frac{1}{x}} = \frac{17x + 2}{8x + 1}.$$

$\therefore x^2 - 2x = \frac{1}{4}, \quad \therefore x = \frac{1}{2}(2 \pm \sqrt{5}).$

Hence,
$$2 + \frac{1}{*} \frac{1}{8 + \frac{1}{2 + \dots}} = \frac{1}{2}(2 + \sqrt{5}),$$

using only the positive value of the radical.

10. Let
$$x = 1 + \frac{1}{*} \frac{1}{10 + \frac{1}{10 + \dots}}.$$

Then,
$$x - 1 = \frac{1}{10 + \frac{1}{*} \frac{1}{10 + \dots}} = \frac{1}{10 + (x - 1)} = \frac{1}{x + 9}.$$

$\therefore x^2 + 8x = 10, \quad \therefore x = -4 \pm \sqrt{26}.$

Hence,
$$1 + \frac{1}{*} \frac{1}{10 + \frac{1}{10 + \dots}} = -4 + \sqrt{26},$$

using only the positive value of the radical.

11. Let
$$x = 2 + \frac{1}{*} \frac{1}{2 + \dots}. \quad (1)$$

Then,
$$x = 2 + \frac{1}{*} \frac{1}{x} = \frac{2x + 1}{x}.$$

$\therefore x^2 - 2x = 1, \quad \therefore x = 1 \pm \sqrt{2}. \quad (2)$

Next, let
$$y = 1 + \frac{1}{3 + \frac{1}{2 + \frac{1}{2 + \dots}}}.$$

By (1),
$$y = 1 + \frac{1}{3 + \frac{1}{*} \frac{1}{3x + 1}} = \frac{4x + 1}{3x + 1}. \quad (3)$$

Substituting the positive root $1 + \sqrt{2}$ for x in (3),

$$y = \frac{5 + 4\sqrt{2}}{4 + 3\sqrt{2}} = \frac{20 + \sqrt{2} - 24}{16 - 18} = \frac{1}{2}(4 - \sqrt{2}).$$

Hence,
$$1 + \frac{1}{3 + \frac{1}{*} \frac{1}{2 + \frac{1}{*} \frac{1}{2 + \dots}}} = \frac{1}{2}(4 - \sqrt{2}).$$

12. Let
$$x = 1 + \frac{1}{*} \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \dots}}}}. \quad (1)$$

Then,
$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{*} \frac{1}{x}}}}}$$

Convergents,
$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{33}{20}, \frac{33x + 5}{20x + 3},$$

that is,
$$x = \frac{33x + 5}{20x + 3}.$$

$\therefore x^2 - \frac{3}{2}x = \frac{1}{4}, \quad \therefore x = \frac{1}{4}(3 \pm \sqrt{13}). \quad (2)$

Next, let
$$y = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{*} \frac{1}{1 + \dots}}}}}. \quad (3)$$

By (1),
$$y = 3 + \frac{1}{*} \frac{1}{x}.$$

Substituting the positive root of (1) in (3),

$$y = 3 + \frac{4}{3 + \sqrt{13}} = 3 + \frac{4(3 - \sqrt{13})}{9 - 13} = \sqrt{13}.$$

Hence,

$$3 + \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{6+} \frac{1}{1+} \dots = \sqrt{13}.$$

THEORY OF NUMBERS

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$$\begin{array}{r} 2. \quad 8 \overline{)50} \\ \quad \quad 6 \dots 2 \\ \hline \therefore 50 = 62_8. \end{array}$$

$$\begin{array}{r} 8 \overline{)128} \\ \quad \quad 16 \dots 0 \\ \quad \quad \quad 2 \dots 0 \\ \hline \therefore 128 = 200_8. \end{array}$$

$$\begin{array}{r} 8 \overline{)5283} \\ \quad \quad 660 \dots 3 \\ \quad \quad \quad 82 \dots 4 \\ \quad \quad \quad \quad 10 \dots 2 \\ \quad \quad \quad \quad \quad 1 \dots 2 \\ \hline \therefore 5283 = 12243_8. \end{array}$$

$$\begin{array}{r} 3. \quad 5 \overline{)12} \\ \quad \quad 2 \dots 2 \\ \hline \therefore 12 = 22_5. \end{array}$$

$$\begin{array}{r} 5 \overline{)342} \\ \quad \quad 68 \dots 2 \\ \quad \quad \quad 13 \dots 3 \\ \quad \quad \quad \quad 2 \dots 3 \\ \hline \therefore 342 = 2332_5. \end{array}$$

$$\begin{array}{r} 5 \overline{)6627} \\ \quad \quad 1325 \dots 2 \\ \quad \quad \quad 265 \dots 0 \\ \quad \quad \quad \quad 53 \dots 0 \\ \quad \quad \quad \quad \quad 10 \dots 3 \\ \quad \quad \quad \quad \quad \quad 2 \dots 0 \\ \hline \therefore 6627 = 203002_5. \end{array}$$

$$\begin{array}{r} 4. \quad 12 \overline{)15} \\ \quad \quad 1 \dots 3 \\ \hline \therefore 15 = 13_{12}. \end{array}$$

$$\begin{array}{r} 12 \overline{)100} \\ \quad \quad 8 \dots 4 \\ \hline \therefore 100 = 84_{12}. \end{array}$$

$$\begin{array}{r} 12 \overline{)6053} \\ \quad \quad 504 \dots 5 \\ \quad \quad \quad 42 \dots 0 \\ \quad \quad \quad \quad 3 \dots 6 \\ \hline \therefore 6053 = 3605_{12}. \end{array}$$

5. $1 = 1_2$; $2 = 10_2$; $3 = 11_2$; $4 = 100_2$; $5 = 101_2$; $6 = 110_2$; $7 = 111_2$;
 $8 = 1000_2$; $9 = 1001_2$; $10 = 1010_2$.

$$\begin{array}{r} 2. \\ 42_7 \\ \hline 7 \\ 30 \end{array}$$

$$\begin{array}{r} 3. \\ 6654_8 \\ \hline 8 \\ 54 \\ \hline 8 \\ 437 \\ \hline 8 \\ 3500 \end{array}$$

$$\begin{array}{r} 4. \\ 4t6_{11} \\ \hline 11 \\ 54 \\ \hline 11 \\ 600 \end{array}$$

$$\begin{array}{r} 5. \\ 6e4_{12} \\ \hline 12 \\ 83 \\ \hline 12 \\ 1000 \end{array}$$

$$\begin{array}{r} 6. \\ 1111001_2 \\ \hline 2 \\ 3 \\ \hline 2 \\ 7 \\ \hline 2 \\ 15 \\ \hline 8 \\ 121 \end{array}$$

$$\begin{array}{r} 1. \\ 38_{12} \\ 45_{12} \\ \hline e6_{12} \\ 177_{12} \end{array}$$

$$\begin{array}{r} 2. \\ 101_2 \\ 110_2 \\ \hline 101101_2 \\ 111000_2 \end{array}$$

$$\begin{array}{r} 3. \\ 4241_5 \\ 3323_5 \\ \hline 413_5 \end{array}$$

$$\begin{array}{r} 4. \\ 4e5823_{12} \\ 15_{12} \\ \hline 20944e3_{12} \\ 4e5823_{12} \\ \hline 7030723_{12} \end{array}$$

$$\begin{array}{r} 5. \\ 25_7 \overline{) 1304_7} \overline{) 35_7} \\ \underline{111_7} \\ 164_7 \\ \underline{164_7} \end{array}$$

$$\begin{array}{r} 6. \\ 1321_4 \overline{) 23_4} \\ \underline{10_4} \\ 103_4 \overline{) 321_4} \\ \underline{321_4} \end{array}$$

$$\begin{array}{r} 7. \\ 21_{12} = 25 \\ 13_5 = 8 \\ \hline 200 \end{array}$$

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8. Let r be the radix. Then, in any system, 121 stands for $1r^2 + 2r + 1 = r^2 + 2r + 1 = (r+1)^2$, a perfect square.
9. Let r be the radix. Then, $5_r \times 6_r = 36_r = 3r + 6$, or $30_{10} = 3r + 6$. Hence, $r = 8$, the radix of the *octary* scale.
10. Let r be the radix. Then, $\frac{1}{4}$ of $100_r = 30_r$, or $100_r = 30_r \times 4$; that is, $r^2 + 0r + 0 = (3r + 0)4$, or $r^2 = 12r$. Hence, $r = 12$, the radix of the *duo-decimal* scale.
11. Expressing 75 as the sum of powers of 2, or in the binary scale, $75 = 2^6 + 2^3 + 2 + 1 = 1001011_2$.
Hence, the weights used are 64 lb., 8 lb., 2 lb., and 1 lb.

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4. By the corollary to Prin. 2, 431 and 211 are prime.
 $253 = 11 \times 23$; $301 = 7 \times 43$; $623 = 7 \times 89$; $323 = 17 \times 19$.
5. 1001 is divisible by 11; 1002 by 2; 1003 by 17; 1004 by 2; 1005 by 5; 1006 by 2; 1007 by 19; 1008 by 2; 1009 is not divisible by any number less than its square root, which lies between 31 and 32, and consequently is prime; 1010 is divisible by 10; 1011 is divisible by 3.
Hence, all are composite except 1009.
6. Let $2a, 2b, 2c, 2d, \dots$ be even numbers.
Then, $2a + 2b + 2c + 2d + \dots = 2(a + b + c + d)$, an even number.
7. Let $2n + 1$ and $2p + 1$ be two odd numbers, $2n + 1$ the greater.
Then, their sum, $2n + 2p + 2$, or $2(n + p + 1)$, is even, and their difference, $2n - 2p$, or $2(n - p)$, is even.
8. Let $2n + 1$ and $2p + 1$ be two odd numbers, $2n + 1$ the greater.
Then, $(2n + 1)^2 - (2p + 1)^2 = 4n^2 + 4n - 4p^2 - 4p = 4(n - p)(n + p + 1)$.
Since $n + p + 1$ exceeds $n - p$ by $2p + 1$, an odd number, one of the factors $n - p$ and $n + p + 1$ must be even.
Hence, the product is divisible by 4×2 , or 8.
9. Let n be the number. Then, $n^3 - n$ denotes the difference between the number and its cube, n^3 being greater than n .
 $n^3 - n = n(n^2 - 1) = (n - 1)(n)(n + 1)$,
which, Prin. 3, is divisible by $\underline{3}$, or 6.
10. $n^5 - 5n^3 + 4n = n(n^2 - 1)(n^2 - 4) = (n - 2)(n - 1)(n)(n + 1)(n + 2)$,
which, Prin. 3, is divisible by $\underline{5}$, or 120.
11. Every number must belong to one of the forms $3p, 3p + 1, 3p - 1$, the last of which is equivalent to the form $3p + 2$, inasmuch as $3p + 2 = 3(p + 1) - 1$.
 $(3p)^2 = 9p^2 = 3(3p^2) = 3n$.
 $(3p \pm 1)^2 = 9p^2 \pm 6p + 1 = 3(3p^2 \pm 2p) + 1 = 3n + 1$.
Hence, every perfect square has one of the forms $3n, 3n + 1$.

12. Every number must belong to one of the forms $4p$, $4p \pm 1$, $4p + 2$; for $4p + 3 = 4(p + 1) - 1$, which has the same form as $4p - 1$, and $4p - 2 = 4(p - 1) + 2$, which has the same form as $4p + 2$.

$$(4p)^2 = 16p^2 = 4(4p^2) = 4n.$$

$$(4p \pm 1)^2 = 16p^2 \pm 8p + 1 = 4(4p^2 \pm 2p) + 1 = 4n + 1.$$

$$(4p + 2)^2 = 16p^2 + 16p + 4 = 4(4p^2 + 4p + 1) = 4n.$$

Hence, every perfect square has one of the forms $4n$, $4n + 1$.

13. Every number belongs to one of the forms $6n$, $6n \pm 1$, $6n \pm 2$, $6n + 3$; for $6n + 4 = 6(n + 1) - 2$, which has the same form as $6n - 2$; also $6n + 5 = 6(n + 1) - 1$, which has the same form as $6n - 1$; also $6n - 3 = 6(n - 1) + 3$, which has the same form as $6n + 3$.

Now $6n$, $6n \pm 2$, and $6n + 3$ are composite numbers, except when $n = 0$, in which case $6n + 2 = 2$ and $6n + 3 = 3$; and therefore the forms $6n \pm 1$ must contain all the primes except 2 and 3.

14. Let
$$x^2 + 5x + 2 = (x + m)^2.$$

Solving,
$$x = \frac{m^2 - 2}{5 - 2m},$$

which is rational for all rational values of m .

To illustrate, substituting for m the values

$$\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots,$$

$$x = \dots, \frac{14}{3}, \frac{7}{11}, \frac{2}{3}, -\frac{1}{7}, -\frac{2}{3}, -\frac{1}{3}, 2, -7, -\frac{14}{3}, \dots$$

15. Let
$$\sqrt{ax + b} = m.$$

Squaring,
$$ax + b = m^2.$$

$$\therefore x = \frac{m^2 - b}{a}.$$

To illustrate, if $m = 2$, $x = \frac{4 - b}{a}$, and $\sqrt{ax + b} = \sqrt{(4 - b) + b} = \sqrt{4} = 2.$

16. $33957 = 3 \cdot 3 \cdot 7 \cdot 7 \cdot 7 \cdot 11 = 3^2 \cdot 7^3 \cdot 11.$

Hence, the least multiplier that will make 33957 a perfect square is $7 \cdot 11$, or 77; and the least multiplier that will make it a perfect cube is $3 \cdot 11 \cdot 11$, or 363.

DETERMINANTS

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$$\begin{aligned} 3. \quad \begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix} &= 4 \begin{vmatrix} 5 & 7 \\ 1 & 6 \end{vmatrix} - 3 \begin{vmatrix} 9 & 2 \\ 1 & 6 \end{vmatrix} + 8 \begin{vmatrix} 9 & 2 \\ 5 & 7 \end{vmatrix} \\ &= 4(30 - 7) - 3(54 - 2) + 8(63 - 10) = 360. \end{aligned}$$

$$\begin{aligned} 4. \quad \begin{vmatrix} 1 & 7 & 1 \\ 3 & 3 & 3 \\ 5 & 1 & 5 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 3 \\ 1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 7 & 1 \\ 1 & 5 \end{vmatrix} + 5 \begin{vmatrix} 7 & 1 \\ 3 & 3 \end{vmatrix} \\ &= (15 + 3) - 3(35 + 1) + 5(21 - 3) = 0. \end{aligned}$$

$$\begin{aligned} 5. \quad \begin{vmatrix} 3 & 2 & \bar{1} \\ 4 & 8 & \bar{3} \\ 5 & \bar{2} & 1 \end{vmatrix} &= 3 \begin{vmatrix} 8 & \bar{3} \\ \bar{2} & 1 \end{vmatrix} - 4 \begin{vmatrix} 2 & \bar{1} \\ \bar{2} & 1 \end{vmatrix} + 5 \begin{vmatrix} 2 & \bar{1} \\ 8 & \bar{3} \end{vmatrix} \\ &= 3(8 - 6) - 4(2 - 2) + 5(-6 + 8) = 16. \end{aligned}$$

6. Using the elements of the first row as multipliers,

$$\begin{vmatrix} 3 & 2 & 0 & 0 \\ 6 & 4 & 1 & \bar{1} \\ 1 & \bar{2} & 2 & 3 \\ 4 & 3 & 2 & \bar{2} \end{vmatrix} = 3 \begin{vmatrix} 4 & 1 & \bar{1} \\ 2 & 2 & 3 \\ 3 & 2 & \bar{2} \end{vmatrix} - 2 \begin{vmatrix} 6 & 1 & \bar{1} \\ 1 & 2 & 3 \\ 4 & 2 & \bar{2} \end{vmatrix} + 0 - 0$$

$$= 12 \begin{vmatrix} 2 & 3 \\ 2 & \bar{2} \end{vmatrix} + 6 \begin{vmatrix} 1 & \bar{1} \\ 2 & \bar{2} \end{vmatrix} + 9 \begin{vmatrix} 1 & \bar{1} \\ 2 & 3 \end{vmatrix}$$

$$- 12 \begin{vmatrix} 2 & 3 \\ 2 & \bar{2} \end{vmatrix} + 2 \begin{vmatrix} 1 & \bar{1} \\ 2 & \bar{2} \end{vmatrix} - 8 \begin{vmatrix} 1 & \bar{1} \\ 2 & 3 \end{vmatrix}$$

Adding like determinants, $= 0 + 8 \begin{vmatrix} 1 & \bar{1} \\ 2 & \bar{2} \end{vmatrix} + \begin{vmatrix} 1 & \bar{1} \\ 2 & 3 \end{vmatrix}$
 $= 8(-2 + 2) + (3 + 2) = 5.$

7. $\begin{vmatrix} 3 & 4 & 2 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & \bar{1} & \bar{2} & \bar{1} \\ 2 & 0 & 2 & 7 \end{vmatrix} = 3 \begin{vmatrix} 3 & 1 & 2 \\ \bar{1} & \bar{2} & \bar{1} \\ 0 & 2 & 7 \end{vmatrix} - 0 + 0 - 2 \begin{vmatrix} 4 & 2 & 5 \\ 3 & 1 & 2 \\ \bar{1} & \bar{2} & \bar{1} \end{vmatrix}$

$$= 9 \begin{vmatrix} \bar{2} & \bar{1} \\ 2 & 7 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} + 0 - 8 \begin{vmatrix} 1 & 2 \\ \bar{2} & \bar{1} \end{vmatrix}$$

$$+ 6 \begin{vmatrix} 2 & 5 \\ \bar{2} & \bar{1} \end{vmatrix} + 2 \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix}$$

$$= 9(-14 + 2) + 3(7 - 4) - 8(-1 + 4) + 6(-2 + 10) + 2(4 - 5)$$

$$= -77.$$

8. $\begin{vmatrix} 1 & \bar{1} & 1 & \bar{1} \\ 0 & 4 & 2 & 7 \\ 0 & 3 & 2 & 7 \\ 0 & 2 & \bar{2} & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 2 & 7 \\ 3 & 2 & 7 \\ 2 & \bar{2} & 1 \end{vmatrix} - 0 + 0 - 0$

$$= 4 \begin{vmatrix} 2 & 7 \\ \bar{2} & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 7 \\ \bar{2} & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 7 \\ 2 & 7 \end{vmatrix}$$

$$= 4(2 + 14) - 3(2 + 14) + 2(14 - 14) = 16.$$

9. Using the elements of the first row as multipliers,

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 \\ 2 & \bar{1} & \bar{2} & \bar{1} \\ 1 & 3 & 5 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 & 1 \\ \bar{1} & \bar{2} & \bar{1} \\ 3 & 5 & 1 \end{vmatrix} - 0 + 0 - 0$$

$$= 4 \begin{vmatrix} \bar{2} & \bar{1} \\ 3 & 1 \end{vmatrix} - 4 \begin{vmatrix} \bar{1} & \bar{1} \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} \bar{1} & \bar{2} \\ 3 & 5 \end{vmatrix}$$

$$= 4(-2 + 5) - 4(-1 + 3) + 2(-5 + 6) = 6.$$

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11. $25 - 21 = 5 \cdot 5 - 3 \cdot 7 = \begin{vmatrix} 5 & 7 \\ 3 & 5 \end{vmatrix}.$

NOTE. — Many other determinants may be formed from 25 - 21, as

$$\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}, \begin{vmatrix} \bar{5} & 7 \\ 3 & \bar{5} \end{vmatrix}, \begin{vmatrix} \bar{5} & 3 \\ 7 & \bar{5} \end{vmatrix}, \begin{vmatrix} 25 & 1 \\ 21 & 1 \end{vmatrix}, \begin{vmatrix} \bar{1} & \bar{21} \\ \bar{1} & \bar{25} \end{vmatrix}, \text{etc.}$$

The student may select and arrange the elements in any way that will give the proper development.

$$12. \quad 42 + 33 = 6 \cdot 7 - 3(-11) = \begin{vmatrix} 6 & \overline{11} \\ 3 & 7 \end{vmatrix}.$$

See Note, Ex. 11.

$$13. \quad ab - cd = a \cdot b - c \cdot d = \begin{vmatrix} a & d \\ c & b \end{vmatrix}.$$

See Note, Ex. 11.

$$14. \quad a^2 - b^2 = a \cdot a - b \cdot b = \begin{vmatrix} a & b \\ b & a \end{vmatrix}.$$

See Note, Ex. 11.

$$15. \quad a + x = a \cdot 1 - x(-1) = \begin{vmatrix} a & \overline{1} \\ x & 1 \end{vmatrix}.$$

See Note, Ex. 11.

$$16. \quad b^2 + 1 = b \cdot b - 1(-1) = \begin{vmatrix} b & \overline{1} \\ 1 & b \end{vmatrix}.$$

See Note, Ex. 11.

$$17. \quad (a^2 - b^2) - c^2 = (a + b)(a - b) - c \cdot c = \begin{vmatrix} a + b & c \\ c & a - b \end{vmatrix}.$$

See Note, Ex. 11.

$$18. \quad 1 - (x^3 + 1) = 1 \cdot 1 - (x + 1)(x^2 - x + 1) = \begin{vmatrix} 1 & x^2 - x + 1 \\ x + 1 & 1 \end{vmatrix}.$$

See Note, Ex. 11.

$$19. \quad m^3 - (n^3 - n) = m \cdot m^2 - n(n^2 - 1) = \begin{vmatrix} m & n^2 - 1 \\ n & m^2 \end{vmatrix}.$$

See Note, Ex. 11.

$$20. \quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

$$= x_1 \begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

See Note, Ex. 11.

$$21. \quad a^3 - abc - abc + b^3 + c^2x - abx$$

$$= a \begin{vmatrix} a & c \\ b & a \end{vmatrix} - b \begin{vmatrix} c & b \\ b & a \end{vmatrix} + x \begin{vmatrix} c & b \\ a & c \end{vmatrix} = \begin{vmatrix} a & c & b \\ b & a & c \\ x & b & a \end{vmatrix}.$$

See Note, Ex. 11.

$$22. \quad abc - axy - acx + xyz + abx - b^2z$$

$$= a \begin{vmatrix} b & x \\ y & c \end{vmatrix} - x \begin{vmatrix} a & z \\ y & c \end{vmatrix} + b \begin{vmatrix} a & z \\ b & x \end{vmatrix} = \begin{vmatrix} a & a & z \\ x & b & x \\ b & y & c \end{vmatrix}.$$

See Note, Ex. 11.

$$23. \quad a \begin{vmatrix} 3 & a & c \\ 4 & 2 & c \\ 5 & a & b \end{vmatrix} - b \begin{vmatrix} 2 & 1 & b \\ 4 & 2 & c \\ 5 & a & b \end{vmatrix} + c \begin{vmatrix} 2 & 1 & b \\ 3 & a & c \\ 5 & a & b \end{vmatrix} - a \begin{vmatrix} 2 & 1 & b \\ 3 & a & c \\ 4 & 2 & c \end{vmatrix}.$$

If a , b , c , and a are taken for the elements of the first column of a determinant of the *fourth* order, it will be found on trial that this determinant is completed by writing the given determinants of the third order as the *co-factors* of their respective elements, thus:

$$\begin{vmatrix} a & 2 & 1 & b \\ b & 3 & a & c \\ c & 4 & 2 & c \\ a & 5 & a & b \end{vmatrix}.$$

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$$4. \quad \begin{vmatrix} 8 & 4 & 6 \\ 2 & 2 & 4 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 4 & 6 \\ 0 & 5 & 0 \\ 2 & 3 & 4 \end{vmatrix} = -5 \begin{vmatrix} 8 & 6 \\ 2 & 4 \end{vmatrix} = -5 \cdot 20 = -100.$$

$$5. \begin{vmatrix} 4 & 2 & 1 & 2 \\ * & 2 & 3 & 2 & 5 \\ * & 3 & 2 & 1 & 2 \\ 5 & 6 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 5 \\ 3 & 2 & 1 & 2 \\ 3 & 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1.$$

$$6. \begin{vmatrix} 5 & 2 & 7 & 5 \\ 6 & 3 & 1 & 4 \\ 4 & 2 & 1 & 3 \\ 6 & 3 & 2 & 5 \\ * \end{vmatrix} = \begin{vmatrix} 1 & 2 & 7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 \end{vmatrix} = 1^* \begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

$$7. \begin{vmatrix} 4 & \bar{1} & 2 & 1 \\ 5 & \bar{2} & 3 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 4 & 6 \\ * \end{vmatrix} = \begin{vmatrix} 4 & \bar{1} & 1 & 1 \\ 5 & \bar{2} & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 6 & 6 \end{vmatrix} = 0 \text{ (Prin. 5).}$$

$$8. \begin{vmatrix} 2 & 4 & 4 & 6 & 1 \\ 2 & 5 & 3 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \\ 2 & 1 & \bar{1} & 1 & 2 \\ 5 & 2 & 2 & 3 & 1 \\ * \quad * \end{vmatrix} = \begin{vmatrix} 1 & 0 & 4 & 5 & 1 \\ 0 & 2 & 3 & \bar{1} & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \bar{1} & 2 \\ 4 & 0 & 2 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 4 & 5 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & \bar{1} & 2 \\ 4 & 2 & 2 & 1 \\ * \end{vmatrix} = 2 \begin{vmatrix} \bar{1} & 3 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 4 & \bar{1} & \bar{3} & 2 \\ 2 & 1 & 1 & 1 \end{vmatrix} \\ = 2 \begin{vmatrix} \bar{1} & 3 & 4 \\ 4 & \bar{1} & \bar{3} \\ 2 & 1 & 1 \\ * \quad * \end{vmatrix} = 2 \begin{vmatrix} \bar{8} & 3 & 4 \\ 0 & \bar{1} & \bar{3} \\ 0 & 1 & 1 \end{vmatrix} = -16 \begin{vmatrix} \bar{1} & \bar{3} \\ 1 & 1 \end{vmatrix} = -32.$$

$$9. \begin{vmatrix} 2 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 4 \\ 3 & 2 & 1 & 3 & 2 \\ 2 & 3 & 4 & 0 & 2 \\ 3 & 3 & 2 & 3 & 0 \\ * \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ \bar{3} & 2 & \bar{2} & 4 & \bar{2} \\ \bar{1} & 2 & \bar{3} & \bar{3} & 4 \\ 4 & 3 & 2 & 9 & \bar{7} \\ \bar{3} & 3 & 4 & 6 & 9 \end{vmatrix} = -1^* \begin{vmatrix} \bar{3} & \bar{2} & 4 & \bar{2} \\ \bar{1} & \bar{3} & \bar{3} & 4 \\ 4 & 2 & 9 & \bar{7} \\ \bar{3} & 4 & 6 & 9 \end{vmatrix} \\ = -1 \begin{vmatrix} 0 & 7 & 5 & 10 \\ \bar{1} & \bar{3} & \bar{3} & 4 \\ 0 & 10 & 3 & 9 \\ 0 & 5 & 3 & 3 \end{vmatrix} = - \begin{vmatrix} 7 & 5 & 10 \\ 10 & 3 & 9 \\ 5 & 3 & 3 \end{vmatrix} = - \begin{vmatrix} 2 & 2 & 7 \\ 0 & \bar{3} & 3 \\ 5 & 3 & 3 \end{vmatrix} \\ = -2 \begin{vmatrix} \bar{3} & 3 \\ 3 & 3 \end{vmatrix} - 5 \begin{vmatrix} 2 & 7 \\ 3 & 3 \end{vmatrix} = 36 - 135 = -99.$$

$$10. \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \\ * \quad * \quad * \end{vmatrix} = \begin{vmatrix} a & 1 & 1 & a+3 \\ 1 & a & 1 & a+3 \\ 1 & 1 & a & a+3 \\ 1 & 1 & 1 & a+3 \end{vmatrix} = (a+3) \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ * & 1 & 1 & 1 \end{vmatrix}$$

$$= (a+3) \begin{vmatrix} a-1 & 0 & 0 & 0 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = (a+3)(a-1) \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ * & 1 & 1 \end{vmatrix} = (a+3)(a-1) \begin{vmatrix} a-1 & 0 & 0 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (a+3)(a-1)^2 \begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} = (a+3)(a-1)^3.$$

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2. In

$$D = \begin{vmatrix} x & 1 & b \\ y & 1 & a \\ x & 1 & a \end{vmatrix},$$

the second and third rows are identical when $x = y$, and the first and third rows are identical when $a = b$. Hence, $x - y$ and $a - b$ are factors of D . Since every constituent of D is of the second degree, no other factor need be sought.

Since the principal diagonal ax is positive, and ax is positive also in the product expanded from $(a - b)(x - y)$,

$$D = (a - b)(x - y).$$

3. In

$$D = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix},$$

the first and second rows are identical when $a = b$, the first and third rows are identical when $a = c$, and the second and third rows are identical when $b = c$. Hence, $a - b$, $a - c$, and $b - c$ are factors of D . Since every constituent of D is of the third degree, no other factors need be sought.

To determine the signs, since the principal diagonal a^2b is positive, and a^2b is positive also in the product expanded from $(a - b)(a - c)(b - c)$,

$$D = (a - b)(a - c)(b - c).$$

4. In

$$D = \begin{vmatrix} 2 & 2 & 5 \\ 2 & x & 5 \\ x & 3 & 5 \end{vmatrix},$$

the first and second rows are identical when $x = 2$, and the first and third columns are multiples of the same column when $x = 2$; also, the determinant has a numerical factor, 5, or -5 .

The secondary diagonal, which is the constituent of the highest degree, is $-5x^2$. Hence,

$$D = -5(x - 2)(x - 2).$$

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$$1. \begin{cases} 2x + 5y = 9, \\ 3x + 2y = 8. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 9 & 5 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix}} = \frac{-22}{-11} = 2,$$

$$\text{and } y = \frac{\begin{vmatrix} 2 & 9 \\ 3 & 8 \end{vmatrix}}{-11} = \frac{-11}{-11} = 1.$$

$$3. \begin{cases} 2x + 7y = 30, \\ x + 4y = 17. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 30 & 7 \\ 17 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 7 \\ 1 & 4 \end{vmatrix}} = \frac{1}{1} = 1,$$

$$\text{and } y = \frac{\begin{vmatrix} 2 & 30 \\ 1 & 17 \end{vmatrix}}{1} = \frac{4}{1} = 4.$$

$$2. \begin{cases} 3x + 2y = 12, \\ 4x + 3y = 17. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 12 & 2 \\ 17 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} = \frac{2}{1} = 2,$$

$$\text{and } y = \frac{\begin{vmatrix} 3 & 12 \\ 4 & 17 \end{vmatrix}}{1} = \frac{3}{1} = 3.$$

$$4. \begin{cases} 5x + y = 12, \\ 2x + 3y = 10. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 12 & 1 \\ 10 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{26}{13} = 2,$$

$$\text{and } y = \frac{\begin{vmatrix} 5 & 12 \\ 2 & 10 \end{vmatrix}}{13} = \frac{26}{13} = 2.$$

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$$5. \begin{cases} 4x - 3y = 8, \\ x + 4y = 21. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 8 & -3 \\ 21 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 1 & 4 \end{vmatrix}} = \frac{95}{19} = 5,$$

$$\text{and } y = \frac{\begin{vmatrix} 4 & 8 \\ 1 & 21 \end{vmatrix}}{19} = \frac{76}{19} = 4.$$

$$7. \begin{cases} ax + by = c, \\ mx + ny = d. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} c & b \\ d & n \end{vmatrix}}{\begin{vmatrix} a & b \\ m & n \end{vmatrix}} = \frac{cn - bd}{an - bm},$$

$$\text{and } y = \frac{\begin{vmatrix} a & c \\ m & d \end{vmatrix}}{an - bm} = \frac{ad - cm}{an - bm}.$$

$$6. \begin{cases} 3x - 2y = -2, \\ 2x - 3y = -5. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} -2 & -2 \\ -5 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}} = \frac{-4}{-5} = \frac{4}{5},$$

$$\text{and } y = \frac{\begin{vmatrix} 3 & -2 \\ 2 & -5 \end{vmatrix}}{-5} = \frac{-11}{-5} = \frac{11}{5}.$$

$$8. \begin{cases} ax - by = r, \\ cx + dy = s. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} r & -b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & -b \\ c & d \end{vmatrix}} = \frac{dr + bs}{ad + bc},$$

$$\text{and } y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{ad + bc} = \frac{as - cr}{ad + bc}.$$

$$9. \begin{cases} 2x + 5y + 2z = 27, \\ 8x + 6y + 3z = 46, \\ 3x + 7y + 5z = 47. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 27 & 5 & 2 \\ 46 & 6 & 3 \\ 47 & 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & 2 \\ 8 & 6 & 3 \\ 3 & 7 & 5 \end{vmatrix}} = \frac{27(30 - 21) - 46(25 - 14) + 47(15 - 12)}{2(30 - 21) - 8(25 - 14) + 3(15 - 12)} = \frac{-122}{-61} = 2;$$

$$y = \frac{\begin{vmatrix} 2 & 27 & 2 \\ 8 & 46 & 3 \\ 3 & 47 & 5 \end{vmatrix}}{-61} = \frac{2(230 - 141) - 8(135 - 94) + 3(81 - 92)}{-61} = \frac{-183}{-61} = 3;$$

$$z = \frac{\begin{vmatrix} 2 & 5 & 27 \\ 8 & 6 & 46 \\ 3 & 7 & 47 \end{vmatrix}}{-61} = \frac{2(282 - 322) - 8(235 - 189) + 3(230 - 162)}{-61} = \frac{-244}{-61} = 4.$$

$$10. \begin{cases} 9x + 2y + z = 25, \\ 5x + y + z = 14, \\ 7x + 3y + 2z = 25. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 25 & 2 & 1 \\ 14 & 1 & 1 \\ 25 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 9 & 2 & 1 \\ 5 & 1 & 1 \\ 7 & 3 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 0 & 1 \\ \overline{11} & \overline{1} & 1 \\ 25 & \overline{1} & 2 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \overline{4} & \overline{1} & 1 \\ \overline{11} & 1 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} \overline{11} & \overline{1} \\ 25 & \overline{1} \end{vmatrix}}{\begin{vmatrix} \overline{4} & \overline{1} \\ \overline{11} & \overline{1} \end{vmatrix}} = \frac{-14}{-7} = 2;$$

$$y = \frac{\begin{vmatrix} 9 & 25 & 1 \\ 5 & 14 & 1 \\ 7 & 25 & 2 \end{vmatrix}^*}{-7} = \frac{\begin{vmatrix} 0 & 0 & 1 \\ \bar{4} & \bar{11} & 1 \\ \bar{11} & \bar{25} & 2 \end{vmatrix}}{-7} = \frac{\begin{vmatrix} \bar{4} & \bar{11} \\ \bar{11} & \bar{25} \end{vmatrix}}{-7} = \frac{-21}{-7} = 3;$$

$$z = \frac{\begin{vmatrix} 9 & 2 & 25 \\ 5 & 1 & 14 \\ 7 & 3 & 25 \end{vmatrix}^*}{-7} = \frac{\begin{vmatrix} \bar{1} & 0 & \bar{3} \\ \bar{5} & 1 & 14 \\ \bar{8} & 0 & \bar{17} \end{vmatrix}}{-7} = \frac{\begin{vmatrix} \bar{1} & \bar{3} \\ \bar{8} & \bar{17} \end{vmatrix}}{-7} = \frac{-7}{-7} = 1.$$

$$11. \begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2a^2 - 2b^2. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 4ab & -a+b \\ 2a^2-2b^2 & a+b \end{vmatrix}}{\begin{vmatrix} a+b & -a+b \\ a-b & a+b \end{vmatrix}} = \frac{2(a+b) \begin{vmatrix} 2ab & -a+b \\ a-b & 1 \end{vmatrix}}{(a+b)^2 + (a-b)^2} \\ = \frac{2(a+b)(a^2+b^2)}{2(a^2+b^2)} = a+b;$$

$$y = \frac{\begin{vmatrix} a+b & 4ab \\ a-b & 2a^2-2b^2 \end{vmatrix}}{2(a^2+b^2)} = \frac{2(a-b) \begin{vmatrix} a+b & 2ab \\ 1 & a+b \end{vmatrix}}{2(a^2+b^2)} = a-b$$

$$12. \begin{cases} 3x + 2y + 3z = 17, \\ 2x + y + 2z = 10, \\ 5x + 5y + z = 29. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} 17 & 2 & 3 \\ 10 & 1 & 2 \\ 29 & 5 & 1 \end{vmatrix}^*}{4} = \frac{\begin{vmatrix} \bar{3} & 0 & \bar{1} \\ 10 & 1 & 2 \\ \bar{21} & 0 & \bar{9} \end{vmatrix}}{4} = \frac{\begin{vmatrix} \bar{3} & \bar{1} \\ \bar{21} & \bar{9} \end{vmatrix}}{4} = \frac{9}{4} = \frac{9}{4};$$

$$y = \frac{\begin{vmatrix} 3 & 17 & 3 \\ 2 & 10 & 2 \\ 5 & 29 & 1 \end{vmatrix}^*}{4} = \frac{\begin{vmatrix} \bar{12} & \bar{70} & 3 \\ \bar{8} & \bar{48} & 2 \\ 0 & 0 & 1 \end{vmatrix}}{4} = \frac{\begin{vmatrix} \bar{12} & \bar{70} \\ \bar{8} & \bar{48} \end{vmatrix}}{4} = \frac{16}{4} = 4;$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 17 \\ 2 & 1 & 10 \\ 5 & 5 & 29 \end{vmatrix}^*}{4} = \frac{\begin{vmatrix} \bar{1} & 0 & \bar{3} \\ 2 & 1 & 10 \\ \bar{5} & 0 & \bar{21} \end{vmatrix}}{4} = \frac{\begin{vmatrix} \bar{1} & \bar{3} \\ \bar{5} & \bar{21} \end{vmatrix}}{4} = \frac{6}{4} = \frac{3}{2}.$$

$$13. \begin{cases} 2x + 3y - 4z = 18, \\ x + y + z = 12, \\ 5x - y - z = 12. \end{cases}$$

$$\begin{aligned}
 \therefore x &= \frac{\begin{vmatrix} 18 & 3 & \bar{4} \\ 12 & 1 & 1 \\ 12 & \bar{1} & \bar{1} \end{vmatrix}}{\begin{vmatrix} 2 & 3 & \bar{4} \\ 1 & 1 & 1 \\ 5 & \bar{1} & \bar{1} \end{vmatrix}} = \frac{\begin{vmatrix} 18 & 3 & \bar{4} \\ 12 & 1 & 1 \\ 24 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & \bar{4} \\ 1 & 1 & 1 \\ 6 & 0 & 0 \end{vmatrix}} = \frac{24 \begin{vmatrix} 3 & \bar{4} \\ 1 & 1 \end{vmatrix}}{6 \begin{vmatrix} 3 & \bar{4} \\ 1 & 1 \end{vmatrix}} = 4; \\
 y &= \frac{\begin{vmatrix} 2 & 18 & \bar{4} \\ 1 & 12 & 1 \\ 5 & 12 & \bar{1} \end{vmatrix}}{\begin{vmatrix} 2 & 3 & \bar{4} \\ 1 & 1 & 1 \\ 5 & \bar{1} & \bar{1} \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \bar{6} & \bar{6} \\ 1 & 12 & 1 \\ 0 & \bar{48} & \bar{6} \end{vmatrix}}{6 \cdot 7} = \frac{6 \begin{vmatrix} 1 & 1 \\ \bar{48} & \bar{6} \end{vmatrix}}{6 \cdot 7} = \frac{42}{7} = 6; \\
 z &= \frac{\begin{vmatrix} 2 & 3 & 18 \\ 1 & 1 & 12 \\ 5 & \bar{1} & 12 \end{vmatrix}}{6 \cdot 7} = \frac{\begin{vmatrix} \bar{1} & 0 & \bar{18} \\ \bar{1} & 1 & 12 \\ 6 & 0 & 24 \end{vmatrix}}{6 \cdot 7} = \frac{\begin{vmatrix} \bar{1} & \bar{18} \\ 6 & 24 \end{vmatrix}}{6 \cdot 7} = \frac{34}{7} = 2.
 \end{aligned}$$

$$14. \begin{cases} x + 2y + z = 0, \\ 2x + y + z = 2a - b, \\ x - y - 2z = 3b. \end{cases}$$

$$\begin{aligned}
 \therefore x &= \frac{\begin{vmatrix} 0 & 2 & 1 \\ 2a-b & 1 & 1 \\ 3b & \bar{1} & \bar{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ * & \bar{1} & \bar{2} \end{vmatrix}} = \frac{-(2a-b)(-3) + 3b}{\begin{vmatrix} 3 & 0 & \bar{3} \\ 3 & 0 & \bar{1} \\ 1 & \bar{1} & \bar{2} \end{vmatrix}} \\
 &= \frac{6a}{\begin{vmatrix} 3 & \bar{3} \\ 3 & \bar{1} \end{vmatrix}} = \frac{6a}{6} = a;
 \end{aligned}$$

$$\begin{aligned}
 y &= \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2a-b & 1 \\ * & 3b & 2 \end{vmatrix}}{6} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2a-b & \bar{1} \\ 1 & 3b & \bar{3} \end{vmatrix}}{6} \\
 &= \frac{\begin{vmatrix} 2a-b & \bar{1} \\ 3b & \bar{3} \end{vmatrix}}{6} = \frac{-6(a-b)}{6} = b - a.
 \end{aligned}$$

$$\begin{aligned}
 z &= \frac{\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2a-b \\ * & \bar{1} & 3b \end{vmatrix}}{6} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & \bar{3} & 2a-b \\ 1 & \bar{3} & 3b \end{vmatrix}}{6} \\
 &= \frac{\begin{vmatrix} \bar{3} & 2a-b \\ \bar{3} & 3b \end{vmatrix}}{6} = a - 2b.
 \end{aligned}$$

$$15. \begin{cases} x + y + 0z = 2a, \\ 0x + y + z = 3a - b, \\ x + 0y + z = 3a. \end{cases}$$

$$\therefore x = \frac{\begin{vmatrix} * & 2a & 1 & 0 \\ 3a - b & 1 & 1 \\ 3a & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 2a & 1 & 0 \\ a - b & 0 & 1 \\ 3a & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ \bar{1} & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix}} = - \frac{\begin{vmatrix} a - b & 1 \\ 3a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{2a + b}{2};$$

$$y = \frac{\begin{vmatrix} * & 1 & 2a & 0 \\ 0 & 3a - b & 1 \\ 1 & 3a & 1 \end{vmatrix}}{2} = \frac{\begin{vmatrix} 1 & 2a & 0 \\ \bar{1} & -b & 0 \\ 1 & 3a & 1 \end{vmatrix}}{2} = \frac{\begin{vmatrix} 1 & 2a \\ \bar{1} & -b \end{vmatrix}}{2} = \frac{2a - b}{2};$$

$$z = \frac{\begin{vmatrix} * & 1 & 1 & 2a \\ 0 & 1 & 3a - b \\ 1 & 0 & 3a \end{vmatrix}}{2} = \frac{\begin{vmatrix} 1 & 1 & 2a \\ 0 & 1 & 3a - b \\ 0 & \bar{1} & a \end{vmatrix}}{2} = \frac{\begin{vmatrix} 1 & 3a - b \\ \bar{1} & a \end{vmatrix}}{2} = \frac{4a - b}{2}.$$

$$16. \begin{cases} u - x + 2y - 3z = -5, \\ 3u - x + y - 2z = 2, \\ 2u + x + y - z = 9, \\ -5u + 2x - 7y + z = -12. \end{cases}$$

$$\therefore u = \frac{\begin{vmatrix} 5 & \bar{1} & 2 & 3 \\ 2 & \bar{1} & 1 & \bar{2} \\ 9 & 1 & 1 & \bar{1} \\ \bar{12} & 2 & \bar{7} & 1 \end{vmatrix}}{\begin{vmatrix} * & 1 & 1 & 2 \\ 3 & 1 & 1 & \bar{2} \\ 2 & 1 & 1 & \bar{1} \\ \bar{5} & 2 & \bar{7} & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \bar{1} & 0 & 0 \\ 7 & \bar{1} & \bar{1} & 1 \\ 4 & 1 & 3 & \bar{4} \\ \bar{22} & 2 & \bar{3} & \bar{5} \end{vmatrix}}{\begin{vmatrix} 7 & \bar{1} & 1 \\ 4 & 3 & \bar{4} \\ \bar{22} & \bar{3} & \bar{5} \end{vmatrix}} = \frac{\begin{vmatrix} * & 2 & \bar{1} & 1 \\ 3 & 3 & \bar{4} \\ \bar{3} & \bar{3} & \bar{5} \end{vmatrix}}{\begin{vmatrix} 2 & \bar{1} & 1 \\ 3 & 3 & \bar{4} \\ \bar{3} & \bar{3} & \bar{5} \end{vmatrix}} =$$

$$= \frac{\begin{vmatrix} 0 & \bar{1} & 0 \\ 25 & 3 & \bar{1} \\ 43 & \bar{3} & \bar{8} \end{vmatrix}}{\begin{vmatrix} 25 & \bar{1} \\ 43 & \bar{8} \end{vmatrix}} = \frac{-243}{-81} = 3;$$

$$x = \frac{\begin{vmatrix} * & 1 & \bar{5} & 2 & \bar{3} \\ 3 & 2 & 1 & \bar{2} \\ 2 & 9 & 1 & \bar{1} \\ \bar{5} & \bar{12} & \bar{7} & 1 \end{vmatrix}}{-81} = \frac{\begin{vmatrix} \bar{5} & \bar{32} & \bar{1} & 0 \\ \bar{1} & \bar{16} & \bar{1} & 0 \\ 2 & 9 & 1 & \bar{1} \\ \bar{3} & \bar{3} & \bar{6} & 0 \end{vmatrix}}{-81} = \frac{\begin{vmatrix} \bar{5} & \bar{32} & \bar{1} \\ \bar{1} & \bar{16} & \bar{1} \\ \bar{3} & \bar{3} & \bar{6} \end{vmatrix}}{-81} = \frac{\begin{vmatrix} \bar{5} & \bar{32} & \bar{1} \\ 1 & \bar{16} & \bar{1} \\ 1 & 1 & 2 \end{vmatrix}}{27} =$$

$$= \frac{\begin{vmatrix} 0 & \bar{27} & 9 \\ 0 & \bar{15} & 1 \\ 1 & 1 & 2 \end{vmatrix}}{27} = \frac{\begin{vmatrix} \bar{27} & 9 \\ \bar{15} & 1 \end{vmatrix}}{27} = \frac{\begin{vmatrix} \bar{3} & 1 \\ \bar{15} & 1 \end{vmatrix}}{3} = \frac{\begin{vmatrix} \bar{1} & 1 \\ \bar{5} & 1 \end{vmatrix}}{1} = 4;$$

$$y = \frac{\begin{vmatrix} 1 & \bar{1} & \bar{5} & \bar{3} \\ 3 & \bar{1} & 2 & \bar{2} \\ 2 & 1 & 9 & \bar{1} \\ \bar{5} & 2 & \bar{12} & \bar{1} \end{vmatrix}}{-81} = \frac{\begin{vmatrix} 0 & \bar{1} & 0 & 0 \\ 2 & \bar{1} & 7 & 1 \\ 3 & 1 & 4 & \bar{4} \\ 3 & 2 & \bar{22} & 5 \end{vmatrix}}{-81} = \frac{\begin{vmatrix} 2 & 7 & 1 \\ 3 & 4 & \bar{4} \\ 3 & \bar{22} & \bar{5} \end{vmatrix}}{-81} = \frac{\begin{vmatrix} 2 & 7 & 1 \\ 11 & 32 & 0 \\ 7 & 13 & 0 \end{vmatrix}}{-81}$$

$$= \frac{\begin{vmatrix} 11 & 32 \\ 7 & 13 \end{vmatrix}}{-81} = \frac{-81}{-81} = 1;$$

$$z = \frac{\begin{vmatrix} 1 & \bar{1} & 2 & \bar{5} \\ 3 & \bar{1} & 1 & 2 \\ 2 & 1 & 1 & 9 \\ \bar{5} & 2 & \bar{7} & \bar{12} \end{vmatrix}}{-81} = \frac{\begin{vmatrix} 0 & \bar{1} & 0 & 0 \\ 2 & \bar{1} & \bar{1} & 7 \\ 3 & 1 & 3 & 4 \\ 3 & 2 & 3 & \bar{22} \end{vmatrix}}{-81} = \frac{\begin{vmatrix} 2 & \bar{1} & 7 \\ 3 & 3 & 4 \\ 3 & 3 & \bar{22} \end{vmatrix}}{-81} = \frac{\begin{vmatrix} 0 & \bar{1} & 0 \\ 9 & 3 & 25 \\ 9 & 3 & 43 \end{vmatrix}}{-81}$$

$$= \frac{\begin{vmatrix} 9 & 25 \\ 9 & 43 \end{vmatrix}}{-81} = \frac{-162}{-81} = 2.$$

$$17. \begin{cases} 2u + 3v - 4x + y = 0, \\ u - v + x - y = -2, \\ 7u + 2v - 3x + y = 6, \\ 5u + 8v - 10x + 3y = 3. \end{cases}$$

$$\therefore u = \frac{\begin{vmatrix} 0 & 3 & \bar{4} & 1 \\ \bar{2} & \bar{1} & 1 & \bar{1} \\ 6 & 2 & \bar{3} & 1 \\ 3 & 8 & \bar{10} & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & \bar{4} & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 7 & 2 & \bar{3} & 1 \\ 5 & 8 & \bar{10} & 3 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ \bar{2} & 2 & \bar{3} & \bar{1} \\ 6 & \bar{1} & 1 & 1 \\ 3 & \bar{1} & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3 & 2 & \bar{3} & \bar{1} \\ 5 & \bar{1} & 1 & 1 \\ \bar{1} & \bar{1} & 2 & 3 \end{vmatrix}} = \frac{\begin{vmatrix} \bar{2} & 2 & \bar{3} \\ 6 & \bar{1} & 1 \\ 3 & \bar{1} & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & \bar{3} \\ 5 & \bar{1} & 1 \\ \bar{1} & \bar{1} & 2 \end{vmatrix}} = \frac{\begin{vmatrix} 4 & 0 & 1 \\ 3 & 0 & \bar{1} \\ 3 & \bar{1} & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & \bar{1} \\ \bar{1} & \bar{1} & 2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 4 & 1 \\ 3 & \bar{1} \\ \bar{1} & 1 \\ 6 & \bar{1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 6 & \bar{1} \end{vmatrix}} = \frac{7}{7} = 1;$$

$$v = \frac{\begin{vmatrix} 2 & 0 & \bar{4} & 1 \\ 1 & \bar{2} & 1 & 1 \\ 7 & 6 & \bar{3} & 1 \\ 5 & 3 & \bar{10} & 3 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3 & 2 & \bar{3} & \bar{1} \\ 5 & 6 & 1 & 1 \\ \bar{1} & 3 & 2 & 3 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 3 & \bar{2} & \bar{3} \\ 5 & 6 & 1 \\ \bar{1} & 3 & 2 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 0 & 7 & 3 \\ 0 & 21 & 11 \\ \bar{1} & 3 & 2 \end{vmatrix}}{7}$$

$$= \frac{\begin{vmatrix} 7 & 3 \\ 21 & 11 \end{vmatrix}}{7} = \frac{14}{7} = 2;$$

$$x = \frac{\begin{vmatrix} 2 & 3 & 0 & 1 \\ 1 & \bar{1} & \bar{2} & \bar{1} \\ 7 & 2 & \bar{6} & 1 \\ 5 & 8 & 3 & 3 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3 & 2 & \bar{2} & \bar{1} \\ 5 & \bar{1} & 6 & 1 \\ \bar{1} & \bar{1} & 3 & 3 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 3 & 2 & \bar{2} \\ 5 & \bar{1} & 6 \\ \bar{1} & \bar{1} & 3 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 0 & \bar{1} & 7 \\ 0 & \bar{6} & 21 \\ \bar{1} & \bar{1} & 3 \end{vmatrix}}{7}$$

$$= \frac{\begin{vmatrix} \bar{1} & 7 \\ \bar{6} & 21 \end{vmatrix}}{7} = \frac{\begin{vmatrix} \bar{1} & 1 \\ \bar{6} & 3 \end{vmatrix}}{1} = 3;$$

$$y = \frac{\begin{vmatrix} 2 & 3 & \bar{4} & 0 \\ 1 & \bar{1} & 1 & \bar{2} \\ 7 & 2 & \bar{3} & 6 \\ 5 & 8 & \bar{10} & 3 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 0 & 5 & \bar{6} & 4 \\ 1 & \bar{1} & 1 & \bar{2} \\ 0 & 9 & \bar{10} & 20 \\ 0 & 13 & \bar{15} & 13 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 5 & \bar{6} & 4 \\ 9 & \bar{10} & 20 \\ 13 & \bar{15} & 13 \end{vmatrix}}{7} = \frac{\begin{vmatrix} 1 & \bar{2} & 4 \\ \bar{11} & 10 & 20 \\ 0 & \bar{2} & 13 \end{vmatrix}}{7}$$

$$= \frac{\begin{vmatrix} 1 & \bar{2} & 4 \\ 0 & \bar{12} & 64 \\ 0 & \bar{2} & 13 \end{vmatrix}}{7} = \frac{\begin{vmatrix} \bar{12} & 64 \\ \bar{2} & 13 \end{vmatrix}}{7} = \frac{28}{7} = 4.$$

CONVERGENCY OF SERIES

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15. $1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \dots$
 $= 1 + \frac{1}{3} + (\frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + (\text{next 8 terms}) + \dots$
 $> 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \dots$
 term by term after the second.

Hence, Prin. 3, the series is *divergent*.

16. $1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \dots$
 $u_n = (-1)^{n-1} \frac{1}{2n-1} \therefore \lim (u_n)_{n \rightarrow \infty} = 0.$

Hence, 3d test, the series is *convergent*.

By the solution of example 15, it is seen that the series does not remain convergent when all the terms are made positive, but becomes divergent. Hence, the series is not absolutely convergent but only *conditionally* convergent.

17. $1 + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \dots$
 $\frac{u_{n+1}}{u_n} = \frac{2n+1}{2^n} \div \frac{2n-1}{2^{n-1}} = \frac{2n+1}{2(2n-1)} = \frac{1}{2} \left(1 + \frac{2}{2n-1} \right).$

Therefore, the limit of the ratio of convergency as $n \rightarrow \infty$ is $\frac{1}{2}$.

Hence, 2d test, the series is *absolutely convergent*.

18. $\frac{3}{2} + \frac{3}{2} + \frac{4}{3} + \frac{5}{3} + \dots$
 $\frac{u_{n+1}}{u_n} = \frac{n+2}{2^{n+1}} \div \frac{n+1}{2^n} = \frac{n+2}{2(n+1)} = \frac{1}{2} \left(1 + \frac{1}{n+1} \right).$

Therefore, the limit of the ratio of convergency as $n \rightarrow \infty$ is $\frac{1}{2}$.

Hence, 2d test, the series is *absolutely convergent*.

$$19. \quad 1 + \frac{2^2}{2} + \frac{3^2}{2^2} + \frac{4^2}{2^3} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^n} \div \frac{n^2}{2^{n-1}} = \frac{(n+1)^2}{2n^2} = \frac{1}{2} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right).$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is $\frac{1}{2}$.
Hence, 2d test, the series is *absolutely convergent*.

$$20. \quad \frac{1}{2^0 \cdot 3} + \frac{1}{2 \cdot 3^2} + \frac{1}{2^2 \cdot 3^3} + \dots$$

This is a geometrical series whose ratio is $\frac{1}{6}$.

Hence, 1st test, the series is *absolutely convergent*.

$$21. \quad 1 + \frac{2 \cdot 2}{3} + \frac{3 \cdot 2^2}{3^2} + \frac{4 \cdot 2^3}{3^3} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)(\frac{2}{3})^n}{n(\frac{2}{3})^{n-1}} = \frac{2}{3} \cdot \frac{n+1}{n} = \frac{2}{3} \left(1 + \frac{1}{n} \right).$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is $\frac{2}{3}$.
Hence, 2d test, the series is *absolutely convergent*.

$$22. \quad \frac{1}{\sqrt{1}+1} + \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{1}{\sqrt{n+1}+1} \div \frac{1}{\sqrt{n}+1} = \frac{\sqrt{n}+1}{\sqrt{n+1}+1}.$$

Now, by § 585, the auxiliary series $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$ is divergent, the value of p being $\frac{1}{2}$. The ratio of convergency of this series is

$$\frac{1}{\sqrt{n+1}} \div \frac{1}{\sqrt{n}}, \text{ or } \frac{\sqrt{n}}{\sqrt{n+1}}.$$

But $\frac{\sqrt{n}+1}{\sqrt{n+1}+1} > \frac{\sqrt{n}}{\sqrt{n+1}}$ and their common limit as $n \doteq \infty$ is 1.

Hence, Prin. 4, the series is *divergent*.

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$$23. \quad 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{(-1)^n(n+1)x^n}{(-1)^{n-1}nx^{n-1}} = -\frac{n+1}{n}x = -\left(1 + \frac{1}{n}\right)x.$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is $-x$.

Hence, 2d test, the series is absolutely convergent when x is numerically less than 1, and divergent when x is numerically greater than 1. When $x = \pm 1$ this test fails.

When $x = 1$,

$$S_1 = 1; S_2 = -1; S_3 = 2; S_4 = -2; S_5 = 3; S_6 = -3; \text{ etc.}$$

$$\therefore S_{2n-1} = +n \text{ and } S_{2n} = -n.$$

Since the sum of $2n-1$ or of $2n$ terms can be made as large as we please by taking n sufficiently great, the series is divergent when $x = 1$.

Much more is the series divergent when $x = -1$.

$$24. \quad 1 + 2x + 4x^2 + 6x^3 + 8x^4 + 10x^5 + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{2nx^n}{2(n-1)x^{n-1}} = \frac{n}{n-1}x = \left(1 + \frac{1}{n-1}\right)x.$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is x .

Hence, 2d test, the series is absolutely convergent when x is numerically less than 1, and divergent when x is numerically greater than 1.

When $x = 1$, the series is evidently divergent.

When $x = -1$, $S_1 = 1$; $S_2 = -1$; $S_3 = 3$; $S_4 = -3$; $S_5 = 5$; $S_6 = -5$; etc.
 $\therefore S_{2n-1} = +n$ and $S_{2n} = -n$.

Since the sum of $2n - 1$ terms or of $2n$ terms can be made as large as we please by taking n sufficiently great, the series is divergent when $x = -1$.

$$25. \quad \frac{1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{17}x^4 + \frac{1}{28}x^5 + \dots}{\frac{u_{n+1}}{u_n} = \frac{x^n}{n^2 + 1} \div \frac{x^{n-1}}{(n-1)^2 + 1} = \frac{(n-1)^2 + 1}{n^2 + 1} x = \left(1 - \frac{2n-1}{n^2 + 1}\right)x}.$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is x .

Hence, 2d test, the series is absolutely convergent when x is numerically less than 1, and divergent when x is numerically greater than 1.

When x is numerically equal to 1, each term of the series

$$\frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{17}x^4 + \frac{1}{28}x^5 + \dots$$

is numerically less than the corresponding term of the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots,$$

which is absolutely convergent (§ 585).

Hence, Prin. 1 and 3, the given series is absolutely convergent also when x is numerically equal to 1.

$$26. \quad 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{x^n}{n} \div \frac{x^{n-1}}{n-1} = \frac{x}{n}.$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is 0, since, however large x is, n may be taken so much larger that as n increases the ratio $\doteq 0$.

Hence, 2d test, the series is absolutely convergent for all finite values of x .

$$27. \quad 1 + \frac{1 \cdot 2}{x} + \frac{2 \cdot 3}{x^2} + \frac{3 \cdot 4}{x^3} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{n(n+1)}{x^n} \div \frac{(n-1)n}{x^{n-1}} = \frac{n+1}{n-1} \cdot \frac{1}{x} = \left(1 + \frac{2}{n-1}\right) \frac{1}{x}.$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is $\frac{1}{x}$.

Hence, 2d test, the series is absolutely convergent when x is numerically greater than 1, and divergent when x is numerically less than 1.

The series is evidently divergent when $x = 1$ or -1 .

$$28. \quad \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)} \div \frac{x^n}{n(n+1)} = \frac{n}{n+2} x = \left(1 - \frac{2}{n+2}\right)x.$$

Therefore, the limit of the ratio of convergency as $n \doteq \infty$ is x .

Hence, 2d test, the series is absolutely convergent when x is numerically less than 1, and divergent when x is numerically greater than 1.

When $x = 1$ or -1 , each term of the series after the first is numerically less than the corresponding term of the auxiliary series,

$$\frac{1}{1 \cdot 1} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \dots,$$

which is absolutely convergent (§ 585).

Hence, Prin. 3, the series is absolutely convergent when $x = \pm 1$.

$$29. \quad a + (a + d)r + (a + 2d)r^2 + \dots + [a + (n-1)d]r^{n-1} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{(a + nd)r^n}{[a + (n-1)d]r^{n-1}} = \frac{a + nd}{a + (n-1)d} r.$$

Therefore, the limit of the ratio of convergency as $n \rightarrow \infty$ is r .

Hence, 2d test, the series is absolutely convergent when r is numerically less than 1, and divergent when r is numerically greater than 1.

When $r = 1$, the series is divergent, being an arithmetical series.

When $r = -1$, $S_1 = a$; $S_2 = -d$; $S_3 = a + d$; $S_4 = -2d$; $S_5 = a + 2d$; etc.

$$\therefore S_{2n} = -nd \text{ and } S_{2n-1} = a + nd.$$

Since the sum of $2n$ terms or of $2n-1$ terms can be made as large as we please by taking n sufficiently great, the series is divergent when $r = -1$.

UNDETERMINED COEFFICIENTS

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3-23. See pp. 382-387 Key.

2-13. See pp. 387-390 Key.

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4-17. See pp. 390-393 Key.

10-11. See p. 394 Key.

EXPONENTIAL AND LOGARITHMIC SERIES

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$$1. \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots \quad (1)$$

$$\therefore e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \dots \quad (2)$$

Subtracting (2) from (1), and dividing the result by 2,

$$\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

2. Substituting 1 and -1 successively for x in the exponential series,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \dots,$$

$$\text{we have} \quad e = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots, \quad (1)$$

$$\text{and} \quad e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots \quad (2)$$

Adding (1) and (2), and dividing the result by 2,

$$\frac{1}{2}(e + e^{-1}) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$$

3. Substituting 1 for x in the exponential series,

$$\begin{aligned}
 e &= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \\
 &= (1 + 1) + \left(\frac{3}{3 \cdot 2} + \frac{1}{3} \right) + \left(\frac{5}{5 \cdot 4} + \frac{1}{5} \right) + \dots \\
 &= 2 + \frac{3+1}{3} + \frac{5+1}{5} + \frac{7+1}{7} + \dots \\
 \therefore \frac{1}{2}e &= 1 + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots
 \end{aligned}$$

4. Substitute -1 for x in the exponential series. When $x = -1$, $e^x = e^{-1} = \frac{1}{e}$.

$$\begin{aligned}
 \therefore \frac{1}{e} &= 1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots \\
 &= (1 - 1) + \left(\frac{3}{3 \cdot 2} - \frac{1}{3} \right) + \left(\frac{5}{5 \cdot 4} - \frac{1}{5} \right) + \left(\frac{7}{7 \cdot 6} - \frac{1}{7} \right) + \dots \\
 &= \frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \frac{8}{9} + \dots
 \end{aligned}$$

5. Substituting ix and $-ix$ successively for x in the exponential series,

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2} + \frac{i^3x^3}{3} + \frac{i^4x^4}{4} + \frac{i^5x^5}{5} + \frac{i^6x^6}{6} + \dots, \quad (1)$$

and $e^{-ix} = 1 - ix + \frac{i^2x^2}{2} - \frac{i^3x^3}{3} + \frac{i^4x^4}{4} - \frac{i^5x^5}{5} + \frac{i^6x^6}{6} - \dots, \quad (2)$

Adding, and dividing by 2,

$$\frac{1}{2}(e^{ix} + e^{-ix}) = 1 + \frac{i^2x^2}{2} + \frac{i^4x^4}{4} + \frac{i^6x^6}{6} + \dots$$

But $i^2 = -1$, $i^4 = 1$, $i^6 = -1$, etc.

$$\therefore \frac{1}{2}(e^{ix} + e^{-ix}) = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$$

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2. Substituting 2 for n in the formula for $\log_e(n+1)$,

$$\log_e 3 = \log_e 2 + 2 \left[\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \dots \right].$$

| | | | |
|----|------------|----------|-------------------|
| 5 | 2.00000000 | | |
| 25 | .40000000 | $\div 1$ | = 0.40000000 |
| 25 | .01600000 | $\div 3$ | = .00533333 |
| 25 | .00064000 | $\div 5$ | = .00012800 |
| 25 | .00002560 | $\div 7$ | = .00000366 |
| | .00000102 | $\div 9$ | = .00000011 |
| | | | <u>0.40546510</u> |

Adding,

The error committed in omitting the rest of the series is less than

$$\frac{2}{11 \cdot 5^{11}} \left(1 + \frac{1}{25} + \frac{1}{25^2} + \dots \right) = \frac{2}{11 \cdot 5^{11}} \times \frac{25}{24} = \frac{2}{5^9} \times \frac{1}{11 \cdot 24} = \frac{.00000102}{11 \cdot 24},$$

which is much too small to affect the sixth decimal place.

By Ex. 1, $\log_e 2 = 0.693147$, to the nearest sixth decimal place.

$$\therefore \log_e 3 = 0.693147 + 0.405465 = 1.098612,$$

to the nearest sixth decimal place.

3. Since $4 = 2^2$, $\log_e 4 = 2 \log_e 2 = 1.38629436$, or 1.386294 to the nearest sixth decimal place. (Ex. 1.)

4. Substituting 4 for n in the formula for $\log_e(n+1)$,

$$\log_e 5 = \log_e 4 + 2 \left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \dots \right).$$

$$\begin{array}{r} 9 \overline{) 2.00000000} \\ 9 \overline{) .22222222} \div 1 = 0.22222222 \\ 9 \overline{) .02469136} \\ 9 \overline{) .00274348} \div 3 = .00091449 \\ 9 \overline{) .00030483} \\ 9 \overline{) .00003387} \div 5 = .00000677 \\ 9 \overline{) .00000376} \\ \hline .00000042 \div 7 = .00000006 \\ 0.22314354 \end{array}$$

Adding,

The error committed in omitting the rest of the series is less than

$$\frac{2}{9 \cdot 9^9} \left(1 + \frac{1}{9^2} + \frac{1}{9^4} + \dots \right) = \frac{2}{9 \cdot 9^9} \times \frac{81}{80} = \frac{2}{9^7} \times \frac{1}{720} = \frac{.00000042}{720},$$

which is much too small to affect the sixth decimal place.

By Ex. 3, $\log_e 4 = 1.386294$, to the nearest sixth decimal place.

$$\therefore \log_e 5 = 1.386294 + 0.223144 = 1.609438,$$

to the nearest sixth decimal place.

5. Since $6 = 2 \cdot 3$, $\log_e 6 = \log_e 2 + \log_e 3$.

Ex. 1, $\log_e 2 = 0.693147$ [18 ...]

Ex. 2, $\log_e 3 = 1.098612$ [10 ...]

$$\therefore \log_e 6 = 1.791759, \text{ to the nearest sixth decimal place.}$$

6. Substituting 6 for n in the formula for $\log_e(n+1)$,

$$\log_e 7 = \log_e 6 + 2 \left(\frac{1}{13} + \frac{1}{3 \cdot 13^3} + \frac{1}{5 \cdot 13^5} + \dots \right).$$

$$\begin{array}{r} 13 \overline{) 2.00000000} \\ 13 \overline{) .15384615} \div 1 = 0.15384615 \\ 13 \overline{) .01183432} \\ 13 \overline{) .00091033} \div 3 = .00030344 \\ 13 \overline{) .00007003} \\ \hline .00000539 \div 5 = .00000108 \\ 0.15415067 \end{array}$$

Adding,

The error committed in omitting the rest of the series is less than

$$\frac{2}{7 \cdot 13^7} \left(1 + \frac{1}{13^2} + \frac{1}{13^4} + \dots \right) = \frac{2}{7 \cdot 13^7} \times \frac{13^2}{168} = \frac{2}{13^5} \times \frac{1}{7 \cdot 168} = \frac{.00000539}{7 \cdot 168},$$

which is much too small to affect the sixth decimal place.

By Ex. 5, $\log_e 6 = 1.791759$, to the nearest sixth decimal place.

$\therefore \log_e 7 = 1.791759 + 0.154151 = 1.945910$,
to the nearest sixth decimal place.

7. Since $8 = 2^3$, $\log_e 8 = 3 \log_e 2$.
Ex. 1, $\log_e 2 = 0.693147$ [18 ...]
 $\therefore \log_e 8 = 2.079442$, to the nearest sixth decimal place.

8. Since $9 = 3^2$, $\log_e 9 = 2 \log_e 3$.
Ex. 2, $\log_e 3 = 1.098612$ [28 ...]
 $\therefore \log_e 9 = 2.197225$, to the nearest sixth decimal place.

9. Since $10 = 2 \cdot 5$, $\log_e 10 = \log_e 2 + \log_e 5$.
Ex. 1, $\log_e 2 = 0.693147$ [18 ...]
Ex. 4, $\log_e 5 = 1.609438$ [-10 ...]
 $\therefore \log_e 10 = 2.302585$, to the nearest sixth decimal place.

10. By (2), § 605, $\log_e (1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right)$.

Substituting $\frac{b}{a}$ for $1 - x$, whence $x = \frac{a - b}{a}$,

$$\log_e \frac{b}{a} = \log_e b - \log_e a = -\left[\frac{a - b}{a} + \frac{1}{2}\left(\frac{a - b}{a}\right)^2 + \frac{1}{3}\left(\frac{a - b}{a}\right)^3 + \dots\right].$$

Changing signs, $\log_e a - \log_e b = \frac{a - b}{a} + \frac{1}{2}\left(\frac{a - b}{a}\right)^2 + \frac{1}{3}\left(\frac{a - b}{a}\right)^3 + \dots$.

$$\begin{aligned} 11. \quad \log_e \sqrt{x^2 - 1} &= \log_e \left(x \sqrt{1 - \frac{1}{x^2}}\right) \\ &= \log_e x + \frac{1}{2} \log_e \left(1 - \frac{1}{x^2}\right). \end{aligned}$$

When $x > 1$, $-\frac{1}{x^2}$ is numerically less than 1 and may be substituted for x in formula (1), § 605, giving a *convergent* series.

Hence, when $x > 1$,

$$\begin{aligned} \log_e \sqrt{x^2 - 1} &= \log_e x + \frac{1}{2} \left[\left(-\frac{1}{x^2}\right) - \frac{1}{2} \left(-\frac{1}{x^2}\right)^2 + \frac{1}{3} \left(-\frac{1}{x^2}\right)^3 - \dots \right] \\ &= \log_e x + \frac{1}{2} \left(-\frac{1}{x^2} - \frac{1}{2x^4} - \frac{1}{3x^6} - \dots \right) \\ &= \log_e x - \left(\frac{1}{2x^2} + \frac{1}{4x^4} + \frac{1}{6x^6} + \dots \right). \end{aligned}$$

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2. By (2), the formula for the common logarithm of $n + 1$,

$$\log 2 = \log 1 + .86858896 \left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \dots \right);$$

$$\log 3 = \log 2 + .86858896 \left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \dots \right);$$

and

$$\log 7 = \log 3 + \log 2 + .86858896 \left(\frac{1}{13} + \frac{1}{3 \cdot 13^3} + \frac{1}{5 \cdot 13^5} + \dots \right).$$

The computation of the above is as follows :

$$\begin{array}{r}
 3 \overline{) .86858896} \\
 9 \overline{) .28952965} \div 1 = .28952965 \\
 9 \overline{) .03216996} \div 3 = .01072332 \\
 9 \overline{) .00357444} \div 5 = .00071489 \\
 9 \overline{) .00039716} \div 7 = .00005674 \\
 9 \overline{) .00004413} \div 9 = .00000490 \\
 9 \overline{) .00000490} \div 11 = .00000045 \\
 \hline
 .00000054 \div 13 = .00000004
 \end{array}$$

Adding, $\log 2 = 0.30102999$

$$\begin{array}{r}
 5 \overline{) .86858896} \\
 25 \overline{) .17371779} \div 1 = .17371779 \\
 25 \overline{) .00694871} \div 3 = .00231624 \\
 25 \overline{) .00027795} \div 5 = .00005559 \\
 25 \overline{) .00001112} \div 7 = .00000159 \\
 \hline
 .00000044 \div 9 = .00000005
 \end{array}$$

Adding, $\log 2 = 0.30102999$

Adding, $\therefore \log 3 = 0.47712125$

$$\begin{array}{r}
 13 \overline{) .86858896} \\
 13 \overline{) .06681454} \div 1 = .06681454 \\
 13 \overline{) .00513958} \\
 13 \overline{) .00039535} \div 3 = .00013178 \\
 13 \overline{) .00003041} \\
 \hline
 .00000234 \div 5 = .00000047
 \end{array}$$

Adding, $\log 2 = 0.30102999$

$\log 3 = 0.47712125$

Adding, $\log 7 = 0.84509803$

The last figure in each of the above results is uncertain. But to the nearest sixth decimal place,

$$\log 2 = 0.301030, \log 3 = 0.477121, \log 7 = 0.845098.$$

The eight-place decimals first obtained for $\log 2$, $\log 3$, and $\log 7$, may be used to find small multiples of these logarithms, provided the results obtained are cut off at the nearest sixth decimal place, as required. Then,

$$\log 4 = \log 2^2 = 2 \log 2 = 0.60205998 = 0.602060.$$

$$\log 10 = 1 = 1.000000.$$

$$\log 5 = \log 10^{\frac{1}{2}} = \log 10 - \log 2 = 0.69897001 = 0.698970.$$

$$\log 6 = \log(2 \cdot 3) = \log 2 + \log 3 = 0.77815124 = 0.778151.$$

$$\log 8 = \log 2^3 = 3 \log 2 = 0.90308997 = 0.903090.$$

$$\log 9 = \log 3^2 = 2 \log 3 = 0.95424250 = 0.954243.$$

$$\log 11 = 1.041393. \quad (\text{See Ex. 1, solved in the text.})$$

$$\log 12 = \log(2^2 \cdot 3) = 2 \log 2 + \log 3 = 1.07918123 = 1.079181$$

$$3. \quad \log 14 = \log(2 \cdot 7) = \log 2 + \log 7 = 1.1461.$$

$$\log 15 = \log(3 \cdot 5) = \log 3 + \log 5 = 1.1761.$$

$$\log 16 = \log 2^4 = 4 \log 2 = 1.2041.$$

$$\begin{aligned}\log 18 &= \log(2 \cdot 3^2) = \log 2 + 2 \log 3 = 1.2553. \\ \log 20 &= \log(2 \cdot 10) = \log 2 + \log 10 = 1.3010. \\ \log 21 &= \log(3 \cdot 7) = \log 3 + \log 7 = 1.3222. \\ \log 22 &= \log(2 \cdot 11) = \log 2 + \log 11 = 1.3424. \\ \log 24 &= \log(2^3 \cdot 3) = 3 \log 2 + \log 3 = 1.3802. \\ \log 25 &= \log 5^2 = 2 \log 5 = 1.3979.\end{aligned}$$

$$\begin{aligned}4. \quad \log 225 &= \log(3^2 \cdot 5^2) = 2 \log 3 + 2 \log 5 = 2.3522. \\ \log 175 &= \log(7 \cdot 5^2) = \log 7 + 2 \log 5 = 2.2430. \\ \log .014 &= \log\left(\frac{2 \cdot 7}{10^3}\right) = \log 2 + \log 7 - 3 = \bar{2}.1461.\end{aligned}$$

$$\begin{aligned}5. \quad .125 &= \frac{1}{8}. \quad \therefore \log .125 = \log 1 - \log 8. \\ \log 1 &= 0.000000 \\ \log 8 &= 0.903090 \\ \hline \therefore \log .125 &= \bar{1}.096910 = \bar{1}.0969 \text{ to four decimal places.} \\ 46.2 &= 2 \times 3 \times 7 \times 11 \div 10. \\ \therefore \log 46.2 &= \log 2 + \log 3 + \log 7 + \log 11 - \log 10 \\ &= 1.664642 = 1.6646 \text{ to four decimal places.} \\ 1.62 &= 2 \times 9^2 \div 10^2. \\ \therefore \log 1.62 &= \log 2 + 2 \log 9 - 2 \log 10 \\ &= 0.209516 = 0.2095 \text{ to four decimal places.} \\ .0625 &= \frac{1}{16} = 1 \div 4^2. \quad \therefore \log .0625 = \log 1 - 2 \log 4. \\ \log 1 &= 0.000000 \\ 2 \log 4 &= 1.204120 \\ \hline \therefore \log .0625 &= \bar{2}.795880 = \bar{2}.7959 \text{ to four decimal places.} \\ \frac{1}{15} &= 1 \div (3 \times 5). \quad \therefore \log \frac{1}{15} = \log 1 - (\log 3 + \log 5). \\ \log 1 &= 0.000000 \\ \log 3 + \log 5 &= 1.176091 \\ \hline \therefore \log \frac{1}{15} &= \bar{2}.823909 = \bar{2}.8239 \text{ to four decimal places.} \\ 9\frac{7}{8} &= \frac{77}{8} = 7 \times 11 \div 8. \\ \therefore \log 9\frac{7}{8} &= \log 7 + \log 11 - \log 8 \\ &= 0.983401 = 0.9834 \text{ to four decimal places.} \\ 1.\dot{1} &= 1\frac{1}{9} = \frac{10}{9}. \\ \therefore \log 1.\dot{1} &= \log 10 - \log 9 \\ &= 0.045757 = 0.0458 \text{ to four decimal places.} \\ \frac{3}{198} &= 3 \div (2 \times 2 \times 7 \times 7) = 3 \div (2 \times 7)^2. \\ \therefore \log \frac{3}{198} &= \log 3 - 2(\log 2 + \log 7). \\ \log 3 &= 0.477121 \\ 2(\log 2 + \log 7) &= 2.292256 \\ \hline \therefore \log \frac{3}{198} &= \bar{2}.184865 = \bar{2}.1849 \text{ to four decimal places.}\end{aligned}$$

$$\begin{aligned}6. \text{ By § 607, } \log_{12} 18 &= (\log_e 18) \times \frac{1}{\log_e 12} = \frac{\log_e 18}{\log_e 12} \\ &= \frac{\log_e (2 \cdot 3^2)}{\log_e (2^2 \cdot 3)} = \frac{\log_e 2 + 2 \log_e 3}{2 \log_e 2 + \log_e 3}.\end{aligned}$$

7. By the definition of the modulus of a system of logarithms, § 607,

$$\log_2 7 = \log_e 7 \cdot \frac{1}{\log_e 2}, \text{ and } \log_2 8 = \log_e 8 \cdot \frac{1}{\log_e 2} = \frac{\log_e 8}{\log_e 2}.$$

Also, since $8 = 2^3$, $\log_2 8 = 3$.

$$\begin{aligned} \therefore \text{ by (5), § 605, } 3 &= \frac{\log_e 8}{\log_e 2} = \frac{1}{\log_e 2} \left[\log_e 7 + 2 \left(\frac{1}{15} + \frac{1}{3 \cdot 15^3} + \frac{1}{5 \cdot 15^5} + \dots \right) \right] \\ &= \log_2 7 + \frac{2}{\log_e 2} \left(\frac{1}{15} + \frac{1}{3 \cdot 15^3} + \frac{1}{5 \cdot 15^5} + \dots \right), \\ \therefore \log_2 7 &= 3 - \frac{2}{\log_e 2} \left(\frac{1}{15} + \frac{1}{3 \cdot 15^3} + \frac{1}{5 \cdot 15^5} + \dots \right). \end{aligned}$$

Since $\log_e 2 = 0.69314718 \dots$, by division $\frac{2}{\log_e 2} = 2.88539008 \dots$.

The rest of the computation, then, is as follows :

$$\begin{array}{r} 15 \overline{) 2.88539008} \\ 15 \overline{) .19235934} \div 1 = 0.19235934 \\ 15 \overline{) .01282396} \\ 15 \overline{) .00085493} \div 3 = .00028498 \\ 15 \overline{) .00005700} \\ \hline .00000380 \div 5 = .00000076 \\ \hline 0.19264508 \end{array}$$

$\therefore \log_2 7 = 3 - 0.192645 = 2.807355$ to six decimal places.

SUMMATION OF SERIES

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3. Since each term after the second is equal to x times the preceding term, the scale of relation is $1 - x$.

4. Assume $1 + Ax + Bx^2$ as the scale of relation.

$$\begin{aligned} \text{Then, } 3x^2 + Ax(2x) + Bx^2(1) &= 0, \text{ or } 3 + 2A + B = 0; \\ 4x^3 + Ax(3x^2) + Bx^2(2x) &= 0, \text{ or } 4 + 3A + 2B = 0; \\ 5x^4 + Ax(4x^3) + Bx^2(3x^2) &= 0, \text{ or } 5 + 4A + 3B = 0; \\ 6x^5 + Ax(5x^4) + Bx^2(4x^3) &= 0, \text{ or } 6 + 5A + 4B = 0. \end{aligned}$$

Each of these equations is satisfied by $A = -2$, $B = 1$.

Hence, the scale of relation is $1 - 2x + x^2$.

5. Assume $1 + Ax + Bx^2$ as the scale of relation.

$$\begin{aligned} \text{Then, } 8x^2 + Ax(3x) + Bx^2(1) &= 0, \text{ or } 8 + 3A + B = 0; \\ 13x^3 + Ax(8x^2) + Bx^2(3x) &= 0, \text{ or } 13 + 8A + 3B = 0; \\ 18x^4 + Ax(13x^3) + Bx^2(8x^2) &= 0, \text{ or } 18 + 13A + 8B = 0; \\ 23x^5 + Ax(18x^4) + Bx^2(13x^3) &= 0, \text{ or } 23 + 18A + 13B = 0. \end{aligned}$$

Each of these equations after the first is satisfied by $A = -2$, $B = 1$.

Hence, the scale of relation is $1 - 2x + x^2$.

6. Assume $1 + Ax + Bx^2 + Cx^3$ as the scale of relation.

$$\begin{aligned} \text{Then, } -2x^3 + Ax(-2x^2) + Bx^2(x) + Cx^3(1) &= 0, \\ \text{or } -2 - 2A + B + C &= 0; \\ \text{or } 7x^4 + Ax(-2x^3) + Bx^2(-2x^2) + Cx^3(x) &= 0, \\ \text{or } 7 - 2A - 2B + C &= 0; \\ \text{or } 7x^5 + Ax(7x^4) + Bx^2(-2x^3) + Cx^3(-2x^2) &= 0, \\ \text{or } 7 + 7A - 2B - 2C &= 0; \\ \text{or } -20x^6 + Ax(7x^5) + Bx^2(7x^4) + Cx^3(-2x^3) &= 0, \\ \text{or } -20 + 7A + 7B - 2C &= 0. \end{aligned}$$

Each of these equations is satisfied by $A = -1$, $B = 3$, $C = -3$.

Hence, the scale of relation is $1 - x + 3x^2 - 3x^3$.

7. Assume $1 + Ax + Bx^2 + Cx^3$ as the scale of relation.

Then, $-x^3 + Ax(x^2) + Bx^2(x) + Cx^3(1) = 0$,

or $-1 + A + B + C = 0$;

$-5x^4 + Ax(-x^3) + Bx^2(x^2) + Cx^3(x) = 0$,

or $-5 - A + B + C = 0$;

$-11x^5 + Ax(-5x^4) + Bx^2(-x^3) + Cx^3(x^2) = 0$,

or $-11 - 5A - B + C = 0$;

$-15x^6 + Ax(-11x^5) + Bx^2(-5x^4) + Cx^3(-x^3) = 0$,

or $-15 - 11A - 5B - C = 0$;

$-9x^7 + Ax(-15x^6) + Bx^2(-11x^5) + Cx^3(-5x^4) = 0$,

or $-9 - 15A - 11B - 5C = 0$.

Each of these equations is satisfied by $A = -2$, $B = 1$, $C = 2$.

Hence, the scale of relation is $1 - 2x + x^2 + 2x^3$.

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1. The scale of relation is found to be $1 - x$.

$$\begin{array}{r} 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \dots \\ 1 - x \\ \hline 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \dots \\ - x - 2x^2 - 2x^3 - 2x^4 - \dots \\ \hline 1 + x \end{array}$$

Hence, the generating function is $\frac{1+x}{1-x}$. This is also the sum for such values of x as make the series convergent.

2. The scale of relation is found to be $1 - x - x^2$.

$$\begin{array}{r} 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots \\ 1 - x - x^2 \\ \hline 1 + 1x + 2x^2 + 3x^3 + 5x^4 + \dots \\ - 1x - 1x^2 - 2x^3 - 3x^4 - \dots \\ \hline 1 \end{array}$$

Hence, the generating function is $\frac{1}{1-x-x^2}$. This is also the sum for such values of x as make the series convergent.

3. The scale of relation is found to be $1 + 2x - x^2$.

$$\begin{array}{r} x - 3x^2 + 7x^3 - 17x^4 + 41x^5 - \dots \\ 1 + 2x - x^2 \\ \hline x - 3x^2 + 7x^3 - 17x^4 + 41x^5 - \dots \\ + 2x + 6x^2 - 14x^3 + 34x^4 - 77x^5 + \dots \\ \hline x - x^2 \end{array}$$

Hence, the generating function is $\frac{x-x^2}{1+2x-x^2}$. This is also the sum for such values of x as make the series convergent.

4. The scale of relation is found to be $1 - 2x - x^2$.

$$\frac{1 + x + x^2 + 3x^3 + 7x^4 + 17x^5 + \dots}{1 - 2x - x^2}$$

| | | | | | |
|-------|-------|--------------------|--------------------|---------------------|----------------------|
| 1 + 1 | x + 1 | x ² + 3 | x ³ + 7 | x ⁴ + 17 | x ⁵ + ... |
| - 2 | - 2 | - 2 | - 6 | - 14 | - ... |
| | - 1 | - 1 | - 1 | - 3 | - ... |

$$1 - x - 2x^2$$

Hence, the generating function is $\frac{1 - x - 2x^2}{1 - 2x - x^2}$. This is also the sum for such values of x as make the series convergent.

5. The scale of relation is found to be $1 - x - x^2 - x^3$.

$$\frac{1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + 24x^6 + 44x^7 + \dots}{1 - x - x^2 - x^3}$$

| | | | | | | | |
|-------|-------|--------------------|--------------------|---------------------|---------------------|---------------------|----------------------|
| 1 + 1 | x + 2 | x ² + 4 | x ³ + 7 | x ⁴ + 13 | x ⁵ + 24 | x ⁶ + 44 | x ⁷ + ... |
| - 1 | - 1 | - 2 | - 4 | - 7 | - 13 | - 24 | - ... |
| | - 1 | - 1 | - 2 | - 4 | - 7 | - 13 | - ... |
| | | - 1 | - 1 | - 2 | - 4 | - 7 | - ... |

$$1$$

Hence, the generating function is $\frac{1}{1 - x - x^2 - x^3}$. This is also the sum for such values of x as make the series convergent.

6. The generating function is found to be $1 + x + 2x^2 - x^3$.

$$\frac{2 - 3x + x^2 + 7x^3 - 12x^4 - x^5 + 32x^6 - 42x^7 - \dots}{1 + x + 2x^2 - x^3}$$

| | | | | | | | |
|-------|-------|--------------------|---------------------|--------------------|---------------------|---------------------|----------------------|
| 2 - 3 | x + 1 | x ² + 7 | x ³ - 12 | x ⁴ - 1 | x ⁵ + 32 | x ⁶ - 42 | x ⁷ - ... |
| 2 | - 3 | + 1 | + 7 | - 12 | - 1 | + 32 | - ... |
| | 4 | - 6 | + 2 | + 14 | - 24 | - 2 | + ... |
| | | - 2 | + 3 | - 1 | - 7 | + 12 | + ... |

$$2 - x + 2x^2$$

Hence, the generating function is $\frac{2 - x + 2x^2}{1 + x + 2x^2 - x^3}$. This is also the sum for such values of x as make the series convergent.

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2. The generating function is found to be $\frac{1}{1 + 2x - 3x^2}$.

Separating this into its partial fractions,

$$\frac{1}{1 + 2x - 3x^2} = \frac{1}{4} \left(\frac{1}{1 - x} \right) + \frac{3}{4} \left(\frac{1}{1 + 3x} \right).$$

Since

$$\frac{1}{1 - x} = 1 + x + x^2 + \dots + x^r + \dots$$

and

$$\frac{1}{1 + 3x} = 1 - 3x + 3^2x^2 - \dots + (-1)^r 3^r x^r + \dots,$$

$$u_{r+1} = \frac{1}{4} x^r + \frac{3}{4} (-3)^r x^r.$$

$$u_{12} = \frac{1}{4} x^{11} + \frac{3}{4} (-3)^{11} x^{11} = -132860 x^{11}.$$

3. The generating function is found to be $\frac{1}{1-3x+2x^2}$.
Separating this into its partial fractions,

$$\frac{1}{1-3x+2x^2} = -\frac{1}{1-x} + \frac{2}{1-2x}.$$

Since $-\frac{1}{1-x} = -(1+x+x^2+\dots+x^r+\dots)$

and $\frac{2}{1-2x} = 2(1+2x+4x^2+\dots+2^rx^r+\dots),$

$$u_{r+1} = -x^r + 2(2^rx^r) = (2^{r+1} - 1)x^r.$$

$$u_{12} = (2^{12} - 1)x^{11} = 4095x^{11}.$$

4. The generating function is found to be $\frac{1}{1-5x+6x^2}$.
Separating this into its partial fractions,

$$\frac{1}{1-5x+6x^2} = -\frac{2}{1-2x} + \frac{3}{1-3x}.$$

Since $-\frac{2}{1-2x} = -2(1+2x+4x^2+\dots+2^rx^r+\dots)$

and $\frac{3}{1-3x} = 3(1+3x+9x^2+\dots+3^rx^r+\dots),$

$$u_{r+1} = -2 \cdot 2^rx^r + 3 \cdot 3^rx^r = (3^{r+1} - 2^{r+1})x^r.$$

$$u_{12} = (3^{12} - 2^{12})x^{11} = 527345x^{11}.$$

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2. Series, 1, 8, 27, 64, 125, ...
1st differences, 7, 19, 37, 61, ...
2d differences, 12, 18, 24, ...
3d differences, 6, 6, ...

By (7), $S_{12} = 12 \cdot 1 + \frac{12 \cdot 11}{2} \cdot 7 + \frac{12 \cdot 11 \cdot 10}{3} \cdot 12 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{4} \cdot 6 = 6084.$

By an inspection of the series it is evident that $a_{12} = 12^3 = 1728.$

3. Series, 1, 16, 81, 256, 625, 1296, ...
1st differences, 15, 65, 175, 369, 671, ...
2d differences, 50, 110, 194, 302, ...
3d differences, 60, 84, 108, ...
4th differences, 24, 24, ...

By (7), $S_{12} = 12 \cdot 1 + \frac{12 \cdot 11}{2} \cdot 15 + \frac{12 \cdot 11 \cdot 10}{3} \cdot 50 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{4} \cdot 60$
 $+ \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5} \cdot 24 = 60710.$

By an inspection of the series it is evident that $a_{12} = 12^4 = 20736.$

4. Series, 1, 8, 21, 40, 65, ...
1st differences, 7, 13, 19, 25, ...
2d differences, 6, 6, 6, ...

By (7), $S_{12} = 12 \cdot 1 + \frac{12 \cdot 11}{2} \cdot 7 + \frac{12 \cdot 11 \cdot 10}{3} \cdot 6 = 1794.$

By (4), $a_{12} = 1 + 11 \cdot 7 + \frac{11 \cdot 10}{2} \cdot 6 = 408.$

| | | | | | | |
|------------------|----|-----|-----|-----|-----|-----|
| 5. Series, | 2, | 8, | 18, | 33, | 54, | ... |
| 1st differences, | 6, | 10, | 15, | 21, | ... | |
| 2d differences, | 4, | 5, | 6, | ... | | |
| 3d differences, | 1, | 1, | ... | | | |

$$\text{By (7), } S_{12} = 12 \cdot 2 + \frac{12 \cdot 11}{2} \cdot 6 + \frac{12 \cdot 11 \cdot 10}{3} \cdot 4 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{4} \cdot 1 = 1795.$$

$$\text{By (4), } a_{12} = 2 + 11 \cdot 6 + \frac{11 \cdot 10}{2} \cdot 4 + \frac{11 \cdot 10 \cdot 9}{3} \cdot 1 = 453.$$

| | | | | | | |
|------------------|----|----|-----|-----|-----|-----|
| 6. Series, | 2, | 7, | 14, | 23, | 34, | ... |
| 1st differences, | 5, | 7, | 9, | 11, | ... | |
| 2d differences, | 2, | 2, | 2, | ... | | |

$$\text{By (7), } S_{12} = 12 \cdot 2 + \frac{12 \cdot 11}{2} \cdot 5 + \frac{12 \cdot 11 \cdot 10}{3} \cdot 2 = 794.$$

$$\text{By (4), } a_{12} = 2 + 11 \cdot 5 + \frac{11 \cdot 10}{2} \cdot 2 = 167.$$

| | | | | | | | | |
|------------------|------|------|-------|-------|-------|-------|--------|-----|
| 7. Series, | 1, | 32, | 243, | 1024, | 3125, | 7776, | 16807, | ... |
| 1st differences, | 31, | 211, | 781, | 2101, | 4651, | 9031, | ... | |
| 2d differences, | 180, | 570, | 1320, | 2550, | 4380, | ... | | |
| 3d differences, | 390, | 750, | 1230, | 1830, | ... | | | |
| 4th differences, | 360, | 480, | 600, | ... | | | | |
| 5th differences, | 120, | 120, | ... | | | | | |

$$\begin{aligned} \text{By (7), } S_{12} = & 12 \cdot 1 + \frac{12 \cdot 11}{2} \cdot 31 + \frac{12 \cdot 11 \cdot 10}{3} \cdot 180 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{4} \cdot 390 \\ & + \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5} \cdot 360 + \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6} \cdot 120 = 630708. \end{aligned}$$

The series is a series of fifth powers. $\therefore a_{12} = 12^5 = 248832.$

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$$1. \text{ By (1), } S_{10} = \frac{1}{6} \cdot 10 \cdot 11 \cdot 12 = 220.$$

2. Since the diameter of each shot is 10 inches, it takes 12 shot to make a row 10 feet long; that is, $n = 12.$

$$\text{By (2), } S_{12} = \frac{1}{6} \cdot 12 \cdot 13 \cdot 25 = 650.$$

$$3. \text{ Substituting 2 for } m \text{ and 7 for } n \text{ in (3), } S_7 = \frac{1}{6} \cdot 7 \cdot 8 \cdot 21 = 196.$$

4. The base of the pile is 18 shot in length and 15 shot in width; therefore, $n = 15$ and $m + n = 18$, whence $m = 3.$

$$\text{Hence, by (3), } S_{15} = \frac{1}{6} \cdot 15 \cdot 16 \cdot 40 = 1600.$$

5. Since there are 7 balls in each side of the top layer of the frustum, there are 6 balls in each side of the bottom layer of the triangular pyramid needed to complete the pile.

$$\text{Hence, by (1), } S_6 = \frac{1}{6} \cdot 6 \cdot 7 \cdot 8 = 56.$$

6. Since there are 10 balls in each side of the top layer of the frustum, there are 9 balls in each side of the bottom layer of the square pyramid needed to complete the pile.

$$\text{Hence, by (2), } S_9 = \frac{1}{6} \cdot 9 \cdot 10 \cdot 19 = 285.$$

7. By (1),
that is,

$$165 = \frac{1}{6} n(n+1)(n+2);$$

$$3 \cdot 5 \cdot 11 = \frac{1}{6} n(n+1)(n+2).$$

Multiplying the first member by 3 · 2, and the second by 6,

$$9 \cdot 10 \cdot 11 = n(n+1)(n+2).$$

∴ $n = 9$, the number of courses.

8. Evidently, the bottom course is 11 shot long and 7 shot wide, and the top course is 7 shot long and 3 shot wide.

If the pile were complete, the number of shot would be

$$S_7 = \frac{1}{6} \cdot 7 \cdot 8 \cdot 27 = 252.$$

The number of shot required to complete the pile is the same as the number of shot in a wedge-shaped pile with a rectangular base 6 shot in length and 2 shot in width, which is

$$S_2 = \frac{1}{6} \cdot 2 \cdot 3 \cdot 17 = 17.$$

Hence, the number of shot in the given pile is $252 - 17$, or 235.

9. Since there are 10 layers, $n = 10$. Substituting 10 for n in

$$605 = \frac{1}{6} n(n+1)(3m+2n+1)$$

and solving for m , $m = 4$.

Hence, § 618, 3, $m + 1 = 5$, the number of shot in the top row.

10. Let n be the number of shot in each side of each base.

Then, by (2) and (1),

$$\frac{1}{6} n(n+1)(2n+1) = \frac{25}{4} \cdot \frac{1}{6} n(n+1)(n+2).$$

Canceling equal factors, $2n+1 = \frac{25}{4}(n+2)$.

$$\therefore n = 12.$$

Hence, the number of shot in the triangular pile is

$$S_{12} = \frac{1}{6} \cdot 12 \cdot 13 \cdot 14 = 364,$$

and the number of shot in the square pile is

$$S_{12} = \frac{1}{6} \cdot 12 \cdot 13 \cdot 25 = 650.$$

11. Let n be the number of layers, or the number of oranges in each side of the bottom layers.

Then, by (2) and (1),

$$\frac{1}{6} n(n+1)(2n+1) = \frac{1}{6} n(n+1)(n+2) + 84.$$

$$n(n+1)(2n+1) = n(n+1)(n+2) + 7 \cdot 8 \cdot 9.$$

$$n(n+1)(2n+1-n-2) = 7 \cdot 8 \cdot 9.$$

$$n(n+1)(n-1) = (n-1)(n)(n+1) = 7 \cdot 8 \cdot 9.$$

$$\therefore n = 8.$$

Hence, the number of oranges in the triangular pile was

$$S_8 = \frac{1}{6} \cdot 8 \cdot 9 \cdot 10 = 120,$$

and the number of oranges in the square pile was

$$120 + 84 = 204, \text{ or } S_8 = \frac{1}{6} \cdot 8 \cdot 9 \cdot 17 = 204.$$

12. The number of oranges in the triangular pyramid is, by (1),

$$S_{2n} = \frac{1}{6} (2n)(2n+1)(2n+2) = \frac{4}{3} n(2n+1)(n+1).$$

Hence, the number of oranges in each square pyramid is

$$\frac{1}{4} S_{2n} = \frac{1}{6} n(2n+1)(n+1) = \frac{1}{6} n(n+1)(2n+1).$$

But the last formula is identical with (2), the formula for the number of shot in a square pyramid having n shot in each side of the base. Hence, the fruit seller must use n^2 oranges for the base of each square pyramid.

Any triangular pyramidal pile of shot having an even number of courses may be divided into four equal square pyramidal piles having half as many courses as the triangular pile.

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2. Series, 3.1416, 12.5664, 28.2744, 50.2656, 78.5400
 1st diff., 9.4248, 15.7080, 21.9912, 28.2744
 2d diff., 6.2832, 6.2832, 6.2832
 Therefore, $a_1 = 3.1416$, $d_1 = 9.4248$, $d_2 = 6.2832$. (1)

Next take the volumes.
 Series, .5236, 4.1888, 14.1372, 33.5104, 65.4500
 1st diff., 3.6652, 9.9484, 19.3732, 31.9396
 2d diff., 6.2832, 9.4248, 12.5664
 3d diff., 3.1416, 3.1416

Therefore, $a_1 = .5236$, $d_1 = 3.6652$, $d_2 = 6.2832$, $d_3 = 3.1416$. (2)

To obtain the surfaces substitute 3.1, 3.2, and 3.7 successively for n , and also the values in (1), in formula (4), § 616. To obtain the volumes, substitute the same values of n , and also the values in (2), in formula (4), § 616.

The results are as follows:

| Diam. | 3.1 | 3.2 | 3.7 |
|-------|---------|---------|---------|
| Surf. | 30.1908 | 32.1700 | 43.0085 |
| Vol. | 15.5986 | 17.1573 | 26.5219 |

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3. Series, 7.0472987, 7.0540041, 7.0606967, 7.0673767, 7.0740440
 1st diff., .00067054, .0006926, .0006800, .0006673
 2d diff., -.0000128, -.0000126, -.0000127

The differences beginning with the third are so small that they may be neglected without affecting the result to seven decimal places.

Regarding $\sqrt[3]{350.6}$ as the 1.6th term of the series, by (4), § 616,

$$\sqrt[3]{350.6} = 7.0472987 + .6(.00067054) + \frac{.6(-.4)}{2}(-.0000128) = 7.0513235, \text{ to seven decimal places.}$$

4. Series, .001379310, .001369863, .001360544, .001351351, .001342282
 1st diff., -.000009447, -.000009319, -.000009193, -.000009069
 2d diff., .000000128, .000000126, .000000124

The differences beginning with the third are so small that they may be neglected without affecting the result to nine decimal places.

Regarding $\pi_{\frac{3}{8}}$ as the $(3\frac{1}{2})$ th term of the series, by (4), § 616,

$$\begin{aligned} \pi_{\frac{3}{8}} &= .001379310 + 2.2(-.000009447) + \frac{(2.2)(1.2)}{2}(.000000128) \\ &= .001358696, \text{ to nine decimal places.} \end{aligned}$$

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2. Assume $1 + 2 + 3 + \dots + n = A + Bn + Cn^2 + Dn^3 + \dots$, (1)

and $1 + 2 + 3 + \dots + n + (n+1) = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + \dots$ (2)

Subtracting (1) from (2),

$$n+1 = B + C(2n+1) + D(3n^2+3n+1) + \dots$$

Equating coefficients of like powers of n ,

$B + C = 1$, $2C = 1$, and D and all succeeding coefficients vanish.

$$\therefore C = \frac{1}{2} \text{ and } B = \frac{1}{2}.$$

Hence, $1 + 2 + 3 + \dots + n = A + \frac{1}{2}n + \frac{1}{2}n^2$.

When $n = 1$, $1 = A + \frac{1}{2} + \frac{1}{2}$. $\therefore A = 0$.

Hence, $1 + 2 + 3 + \dots + n = \frac{1}{2}n + \frac{1}{2}n^2 = \frac{1}{2}n(n+1)$.

3. Assume $1^3 + 2^3 + 3^3 + \dots + n^3 = A + Bn + Cn^2 + Dn^3 + En^4 + \dots$, (1)
 and $1^3 + 2^3 + \dots + n^3 + (n+1)^3 = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \dots$ (2)

Subtracting (1) from (2),

$$n^3 + 3n^2 + 3n + 1 = B + C(2n+1) + D(3n^2 + 3n + 1) + E(4n^3 + 6n^2 + 4n + 1) + \dots$$

Equating coefficients of like powers of n ,

$B + C + D + E = 1$, $2C + 3D + 4E = 3$, $3D + 6E = 3$, $4E = 1$, and all coefficients after E vanish.

$$\therefore E = \frac{1}{4}, D = \frac{1}{2}, C = \frac{1}{4}, B = 0.$$

Hence, $1^3 + 2^3 + 3^3 + \dots + n^3 = A + \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$.

When $n = 1$, $1 = A + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$. $\therefore A = 0$.

Hence, $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4 = \frac{1}{4}n^2(n+1)^2$.

4. Assume $1^4 + 2^4 + \dots + n^4 = A + Bn + Cn^2 + Dn^3 + En^4 + Fn^5 + \dots$, (1)
 and $1^4 + 2^4 + \dots + n^4 + (n+1)^4 = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + F(n+1)^5 + \dots$ (2)

Subtracting (1) from (2),

$$n^4 + 4n^3 + 6n^2 + 4n + 1 = B + C(2n+1) + D(3n^2 + 3n + 1) + E(4n^3 + 6n^2 + 4n + 1) + F(5n^4 + 10n^3 + 10n^2 + 5n + 1) + \dots$$

Equating coefficients of like powers of n ,

$5F = 1$, $4E + 10F = 4$, $3D + 6E + 10F = 6$, $2C + 3D + 4E + 5F = 4$, and $B + C + D + E + F = 1$.

$\therefore F = \frac{1}{5}$, $E = \frac{1}{2}$, $D = \frac{1}{3}$, $C = 0$, $B = -\frac{1}{30}$, and all coefficients after F vanish.

Hence, $1^4 + 2^4 + \dots + n^4 = A - \frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5$.

When $n = 1$, $1 = A - \frac{1}{30} + \frac{1}{3} + \frac{1}{2} + \frac{1}{5}$. $\therefore A = 0$.

Hence, $1^4 + 2^4 + \dots + n^4 = -\frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5$
 $= \frac{1}{30}n(6n^4 + 15n^3 + 10n^2 - 1)$
 $= \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$.

5. Assume $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = A + Bn + Cn^2 + Dn^3 + \dots$, (1)
 and $1^2 + 3^2 + \dots + (2n-1)^2 + (2n+1)^2 = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + \dots$ (2)

Subtracting (1) from (2),

$$4n^2 + 4n + 1 = B + C(2n+1) + D(3n^2 + 3n + 1) + \dots$$

Equating coefficients of like powers of n ,

$3D = 4$, $2C + 3D = 4$, $B + C + D = 1$, and all coefficients after D vanish.

$$\therefore D = \frac{4}{3}, C = 0, B = -\frac{1}{3}.$$

Hence, $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = A - \frac{1}{3}n + \frac{4}{3}n^3$.

When $n = 1$, $1 = A - \frac{1}{3} + \frac{4}{3}$. $\therefore A = 0$.

Hence, $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = -\frac{1}{3}n + \frac{4}{3}n^3 = \frac{1}{3}n(2n+1)(2n-1)$.

2. General term $= \frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right)$.

$$\begin{aligned} \therefore \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{n(n+2)} \\ = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} \right) \\ - \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \right) \end{aligned}$$

$$= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)} \right), \text{ the sum to } n \text{ terms.}$$

As $n \rightarrow \infty$, this sum approaches $\frac{3}{4}$ as a limit.

3. General term $= \frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$.

$$\therefore \frac{3}{1 \cdot 4} + \frac{3}{2 \cdot 5} + \frac{3}{3 \cdot 6} + \dots + \frac{3}{n(n+3)}$$

$$= \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n}$$

$$- \frac{1}{4} - \frac{1}{5} - \dots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{11}{6} - \frac{3n^2 + 12n + 11}{(n+1)(n+2)(n+3)}, \text{ the sum to } n \text{ terms.}$$

As $n \rightarrow \infty$, this sum approaches $\frac{11}{6}$ as a limit.

4. General term $= \frac{4}{(2n-1)(2n+3)} = \frac{1}{2n-1} - \frac{1}{2n+3}$.

$$\therefore \frac{4}{1 \cdot 5} + \frac{4}{3 \cdot 7} + \frac{4}{5 \cdot 9} + \dots + \frac{4}{(2n-1)(2n+3)}$$

$$= \left(\frac{1}{1} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+3} \right)$$

$$= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n-1}$$

$$- \frac{1}{5} - \frac{1}{7} - \dots - \frac{1}{2n-1} - \frac{1}{2n+1} - \frac{1}{2n+3}$$

$$= 1 + \frac{1}{3} - \frac{1}{2n+1} - \frac{1}{2n+3}$$

$$= \frac{4}{3} - \frac{4(n+1)}{(2n-1)(2n+3)}, \text{ the sum to } n \text{ terms.}$$

As $n \rightarrow \infty$, this sum approaches $\frac{4}{3}$ as a limit.

5. General term $= \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right)$.

$$\therefore \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \dots + \frac{1}{(2n+1)(2n+3)}$$

$$= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \frac{1}{2} \left(\frac{1}{5} - \frac{1}{7} \right) + \frac{1}{2} \left(\frac{1}{7} - \frac{1}{9} \right) + \dots + \frac{1}{2} \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right)$$

$$\begin{aligned}
 &= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1} \right) \\
 &- \frac{1}{2} \left(\frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1} + \frac{1}{2n+3} \right) \\
 &= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2n+3} \right) \\
 &= \frac{1}{6} - \frac{1}{2(2n+3)}, \text{ the sum to } n \text{ terms.}
 \end{aligned}$$

As $n \doteq \infty$, this sum approaches $\frac{1}{6}$ as a limit.

$$6. \text{ General term} = \frac{2}{(n+1)(n+4)} = \frac{2}{3} \left(\frac{1}{n+1} - \frac{1}{n+4} \right).$$

$$\begin{aligned}
 \therefore \frac{2}{2 \cdot 5} + \frac{2}{3 \cdot 6} + \frac{2}{4 \cdot 7} + \dots + \frac{2}{(n+1)(n+4)} \\
 &= \frac{2}{3} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{n+1} \right) \\
 &- \frac{2}{3} \left(\frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} \right) \\
 &= \frac{2}{3} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} \right) \\
 &= \frac{2}{3} \left[\frac{13}{12} - \frac{3n^2 + 18n + 26}{(n+2)(n+3)(n+4)} \right].
 \end{aligned}$$

As $n \doteq \infty$, this sum approaches $\frac{13}{18}$ as a limit.

$$7. \text{ General term} = (-1)^{n-1} \frac{2}{n(n+2)} = (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

$$\begin{aligned}
 \therefore \frac{2}{1 \cdot 3} - \frac{2}{2 \cdot 4} + \frac{2}{3 \cdot 5} - \dots + (-1)^{n-1} \frac{2}{n(n+2)} \\
 &= \left(\frac{1}{1} - \frac{1}{3} \right) - \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) - \dots + (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\
 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + (-1)^{n-1} \left(\frac{1}{n} \right) \\
 &\quad - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots + (-1)^n \left(-\frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2} \right) \\
 &= 1 - \frac{1}{2} + \dots + (-1)^{n-1} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\
 &= \frac{1}{2} + (-1)^{n-1} \frac{1}{(n+1)(n+2)}.
 \end{aligned}$$

Hence, when n is even the sum to n terms is $\frac{1}{2} - \frac{1}{(n+1)(n+2)}$; when n is odd the sum to n terms is $\frac{1}{2} + \frac{1}{(n+1)(n+2)}$.

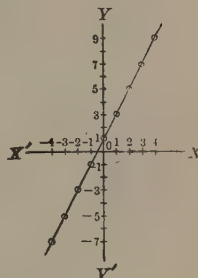
As $n \doteq \infty$, the sum approaches $\frac{1}{2}$ as a limit.

FUNCTIONS OF A SINGLE VARIABLE

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1. Let $2x + 1 = f(x)$.

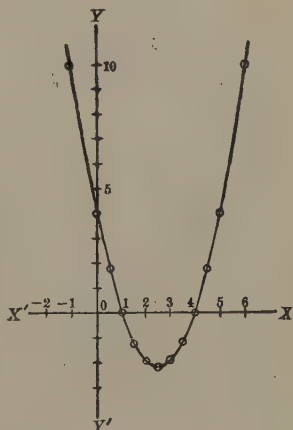
| x | $f(x)$ | x | $f(x)$ |
|-----------|-----------|----------|----------|
| $-\infty$ | $-\infty$ | 1 | 3 |
| -4 | -7 | 2 | 5 |
| -3 | -5 | 3 | 7 |
| -2 | -3 | 4 | 9 |
| -1 | -1 | ∞ | ∞ |
| 0 | 1 | | |



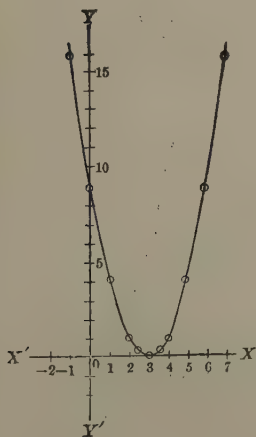
The graph of $2x + 1$ crosses the X -axis between 0 and -1. Hence, the root of $2x + 1 = 0$ lies between 0 and -1. Try $x = -\frac{1}{2}$. When $x = -\frac{1}{2}$, $2x + 1 = 0$. Therefore, the root is $-\frac{1}{2}$.

2. Let $x^2 - 5x + 4 = f(x)$.

| x | $f(x)$ | x | $f(x)$ |
|----------------|-----------------|----------------|-----------------|
| $-\infty$ | ∞ | 3 | -2 |
| -1 | 10 | $3\frac{1}{2}$ | $-1\frac{1}{4}$ |
| 0 | 4 | 4 | 0 |
| $\frac{1}{2}$ | $1\frac{3}{4}$ | $4\frac{1}{2}$ | $1\frac{3}{4}$ |
| 1 | 0 | 5 | 4 |
| $1\frac{1}{2}$ | $-1\frac{1}{4}$ | 6 | 10 |
| 2 | -2 | ∞ | ∞ |
| $2\frac{1}{2}$ | $-2\frac{1}{4}$ | | |



The roots of $x^2 - 5x + 4 = 0$ are evidently 1 and 4.

3. Let $x^2 - 6x + 9 = f(x)$.

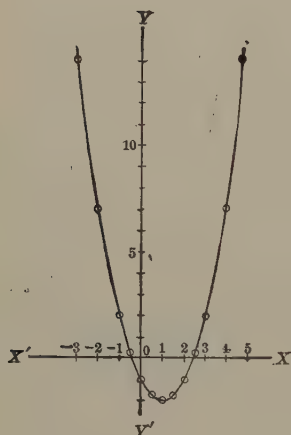
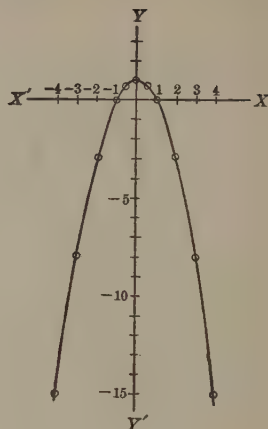
| x | $f(x)$ | x | $f(x)$ |
|----------------|---------------|----------------|---------------|
| $-\infty$ | ∞ | $3\frac{1}{2}$ | $\frac{1}{4}$ |
| -1 | 16 | 4 | 1 |
| 0 | 9 | 5 | 4 |
| 1 | 4 | 6 | 9 |
| 2 | 1 | 7 | 16 |
| $2\frac{1}{2}$ | $\frac{1}{4}$ | ∞ | ∞ |
| 3 | 0 | | |

Since the graph touches the X -axis at $x = 3$ but does not cross it, the equation $x^2 - 6x + 9 = 0$ has two roots each equal to 3.

4. Let $1 - x^2 = f(x)$.

| x | $f(x)$ |
|-------------------|---------------|
| 0 | 1 |
| $\pm \frac{1}{2}$ | $\frac{3}{4}$ |
| ± 1 | 0 |
| ± 2 | -3 |
| ± 3 | -8 |
| ± 4 | -15 |
| $\pm \infty$ | $-\infty$ |

The roots of $1 - x^2 = 0$ are evidently -1 and 1.



5. Let $x^2 - 2x - 1 = f(x)$.

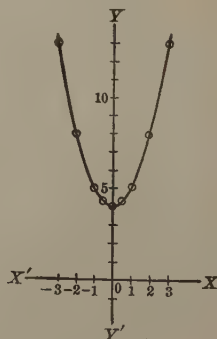
| x | $f(x)$ | x | $f(x)$ |
|----------------|-----------------|----------------|-----------------|
| $-\infty$ | ∞ | $1\frac{1}{2}$ | $-1\frac{3}{4}$ |
| -3 | 14 | 2 | -1 |
| -2 | 7 | $2\frac{1}{2}$ | $\frac{1}{4}$ |
| -1 | 2 | 3 | 2 |
| $-\frac{1}{2}$ | $\frac{1}{4}$ | 4 | 7 |
| 0 | -1 | 5 | 14 |
| $\frac{1}{2}$ | $-1\frac{3}{4}$ | ∞ | ∞ |
| 1 | -2 | | |

The graph of $x^2 - 2x - 1$ shows that the equation $x^2 - 2x - 1 = 0$ has a negative root a little greater than $-\frac{1}{2}$ and a positive root a little less than $2\frac{1}{2}$.

6. Let $x^2 + 4 = f(x)$.

| x | $f(x)$ |
|-------------------|----------------|
| 0 | 4 |
| $\pm \frac{1}{2}$ | $4\frac{1}{4}$ |
| ± 1 | 5 |
| ± 2 | 8 |
| ± 3 | 13 |
| $\pm \infty$ | ∞ |

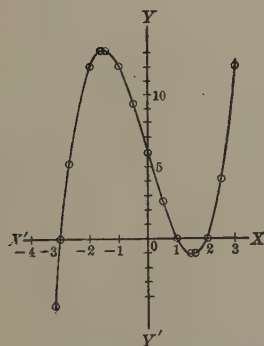
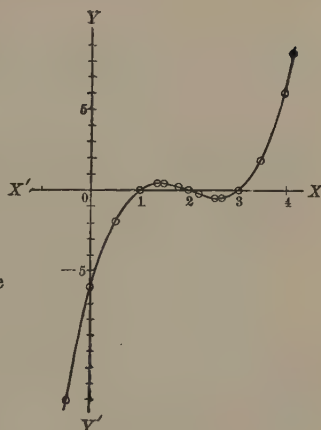
The graph of $x^2 + 4$ does not cross or touch the X -axis. Hence, both roots of $x^2 + 4 = 0$ are imaginary.



7. Let $x^3 - 6x^2 + 11x - 6 = f(x)$.

| x | $f(x)$ | x | $f(x)$ |
|-----------|-----------|----------|----------|
| $-\infty$ | $-\infty$ | 2.2 | -.192 |
| -.5 | -13.125 | 2.5 | -.375 |
| 0 | -6 | 2.6 | -.384 |
| .5 | -1.875 | 3 | 0 |
| 1 | 0 | 3.5 | 1.875 |
| 1.4 | .384 | 4 | 6 |
| 1.5 | .375 | 4.2 | 8.448 |
| 1.8 | .192 | ∞ | ∞ |
| 2 | 0 | | |

The roots of $x^3 - 6x^2 + 11x - 6 = 0$ are evidently 1, 2, and 3.



8. Let $x^3 - 7x + 6 = f(x)$.

| x | $f(x)$ | x | $f(x)$ |
|-----------|-----------|----------|----------|
| $-\infty$ | $-\infty$ | 0 | 6 |
| -3.2 | -4.368 | .5 | 2.625 |
| -3 | 0 | 1 | 0 |
| -2.7 | 5.217 | 1.5 | -1.125 |
| -2 | 12 | 1.6 | -1.104 |
| -1.6 | 13.104 | 2 | 0 |
| -1.5 | 13.125 | 2.5 | 4.125 |
| -1 | 12 | 3 | 12 |
| -.5 | 9.375 | ∞ | ∞ |

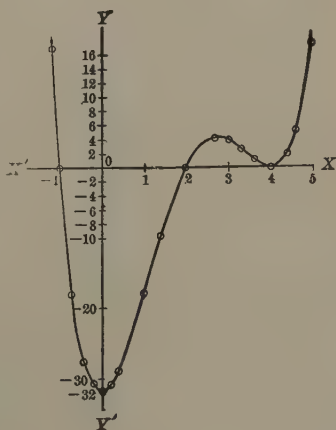
The roots of $x^3 - 7x + 6 = 0$ are evidently -3, 1, and 2.

9. Let $x^4 - 9x^3 + 22x^2 - 32 = f(x)$.

| x | $f(x)$ | x | $f(x)$ |
|-----------|----------|----------|----------|
| $-\infty$ | ∞ | 2 | 0 |
| -1.2 | 17.3056 | 2.7 | 4.3771 |
| -1 | 0 | 3 | 4 |
| -.7 | -17.8929 | 3.3 | 2.7391 |
| -.4 | -27.8784 | 3.6 | 1.1776 |
| -.2 | -31.0464 | 4 | 0 |
| 0 | -32 | 4.4 | 2.0736 |
| .2 | -31.1904 | 4.6 | 5.2416 |
| .4 | -29.0304 | 5 | 18 |
| 1 | -18 | ∞ | ∞ |
| 1.4 | -9.7344 | | |

There are four roots, -1, 2, 4, 4, two of which are equal.

Key Adv. Alg.—29



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2. Let

Then,

$$\begin{aligned} 2x &= f(x). \\ f(x+h) - f(x) &= 2(x+h) - 2x = 2h. \\ \therefore \frac{f(x+h) - f(x)}{h} &= 2. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(x) = 2$.

3. Let

Then,

$$\begin{aligned} 2x+1 &= f(x). \\ f(x+h) - f(x) &= [2(x+h) + 1] - (2x+1) = 2h. \\ \therefore \frac{f(x+h) - f(x)}{h} &= 2. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(x) = 2$.

4. Let

Then,

$$\begin{aligned} x^2 - 8x &= f(x). \\ f(x+h) - f(x) &= [(x+h)^2 - 8(x+h)] - (x^2 - 8x) \\ &= 2hx + h^2 - 8h. \\ \therefore \frac{f(x+h) - f(x)}{h} &= 2x + h - 8. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(x) = 2x - 8$.

Again,

$$\begin{aligned} f'(x+h) - f'(x) &= [2(x+h) - 8] - (2x - 8) = 2h. \\ \therefore \frac{f'(x+h) - f'(x)}{h} &= 2. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f''(x) = 2$.

5. Let

Then,

$$\begin{aligned} x^2 - 8x + 16 &= f(x). \\ f(x+h) - f(x) &= [(x+h)^2 - 8(x+h) + 16] \\ &\quad - (x^2 - 8x + 16) \\ &= 2hx + h^2 - 8h. \end{aligned}$$

The rest of the solution is the same as that of Ex. 4.

Hence,

$$f'(x) = 2x - 8 \text{ and } f''(x) = 2.$$

6. Let

Then,

$$\begin{aligned} 3 - 2x - x^2 &= f(x). \\ f(x+h) - f(x) &= [3 - 2(x+h) - (x+h)^2] \\ &\quad - (3 - 2x - x^2) \\ &= -2h - 2hx - h^2. \\ \therefore \frac{f(x+h) - f(x)}{h} &= -2 - 2x - h. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(x) = -2 - 2x$.

Again,

$$\begin{aligned} f'(x+h) - f'(x) &= [-2 - 2(x+h)] - (-2 - 2x) = -2h. \\ \therefore \frac{f'(x+h) - f'(x)}{h} &= -2. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f''(x) = -2$.

7. Let

Then,

$$\begin{aligned} x^3 + 9x^2 + 23x + 15 &= f(x). \\ f(x+h) - f(x) &= [(x+h)^3 + 9(x+h)^2 + 23(x+h) + 15] \\ &\quad - (x^3 + 9x^2 + 23x + 15) \\ &= 3hx^2 + 3h^2x + h^3 + 18hx + 9h^2 + 23h. \\ \therefore \frac{f(x+h) - f(x)}{h} &= 3x^2 + 3hx + h^2 + 18x + 9h + 23. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(x) = 3x^2 + 18x + 23$.

Again,

$$\begin{aligned} f'(x+h) - f'(x) &= [3(x+h)^2 + 18(x+h) + 23] \\ &\quad - (3x^2 + 18x + 23) \\ &= 6hx + 3h^2 + 18h. \\ \therefore \frac{f'(x+h) - f'(x)}{h} &= 6x + 3h + 18. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f''(x) = 6x + 18$.

Again, $f''(x+h) - f''(x) = [6(x+h) + 18] - (6x + 18) = 6h$

$$\therefore \frac{f''(x+h) - f''(x)}{h} = 6.$$

Taking the limit as $h \rightarrow 0$, $f'''(x) = 6$.

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1. Let

$$(x-1)(x-2)(x-3) = f(x).$$

$$f'(x) = (x-2)(x-3) \frac{d}{dx}(x-1) + (x-1)(x-3) \frac{d}{dx}(x-2) \\ + (x-1)(x-2) \frac{d}{dx}(x-3)$$

$$= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2) \\ = 3x^2 - 12x + 11.$$

Or, expanding $f(x)$,

$$(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

Hence, $f'(x) = 3x^2 - 12x + 11$.

2. Let

$$(x-1)^3(x+2) = f(x).$$

$$\text{Then, } f'(x) = (x+2) \frac{d}{dx}(x-1)^3 + (x-1)^3 \frac{d}{dx}(x+2) \\ = (x+2) 3(x-1)^2 + (x-1)^3 \cdot 1 \\ = (x-1)^2(4x+5).$$

$$\text{Again, } f''(x) = (4x+5) \frac{d}{dx}(x-1)^2 + (x-1)^2 \frac{d}{dx}(4x+5) \\ = (4x+5) 2(x-1) + (x-1)^2 \cdot 4 \\ = (x-1)(12x+6).$$

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1. By Prin. 1, $x^2 - 5x + 6$ has one critical value, $6 - \frac{25}{4}$, or $-\frac{1}{4}$. This is a minimum, and corresponds to $x = \frac{5}{2}$.

2. By Prin. 1, $x^2 + x - 30$ has one critical value, $-30 - \frac{1}{4}$, or $-30\frac{1}{4}$. This is a minimum and corresponds to $x = -\frac{1}{2}$.

3. By Prin. 1, $3x^2 - 4x - 15$ has one critical value, $-15 - \frac{16}{12}$, or $-16\frac{1}{3}$. This is a minimum and corresponds to $x = \frac{2}{3}$.

4. Since x^2 is positive for all real values of x , the least value of $x^2 + 4$ occurs when $x = 0$. Hence, the minimum of $x^2 + 4$ is 4.

Hence, Prin. 3, -4 is a maximum of $-(x^2 + 4)$, or of $-x^2 - 4$.

Or, by Prin. 1, $-x^2 - 4$ has one critical value, $-4 + \frac{0}{4}$, or -4 . This is a maximum and corresponds to $x = 0$.

5. The sum of the factors of $(x-3)(7-x)$ is 4, a constant. Hence, Prin. 6, $(\frac{4}{2})^2$, or 4, is a maximum of the function.

6. The sum of the factors of $(1 + \frac{1}{x})(1 - \frac{1}{x})$ is 2, a constant. Hence, Prin. 6, $(\frac{2}{2})^2$, or 1, is a maximum of the function.

7. The product of the terms of $x + \frac{1}{x}$ is 1, a constant. Hence, Prin. 7, 2 is a minimum and -2 is a maximum of the function. (See figure, § 653.)

8. By Ex. 7, 2 is a minimum, and -2 is a maximum, of $x + \frac{1}{x}$. Hence, Prin. 2, $2 + 2$, or 4 , is a minimum, and $-2 + 2$, or 0 , is a maximum, of $x + \frac{1}{x} + 2$.

Graphically considered, adding 2 to $x + \frac{1}{x}$ is equivalent to moving the X -axis downward 2 units parallel to itself. (See figure, § 653.)

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2. Let $x^3 - 6x^2 + 11x - 6 = f(x)$.

Then, $f'(x) = 3x^2 - 12x + 11$,

and $f''(x) = 6x - 12$.

To find the critical values of $f(x)$, put $f'(x) = 0$.

Then, $3x^2 - 12x + 11 = 0$.

Solving,

$$x = 2 - \frac{1}{3}\sqrt{3}, \text{ or } 2 + \frac{1}{3}\sqrt{3}$$

$$= 1.423, \text{ or } 2.577, \text{ approximately.}$$

The first of these values makes $f''(x)$ negative, and the second makes $f''(x)$ positive. Hence, the first value corresponds to a maximum, and the second to a minimum, of $f(x)$. Substituting these values of x in $f(x)$,

$$f(2 - \frac{1}{3}\sqrt{3}) = \frac{2}{3}\sqrt{3} = .385, \text{ a maximum,}$$

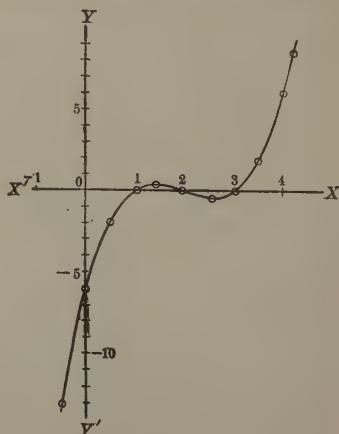
$$\text{and } f(2 + \frac{1}{3}\sqrt{3}) = -\frac{2}{3}\sqrt{3} = -.385, \text{ a minimum.}$$

It is evident that $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. Also, since $f(x)$ is a rational integral function of x , the function is *continuous*.

Hence, the function is an *increasing* function for all values of x up to $2 - \frac{1}{3}\sqrt{3}$, then a *decreasing* function for values of x between $2 - \frac{1}{3}\sqrt{3}$ and $2 + \frac{1}{3}\sqrt{3}$, and thereafter an *increasing* function; and since the function is continuous, its graph must cross the X -axis between $f(-\infty)$, which is negative, and the maximum .385, which is positive, again between the maximum and the minimum, which is negative, and finally between the minimum and $f(+\infty)$, which is positive. These intersections give *three real roots* of the equation $f(x) = 0$. By trial they are found to be 1, 2, and 3.

The following values are suitable for plotting the graph:

| x | $f(x)$ |
|-----------|-----------------------|
| $-\infty$ | $-\infty$ |
| $-.5$ | -13.125 |
| 0 | -6 |
| $.5$ | -1.875 |
| 1 | 0 |
| 1.423 | $.385, \text{ max.}$ |
| 2 | 0 |
| 2.577 | $-.385, \text{ min.}$ |
| 3 | 0 |
| 3.5 | 1.875 |
| 4 | 6 |
| 4.2 | 8.448 |
| ∞ | ∞ |



3. Let $x^2 + 1 = f(x)$.

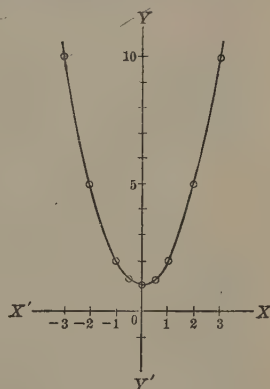
By Prin. 1, this function has only one critical value, $1 - \frac{9}{4}$, or 1, a *minimum* corresponding to $x = 0$.

It is evident that $f(-\infty) = +\infty$ and $f(+\infty) = +\infty$; also that the function is *continuous*, being rational and integral with respect to x .

Hence, the function is a *decreasing* function for negative values of x , and an *increasing* function for positive values of x . But since $f(x)$ is never less than the minimum 1, the graph of $f(x)$ does not cross or touch the X -axis; that is, $f(x) = 0$ has no real roots. Hence, both roots are *imaginary*.

The following values are suitable for plotting the graph:

| x | $f(x)$ |
|-------------------|----------------|
| $\pm \infty$ | ∞ |
| ± 3 | 10 |
| ± 2 | 5 |
| ± 1 | 2 |
| $\pm \frac{1}{2}$ | $1\frac{1}{4}$ |
| 0 | 1, min. |



4. Let

$$x^2 - 10x + 25 = f(x).$$

By Prin. 1, this function has only one critical value, $25 - \frac{100}{4}$, or 0, a *minimum* corresponding to $x = \frac{10}{2}$, or 5.

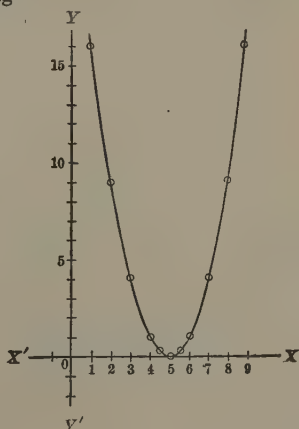
It is evident that $f(-\infty) = +\infty$ and $f(+\infty) = +\infty$; also that the function is *continuous*, being rational and integral with respect to x .

Hence, the function is a *decreasing* function for values of x less than 5 and an *increasing* function for all greater values.

Since $f(5) = 0$ and $f(5)$ is also a minimum, less than either $f(5 - h)$ or $f(5 + h)$ however small h is, the equation $f(x) = 0$ has two roots each equal to 5.

The following values are suitable for plotting the graph:

| x | $f(x)$ |
|----------------|---------------|
| $-\infty$ | ∞ |
| 1 | 16 |
| 2 | 9 |
| 3 | 4 |
| 4 | 1 |
| $4\frac{1}{2}$ | $\frac{1}{4}$ |
| 5 | 0, min. |
| $5\frac{1}{2}$ | $\frac{1}{4}$ |
| 6 | 1 |
| 7 | 4 |
| 8 | 9 |
| 9 | 16 |
| ∞ | ∞ |



5. Let $x^3 - 7x + 6 = f(x)$.

Then, $f'(x) = 3x^2 - 7$,

and $f''(x) = 6x$.

To find the critical values of $f(x)$, put $f'(x) = 0$.

Then, $3x^2 - 7 = 0$.

Solving, $x = -\frac{1}{3}\sqrt{21}$ or $\frac{1}{3}\sqrt{21}$
 $= -1.527$ or 1.528 , approximately.

The first of these values makes $f''(x)$ negative and the second makes $f''(x)$ positive.

Hence, the first value corresponds to a maximum and the second to a minimum of $f(x)$.

Substituting these values of x in $f(x)$,

$$f(-\frac{1}{3}\sqrt{21}) = \frac{14}{9}\sqrt{21} + 6 = 13.128+, \text{ a maximum,}$$

and $f(\frac{1}{3}\sqrt{21}) = -\frac{14}{9}\sqrt{21} + 6 = -1.128+$, a minimum.

It is evident that $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$.

Also, since $f(x)$ is a rational integral function of x , the function is *continuous*.

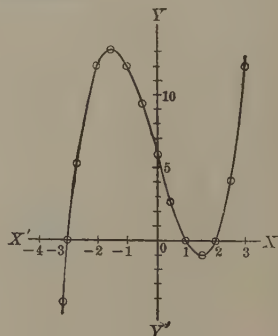
Hence, the function is an *increasing* function for all values of x up to $-\frac{1}{3}\sqrt{21}$, then a *decreasing* function for values of x between $-\frac{1}{3}\sqrt{21}$ and $\frac{1}{3}\sqrt{21}$, and thereafter an *increasing* function; and since the function is continuous, its graph must cross the X -axis between $f(-\infty)$, which is negative, and the maximum $13.128+$, which is positive, again between the maximum and the minimum, which is negative, and finally between the minimum and $f(+\infty)$, which is positive.

These intersections give *three real roots* of the equation $f(x) = 0$.

By trial, they are found to be -3 , 1 , and 2 .

The following values are suitable for plotting the graph:

| x | $f(x)$ | x | $f(x)$ |
|-----------|-----------------|----------|-----------------|
| $-\infty$ | $-\infty$ | 0 | 6 |
| -3.2 | -4.368 | $.5$ | 2.625 |
| -3 | 0 | 1 | 0 |
| -2.7 | 5.217 | 1.528 | -1.128 , min. |
| -2 | 12 | 2 | 0 |
| -1.528 | 13.128 , max. | 2.5 | 4.125 |
| -1 | 12 | 3 | 12 |
| $-.5$ | 9.375 | ∞ | ∞ |



6. Let $x^4 - 5x^2 + 4 = f(x)$.

Then, $f'(x) = 4x^3 - 10x$,

and $f''(x) = 12x^2 - 10$.

To find the critical values of $f(x)$, put $f'(x) = 0$.

Then, $4x^3 - 10x = 0$.

Solving, $x = -\frac{1}{2}\sqrt{10}$, 0 , $\frac{1}{2}\sqrt{10}$
 $= -1.581$, 0 , 1.581 , approximately.

The first and third of these values make $f''(x)$ positive and the second makes $f''(x)$ negative. Hence, the first and third values correspond to minima, and the second to a maximum of $f(x)$.

Substituting these values of x in $f(x)$,

$$f(-\frac{1}{2}\sqrt{10}) = -2.25, \text{ a minimum,}$$

$$f(0) = 4, \text{ a maximum,}$$

and $f(\frac{1}{2}\sqrt{10}) = -2.25, \text{ a minimum.}$

It is evident that $f(-\infty) = +\infty$ and $f(+\infty) = +\infty$. Also, since $f(x)$ is a rational integral function of x , the function is *continuous*.

Hence, the function is a *decreasing* function for all values of x up to $-\frac{1}{2}\sqrt{10}$, then an *increasing* function for values of x between $-\frac{1}{2}\sqrt{10}$ and 0, then a *decreasing* function for values of x between 0 and $\frac{1}{2}\sqrt{10}$, and thereafter an *increasing* function.

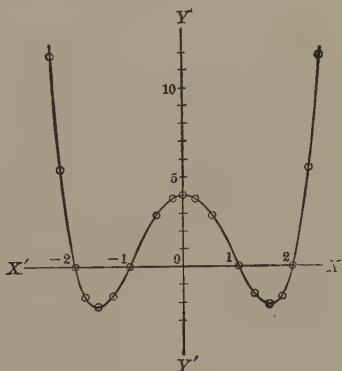
Since the function is continuous, its graph must cross the X -axis between $f(-\infty)$, which is positive, and the minimum -2.25 , which is negative, again between this minimum and the maximum 4, which is positive, again between the maximum and the next minimum -2.25 , which is negative, and finally between this second minimum and $f(+\infty)$, which is positive.

These intersections give *four real roots* of $f(x) = 0$.

By trial they are found to be $-2, -1, 1$, and 2 .

The following values are suitable for plotting the graph:

| x | $f(x)$ |
|--------------|-------------|
| $\pm \infty$ | ∞ |
| ± 2.5 | 11.8125 |
| ± 2.3 | 5.5341 |
| ± 2 | 0 |
| ± 1.8 | -1.7024 |
| ± 1.581 | -2.25, min. |
| ± 1.3 | -1.5939 |
| ± 1 | 0 |
| $\pm .5$ | 2.8125 |
| $\pm .2$ | 3.8016 |
| 0 | 4, max. |



7. Let $x + 4x^{-1} = f(x)$.

Then, $f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$.

Since $f(x)$ is the sum of two variables whose product is 4, a constant, -4 is a *maximum* and 4 is a *minimum* of the function (Prin. 7).

Since $f'(x) \doteq \infty$ as $x \doteq 0$, the function is *discontinuous* for $x = 0$. For all other values of x , however, the function is continuous.

It is evident that $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$.

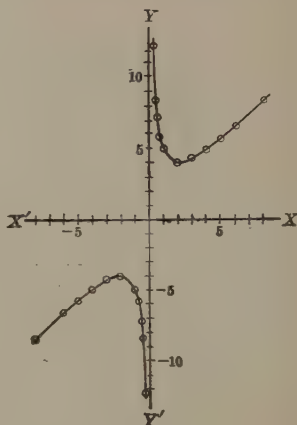
Hence, the graph of $f(x)$ has *two branches*, one in the third quadrant corresponding to negative values of x , the other in the first quadrant corresponding to positive values of x .

In the third quadrant the function *increases* continuously from $-\infty$ to its maximum, -4 , and thereafter *decreases* without limit as $x \doteq 0$; in the first quadrant the function *decreases* continuously from ∞ to its minimum, 4, and thereafter *increases* without limit, as $x \doteq \infty$.

Since $f(x)$ has no values lying between its maximum, -4 , and its minimum, 4, the equation $f(x) = 0$ has *no real roots*.

The following values, in which the upper and lower signs correspond, are suitable for plotting the graph:

| x | $f(x)$ |
|-------------------|----------------------|
| $\pm \infty$ | $\pm \infty$ |
| ± 8 | $\pm 8\frac{1}{2}$ |
| ± 6 | $\pm 6\frac{3}{4}$ |
| ± 5 | ± 5.8 |
| ± 4 | ± 5 |
| ± 3 | $\pm 4\frac{1}{8}$ |
| ± 2 | ± 4 |
| ± 1 | ± 5 |
| $\pm .8$ | ± 5.8 |
| $\pm .6$ | $\pm 7.2\frac{3}{4}$ |
| $\pm .5$ | ± 8.5 |
| $\pm \frac{1}{3}$ | $\pm 12\frac{1}{3}$ |
| ± 0 | $\pm \infty$ |



§. Let

$$f(x) = ax + b$$

be any function of the first degree in x .

Then,

$$f'(x) = a, \text{ a constant.}$$

Hence, § 634, $f(x)$ varies uniformly with x , so that every increase in x produces an increase a times as great in $f(x)$.

There is no value of x , then, at which, as x increases, $f(x)$ ceases to increase and begins to decrease, or *vice versa*; that is, a function of the first degree in x has no critical values.

Let $f(x) = ax + b = y$, and let P_1 , P_2 , and P_3 be any three points on the graph of $ax + b$.

Then,

$$y_1 = ax_1 + b, (1)$$

$$y_2 = ax_2 + b, (2)$$

and

$$y_3 = ax_3 + b. (3)$$

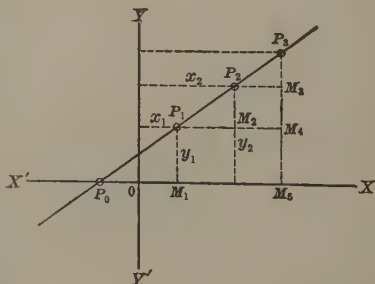
$$\text{By (1) and (2), } \frac{y_2 - y_1}{x_2 - x_1} = a. (4)$$

$$\text{By (1) and (3), } \frac{y_3 - y_1}{x_3 - x_1} = a. (5)$$

Now in the figure, $y_2 - y_1 = M_2P_2$ and $x_2 - x_1 = P_1M_2$, sides of a right triangle $P_1M_2P_2$, if a straight line is drawn from P_1 to P_2 . Also $y_3 - y_1 = M_4P_3$ and $x_3 - x_1 = P_1M_4$, sides of a right triangle $P_1M_4P_3$, if a straight line is drawn from P_1 to P_3 . By (4) and (5), the ratio of the altitude to the base in each right triangle is a , a constant. Hence, these triangles are *similar*, and since P_1M_2 and P_1M_4 lie in the same straight line, P_1P_2 and P_1P_3 lie in the same straight line.

Since P_1 , P_2 , and P_3 lie in the same straight line and are any three points of the graph, the graph is a straight line.

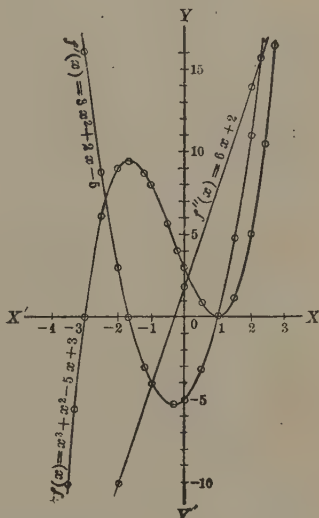
Hence, the graph of any function of the first degree in x is a straight line.



2. $f(x) = x^3 + x^2 - 5x + 3$, $f'(x) = 3x^2 + 2x - 5$, $f''(x) = 6x + 2$.

| x | $f(x)$ | $f'(x)$ | $f''(x)$ | x | $f(x)$ | $f'(x)$ | $f''(x)$ |
|-----------------|-----------------|-----------------|-----------|----------------|----------|----------|----------|
| $-\infty$ | $-\infty$ | ∞ | $-\infty$ | $-.2$ | 4.032 | | |
| -3.5 | -10.125 | | | 0 | 3 | -5 | 2 |
| -3.3 | -5.547 | | | .5 | .875 | -3.25 | |
| -3 | 0 | 16 | | 1 | 0 | 0 | 8 |
| -2.5 | 6.125 | 8.75 | | 1.5 | 1.125 | 4.75 | |
| -2 | 9 | 3 | -10 | 2 | 5 | 11 | 14 |
| $-1\frac{3}{4}$ | $9\frac{13}{7}$ | 0 | -8 | 2.4 | 10.584 | | 16 |
| -1.2 | 8.712 | -3.08 | | $2\frac{1}{3}$ | | 16 | 16 |
| -1 | 8 | -4 | -4 | 2.7 | 16.473 | | |
| $-.5$ | 5.625 | | | ∞ | ∞ | ∞ | ∞ |
| $-\frac{1}{2}$ | | $-5\frac{1}{2}$ | 0 | | | | |

It appears from the graphs of $f(x)$ and its derived functions that $f(x)=0$ has two roots equal to 1 and that $f'(x)=0$ has one root equal to 1. The reason for this is that the roots of $f'(x)=0$ always correspond to critical values of $f(x)$. In this case 1 is a root of $f'(x)=0$, and therefore corresponds to a *minimum* of $f(x)$. Since this minimum is 0, $f(x)=0$ has two equal roots, $x=1$.



THEORY OF EQUATIONS

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$$\begin{array}{r}
 1. \quad 1 + 1 - 3 - 17 - 30 \quad | \quad 3 \quad 1 + 1 - 3 - 17 - 30 \quad | \quad -2 \\
 \hline
 3 + 12 + 27 + 30 \quad -2 + 2 + 2 + 30 \\
 \hline
 1 + 4 + 9 + 10 \quad 1 - 1 - 1 - 15
 \end{array}$$

In the first case the quotient is $x^3 + 4x^2 + 9x + 10$ and in the second it is $x^3 - x^2 - x - 15$. The division is exact in both cases.

$$\begin{array}{r}
 2. \quad 1 + 4 - 6 + 0 - 1 + 2 \quad | \quad 1 \quad 1 + 4 - 6 + 0 - 1 + 2 \quad | \quad -1 \\
 \hline
 1 + 5 - 1 - 1 - 2 \quad -1 - 3 + 9 - 9 + 10 \\
 \hline
 1 + 5 - 1 - 1 - 2 \quad 1 + 3 - 9 + 9 - 10 \quad | \quad 12
 \end{array}$$

In the first case the quotient is $x^4 + 5x^3 - x^2 - x - 2$ and there is no remainder; in the second case the quotient is $x^4 + 3x^3 - 9x^2 + 9x - 10$ and the remainder is 12.

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$$\begin{array}{r}
 3. \quad 2 - 7 - 16 - 1 + 32 - 10 \quad | 5 \quad 2 - 7 - 16 - 1 + 32 - 10 \quad | 1 \\
 \underline{10 + 15 - 5 - 30 + 10} \quad \quad \quad \underline{2 - 5 - 21 - 22 + 10} \\
 2 + 3 - 1 - 6 + 2 \quad \quad \quad 2 - 5 - 21 - 22 + 10
 \end{array}$$

In the first case the quotient is $2x^4 + 3x^3 - x^2 - 6x + 2$ and in the second it is $2x^4 - 5x^3 - 21x^2 - 22x + 10$. The division is exact in both cases.

$$\begin{array}{r}
 4. \quad \begin{array}{r} 1 \quad 2 - 5 + 4 + 0 - 22 + 21 \\ + 2 \quad \quad 4 - 6 \\ - 3 \quad \quad - 2 + 3 \\ \quad \quad - 8 + 12 \\ \quad \quad - 10 + 15 \\ \hline 2 - 1 - 4 - 5 \quad | - 20 + 36 \end{array}
 \end{array}$$

The quotient is $2x^3 - x^2 - 4x - 5$ and the remainder is $-20x + 36$.

$$\begin{array}{r}
 2. \quad \begin{array}{r} 1 - 9 + 23 - 15 \quad | 1 \\ \hline 1 - 8 + 15 \\ \hline 1 - 8 + 15 \end{array}
 \end{array}$$

$$x^2 - 8x + 15 = (x - 3)(x - 5) = 0.$$

Hence, the roots are 1, 3, and 5.

$$\begin{array}{r}
 3. \quad \begin{array}{r} 1 - 10 + 29 - 20 \quad | 1 \\ \hline 1 - 9 + 20 \\ \hline 1 - 9 + 20 \end{array}
 \end{array}$$

$$x^2 - 9x + 20 = (x - 4)(x - 5) = 0.$$

Hence, the roots are 1, 4, and 5.

$$\begin{array}{r}
 4. \quad \begin{array}{r} 1 + 0 - 7 + 6 \quad | 1 \\ \hline 1 + 1 - 6 \\ \hline 1 + 1 - 6 \end{array}
 \end{array}$$

$$x^2 + x - 6 = (x - 2)(x + 3) = 0.$$

Hence, the roots are 1, 2, and -3.

$$\begin{array}{r}
 5. \quad \begin{array}{r} 1 + 0 - 9 + 4 + 12 \quad | -1 \\ \hline -1 + 1 + 8 - 12 \\ \hline 1 - 1 - 8 + 12 \quad | 2 \\ \hline 2 + 2 - 12 \\ \hline 1 + 1 - 6 \end{array}
 \end{array}$$

$$x^2 + x - 6 = (x - 2)(x + 3) = 0.$$

Hence, the roots are -1, 2, 2, and -3.

$$\begin{array}{r}
 6. \quad \begin{array}{r} 1 - 10 + 33 - 36 \quad | 3 \\ \hline 3 - 21 + 36 \\ \hline 1 - 7 + 12 \end{array}
 \end{array}$$

$$x^2 - 7x + 12 = (x - 3)(x - 4) = 0.$$

Hence, the roots are 3, 3, and 4.

$$\begin{array}{r}
 7. \quad \begin{array}{r} 1 - 9 + 21 + 1 - 30 \quad | -1 \\ \hline -1 + 10 - 31 + 30 \\ \hline 1 - 10 + 31 - 30 \quad | 2 \\ \hline 2 - 16 + 30 \\ \hline 1 - 8 + 15 \end{array}
 \end{array}$$

$$x^2 - 8x + 15 = (x - 3)(x - 5) = 0.$$

Hence, the roots are -1, 2, 3, and 5.

$$\begin{array}{r}
 8. \quad 5 - 2 - 35 - 16 + 12 \quad | -1 \\
 \quad \quad - 5 + 7 + 28 - 12 \\
 \quad \quad 5 - 7 - 28 + 12 \quad | -2 \\
 \quad \quad - 10 + 34 - 12 \\
 \quad \quad 5 - 17 + 6
 \end{array}$$

$$5x^2 - 17x + 6 = (x - 3)(5x - 2) = 0.$$

Hence, the roots are -1 , -2 , 3 , and $\frac{2}{5}$.

$$\begin{array}{r}
 9. \quad 2 - 1 - 12 + 7 + 16 - 12 \quad | 1 \\
 \quad \quad 2 + 1 - 11 - 4 + 12 \\
 \quad \quad 2 + 1 - 11 - 4 + 12 \quad | 1 \\
 \quad \quad 2 + 3 - 8 - 12 \\
 \quad \quad 4 + 14 + 12 \\
 \quad \quad 2 + 7 + 6
 \end{array}$$

$$2x^2 + 7x + 6 = (x + 2)(2x + 3) = 0.$$

Hence, the roots are, 1 , 1 , 2 , -2 , and $-\frac{3}{2}$.

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$$1. \quad (x - 1)(x - 2)(x - 3) = 0.$$

The coefficient of x^3 is 1 ; of x^2 , $-1 - 2 - 3$, or -6 ; of x , $(-1)(-2) + (-1)(-3) + (-2)(-3)$, or 11 ; the absolute term is $(-1)(-2)(-3)$, or -6 . Hence, the reduced form of the equation is

$$x^3 - 6x^2 + 11x - 6 = 0.$$

$$2. \quad (x - 2)(x - 3)(x - 5) = 0.$$

The coefficient of x^3 is 1 ; of x^2 , $-2 - 3 - 5$, or -10 ; of x , $(-2)(-3) + (-2)(-5) + (-3)(-5)$, or 31 ; the absolute term is $(-2)(-3)(-5)$, or -30 . Hence, the reduced form of the equation is

$$x^3 - 10x^2 + 31x - 30 = 0.$$

$$3. \quad (x + 1)(x + 3)(x - 4) = 0.$$

The coefficient of x^3 is 1 ; of x^2 , $1 + 3 - 4$, or 0 ; of x , $1 \cdot 3 + 1(-4) + 3(-4)$, or -13 ; the absolute term is $1 \cdot 3(-4)$, or -12 . Hence, the reduced form of the equation is $x^3 - 13x - 12 = 0$.

$$4. \quad (x - 1)(x - 1)(x + \frac{1}{2}) = 0.$$

The coefficient of x^3 is 1 ; of x^2 , $-1 - 1 + \frac{1}{2}$, or $-\frac{3}{2}$; of x , $(-1)(-1) + (-1)\frac{1}{2} + (-1)\frac{1}{2}$, or 0 ; the absolute term is $(-1)(-1)\frac{1}{2}$, or $\frac{1}{2}$. Hence, the reduced form of the equation is

$$x^3 - \frac{3}{2}x^2 + \frac{1}{2} = 0.$$

Multiplying by 2 , the general form of the equation is

$$2x^3 - 3x^2 + 1 = 0.$$

5. Since $(\frac{1}{2} - \sqrt{5}) + (\frac{1}{2} + \sqrt{5}) = 1$ and $(\frac{1}{2} - \sqrt{5})(\frac{1}{2} + \sqrt{5}) = \frac{1}{4} - 5$, or $-\frac{19}{4}$, the quadratic equation whose roots are $-\frac{1}{2} + \sqrt{5}$ and $-\frac{1}{2} - \sqrt{5}$ is

$$x^2 + x - \frac{19}{4} = 0, \text{ or } 4x^2 + 4x - 19 = 0.$$

$$\begin{array}{r}
 4 + 4 - 19 \\
 1 - 2 \\
 4 + 4 - 19 \\
 -8 - 8 + 38 \\
 4 - 4 - 27 + 38
 \end{array}$$

Hence, the equation whose roots are $2, -\frac{1}{2} \pm \sqrt{5}$ is

$$4x^3 - 4x^2 - 27x + 38 = 0.$$

6. Since $(\frac{1}{2} - \frac{1}{2}\sqrt{-3}) + (\frac{1}{2} + \frac{1}{2}\sqrt{-3}) = 1$ and $(\frac{1}{2} - \frac{1}{2}\sqrt{-3})(\frac{1}{2} + \frac{1}{2}\sqrt{-3}) = \frac{1}{4} - (-\frac{3}{4})$, or 1, the quadratic equation whose roots are $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ is $x^2 + x + 1 = 0$.

$$\begin{array}{r} 1 + 1 + 1 \\ 1 - 1 \\ \hline 1 + 1 + 1 \\ - 1 - 1 - 1 \\ \hline 1 + 0 + 0 - 1 \end{array}$$

Hence, the equation whose roots are $1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$ is
 $x^3 - 1 = 0$.

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2. Dividing by 9, $x^3 - \frac{7}{9}x + \frac{2}{9} = 0$.

Represent the roots by $a, 2a$, and b .

Then, § 666,

$$-3a - b = 0, \quad (1)$$

$$2a^2 + 3ab = -\frac{7}{9}, \quad (2)$$

$$-2a^2b = \frac{2}{9}. \quad (3)$$

and

Solving (1) and (3),

$$a = \frac{1}{3} \text{ and } b = -1.$$

These values satisfy (2) also.

Hence, the roots are $\frac{1}{3}, \frac{2}{3}$, and -1 .

3. Dividing by 9, $x^3 - 3x^2 + \frac{23}{9}x - \frac{5}{9} = 0$.

Represent the roots by $a - d, a$, and $a + d$.

Then, § 666,

$$-3a = -3, \text{ whence } a = 1, \quad (1)$$

$$(a - d)a + (a - d)(a + d) + a(a + d) = \frac{23}{9}, \quad (2)$$

and

$$-(a - d)(a)(a + d) = -\frac{5}{9}. \quad (3)$$

Substituting (1) in (3), $d = \pm \frac{2}{3}$.

The values $a = 1, d = \pm \frac{2}{3}$ satisfy (2) also.

Hence, the roots are $\frac{1}{3}, 1$, and $\frac{5}{3}$.

4. $x^3 - 12x + 16 = 0$.

Represent the roots by a, a , and b .

Then, § 666,

$$-2a - b = 0, \quad (1)$$

$$a^2 + 2ab = -12, \quad (2)$$

$$-a^2b = 16. \quad (3)$$

and

Solving (1) and (2),

$$a = 2 \text{ or } -2,$$

and

$$b = -4 \text{ or } 4.$$

The first values of a and b satisfy (3) also, but the second values do not, and are to be rejected.

Hence, the roots are 2, 2, and -4 .

5. Dividing by 2, $x^4 - \frac{7}{2}x^3 - 3x^2 + 22x - 20 = 0$.

Represent the roots by a, a, a , and b .

Then, § 666,

$$-3a - b = -\frac{7}{2}, \quad (1)$$

$$3a^2 + 3ab = -3, \quad (2)$$

$$-a^3 - 3a^2b = 22, \quad (3)$$

and

$$a^3b = -20. \quad (4)$$

Solving (1) and (2),

$$a = 2 \text{ or } -\frac{1}{2},$$

and

$$b = -\frac{5}{2} \text{ or } \frac{17}{4}.$$

The first values of a and b satisfy also (3) and (4), but the second values do not and are to be rejected.

Hence, the roots are 2, 2, 2, and $-\frac{5}{2}$.

6. Dividing by 72, $x^4 + \frac{5}{4}x^3 - \frac{5}{2}x^2 - \frac{5}{6}x - \frac{1}{8} = 0$.

Represent the roots by a , $-a$, b , and c .

Then, § 666, $-b - c = \frac{5}{4}$, (1)

$$-a^2 + bc = -\frac{5}{2}, \quad (2)$$

$$a^2b + a^2c = -\frac{5}{6}, \quad (3)$$

$$\text{and} \quad -a^2bc = -\frac{1}{8}. \quad (4)$$

Solving (1), (3), and (4), $a = \pm \frac{2}{3}, \pm \frac{1}{3}$,

$$b = -\frac{1}{2}, -\frac{3}{4},$$

$$c = -\frac{3}{4}, -\frac{1}{2}.$$

Any set of corresponding values satisfies (2), the remaining equation, but inasmuch as all the sets give the same roots, take the first, $a = \frac{2}{3}$, $b = -\frac{1}{2}$, and $c = -\frac{3}{4}$. Then, the roots are

$$\frac{2}{3}, -\frac{2}{3}, -\frac{1}{2}, \text{ and } -\frac{3}{4}.$$

7. Dividing by 6, $x^3 - \frac{11}{6}x^2 + x - \frac{1}{6} = 0$.

Represent the roots by $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$.

Then, § 666, $-\frac{1}{a-d} - \frac{1}{a} - \frac{1}{a+d} = -\frac{11}{6}$, (1)

$$\frac{1}{a(a-d)} + \frac{1}{a^2-d^2} + \frac{1}{a(a+d)} = 1, \quad (2)$$

$$\text{and} \quad \frac{1}{a(a^2-d^2)} = -\frac{1}{6}. \quad (3)$$

Solving (2) and (3), $a = 2$ and $d = \pm 1$.

These values satisfy (1) also, but all the roots are obtained by taking either value of d .

Then, the roots are $1, \frac{1}{2}$, and $\frac{1}{3}$.

8. $x^3 + px^2 + qx + r = 0$.

Represent the roots by a , $-a$, and b .

Then, § 666, $p = -b$, (1)

$$q = -a^2, \quad (2)$$

$$\text{and} \quad r = a^2b. \quad (3)$$

Hence, $r = pq$

is the desired relation.

The coefficients of $x^3 - 5x^2 - 4x + 20 = 0$ satisfy this relation.

Hence, by (1), (2), and (3),

$$b = -p = -(-5) = 5,$$

$$\text{and} \quad a = \sqrt{-q} = \sqrt{4} = \pm 2.$$

Hence, the roots are $2, -2$, and 5 .

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2. Since $3 + \sqrt{2}$ is a root, $3 - \sqrt{2}$ is a root, and

$$(x - 3 - \sqrt{2})(x - 3 + \sqrt{2}), \text{ or } x^2 - 6x + 7,$$

is the quadratic factor corresponding to these roots. The other factor of $2x^3 - 11x^2 + 8x + 7$ is $2x + 1$, giving $-\frac{1}{2}$ for the corresponding root.

Hence, the roots are $-\frac{1}{2}, 3 \pm \sqrt{2}$.

3. Since $3 - \sqrt{-2}$ is a root, $3 + \sqrt{-2}$ is a root, and

$(x - 3 + \sqrt{-2})(x - 3 - \sqrt{-2})$, or $x^2 - 6x + 11$,
is the quadratic factor corresponding to these roots. The other factor of
 $2x^3 - 11x^2 + 16x + 11$ is $2x + 1$, giving $-\frac{1}{2}$ for the corresponding root.

Hence, the roots are $-\frac{1}{2}$, $3 \pm \sqrt{-2}$.

4. Since $\frac{1}{2}(3 + \sqrt{-3})$ is a root, $\frac{1}{2}(3 - \sqrt{-3})$ is a root, and

$[x - \frac{1}{2}(3 + \sqrt{-3})][x - \frac{1}{2}(3 - \sqrt{-3})]$, or $x^2 - 3x + 3$,
is the quadratic factor corresponding to these roots.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -1 & +6 & -3 \\ +3 & & 3 & -3 & & \\ -3 & & & 3 & -3 & \\ & & & & -3 & +3 \\ \hline & 1 & +1 & -1 & | & 0 & 0 \end{array}$$

Removing the factor $x^2 - 3x + 3$, the depressed equation is
 $x^2 + x - 1 = 0$.

Solving, $x = \frac{1}{2}(-1 \pm \sqrt{5})$.

Hence, the roots are $\frac{1}{2}(3 \pm \sqrt{-3})$, $\frac{1}{2}(-1 \pm \sqrt{5})$.

5. Since $-\frac{1}{2}(1 + \sqrt{-3})$ is a root, $-\frac{1}{2}(1 - \sqrt{-3})$ is a root, and

$[x + \frac{1}{2}(1 + \sqrt{-3})][x + \frac{1}{2}(1 - \sqrt{-3})]$, or $x^2 + x + 1$,
is the quadratic factor corresponding to these roots.

$$\begin{array}{r|rrrr} 1 & 1 & +3 & +5 & +4 & +2 \\ -1 & & -1 & -1 & & \\ -1 & & & -2 & -2 & \\ & & & & -2 & -2 \\ \hline & 1 & +2 & +2 & | & 0 & 0 \end{array}$$

Removing the factor $x^2 + x + 1$, the depressed equation is
 $x^2 + 2x + 2 = 0$.

Solving, $x = -1 \pm \sqrt{-1}$.

Hence, the roots are $-\frac{1}{2}(1 \pm \sqrt{-3})$, $-1 \pm \sqrt{-1}$.

6. Since $\sqrt{-1}$ and $-\sqrt{-1}$ are roots of the equation, $-\sqrt{-1}$ and $-\sqrt{-1}$
are roots also, and the factor corresponding to these four roots is

$$(x - \sqrt{-1})(x + \sqrt{-1})(x - \sqrt{-1})(x + \sqrt{-1}), \text{ or } x^4 + 2x^2 + 1.$$

The other factor of $2x^5 - x^4 + 4x^3 - 2x^2 + 2x - 1$ is $2x - 1$, giving the
root $\frac{1}{2}$.

Hence, the roots are $\frac{1}{2}$, $\pm \sqrt{-1}$, $\pm \sqrt{-1}$.

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6. $4x^4 + 3x^2 - 2x + 9 = 0$.

Dividing by 4, $x^4 + \frac{3}{4}x^2 - \frac{1}{2}x + \frac{9}{4} = 0$.

Multiplying the roots by 2, $y^4 + 3y^2 - 4y + 36 = 0$,

in which $y = 2x$.

7. $16x^3 - 12x^2 + 10x - 7 = 0$.

Dividing by 16, $x^3 - \frac{3}{4}x^2 + \frac{5}{8}x - \frac{7}{16} = 0$.

Multiplying the roots by 4, $y^3 - 3y^2 + 10y - 28 = 0$,

in which $y = 4x$.

$$8. \quad 2x^3 - \frac{14}{3}x^2 + \frac{11}{6}x - \frac{25}{12} = 0.$$

$$\text{Dividing by 2,} \quad x^3 - \frac{7}{3}x^2 + \frac{11}{12}x - \frac{25}{24} = 0.$$

Multiplying the roots by m ,

$$y^3 - \frac{7m}{3}y^2 + \frac{11m^2}{12}y - \frac{25m^3}{24} = 0.$$

Substituting 6 for m , $y^3 - 14y^2 + 33y - 225 = 0$,
in which $y = mx = 6x$.

$$9. \quad x^3 - \frac{1}{2}x^2 + \frac{5}{12}x - \frac{7}{24} = 0.$$

Multiplying the roots by m ,

$$y^3 - \frac{m}{2}y^2 + \frac{5m^2}{12}y - \frac{7m^3}{24} = 0.$$

Substituting 6 for m , $y^3 - 3y^2 + 15y - 28 = 0$,
in which $y = mx = 6x$.

$$10. \quad (x-1)^3(3x-1)^2 = 50.$$

Substituting y for $3x$, or $\frac{y}{3}$ for x ,

$$\left(\frac{y}{3} - 1\right)^3(y-1)^2 = \frac{1}{27}(y-3)^3(y-1)^2 = 50,$$

in which $y = 3x$.

$$\text{Clearing of fractions,} \quad (y-3)^3(y-1)^2 = 1350.$$

$$\text{Reducing,} \quad y^5 - 11y^4 + 46y^3 - 90y^2 + 81y - 1377 = 0.$$

$$11. \quad 5x^4 + .1x^2 + .125x + .2 = 0.$$

$$\text{Dividing by 5,} \quad x^4 + .02x^2 + .025x + .04 = 0.$$

Multiplying the roots by 10, $y^4 + 2y^2 + 25y + 400 = 0$,
in which $y = 10x$.

12. The three cube roots of 1 are the roots of

$$x^3 = 1, \text{ or of } x^3 + 0x^2 + 0x - 1 = 0.$$

Multiplying the roots of this equation by a gives

$$y^3 + 0y^2 + 0y - a^3 = 0, \text{ or } y^3 = a^3,$$

whose roots are the three cube roots of a^3 .

Hence, the three cube roots of a^3 are equal to a times the corresponding cube roots of 1, or to $a, \frac{1}{2}a(-1 \pm \sqrt{-3})$.

Putting 8, 27, -8, and -1 successively for a^3 , the cube roots of these numbers are found to be

| | | | |
|-------------------|-----|---------------------------------|---------------------------------|
| cube roots of 8, | 2, | $-1 + \sqrt{-3}$, | $-1 - \sqrt{-3}$; |
| cube roots of 27, | 3, | $\frac{3}{2}(-1 + \sqrt{-3})$, | $\frac{3}{2}(-1 - \sqrt{-3})$; |
| cube roots of -8, | -2, | $1 - \sqrt{-3}$, | $1 + \sqrt{-3}$; |
| cube roots of -1, | -1, | $\frac{1}{2}(1 - \sqrt{-3})$, | $\frac{1}{2}(1 + \sqrt{-3})$. |

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| | | | | | |
|----|---|-----|-----|-----|---|
| 2. | 1 | -12 | 47 | -60 | 3 |
| | | 3 | -27 | 60 | |
| | | -9 | 20 | 0 | |
| | | 3 | -18 | | |
| | | -6 | 2 | | |
| | | 3 | | | |
| | | -3 | | | |

Hence, the transformed equation, in which $y = x - 3$, is
 $y^3 - 3y^2 + 2y = 0$, or $y(y-1)(y-2) = 0$.

To test this result compare the roots of this equation with those of the given equation. The above process shows that 3 is one root of the given equation and that the other two roots are the roots of $x^2 - 9x + 20 = 0$, which are 4 and 5. The roots of the transformed equation are easily seen to be 0, 1, and 2, each 3 less than the corresponding root of the given equation.

$$\begin{array}{r}
 3. \qquad 1 \qquad -4 \quad -19 \quad 46 \quad 120 \quad | \quad 3 \\
 \qquad \qquad 3 \quad -3 \quad -66 \quad -60 \\
 \hline
 \qquad -1 \quad -22 \quad -20 \quad | \quad 60 \\
 \qquad 3 \quad 6 \quad -48 \\
 \hline
 \qquad 2 \quad -16 \quad | \quad -68 \\
 \qquad 3 \quad 15 \\
 \hline
 \qquad 5 \quad | \quad -1 \\
 \qquad 3 \\
 \hline
 \qquad 8
 \end{array}$$

Hence, the transformed equation, in which $y = x - 3$, is
 $y^4 + 8y^3 - y^2 - 68y + 60 = 0$.

$$\begin{array}{r}
 4. \qquad 3 \qquad -19 \quad 21 \quad 31 \quad 12 \quad | \quad 3 \\
 \qquad \qquad 9 \quad -30 \quad -27 \quad 12 \\
 \hline
 \qquad -10 \quad -9 \quad 4 \quad | \quad 24 \\
 \qquad 9 \quad -3 \quad -36 \\
 \hline
 \qquad -1 \quad -12 \quad | \quad -32 \\
 \qquad 9 \quad 24 \\
 \hline
 \qquad 8 \quad | \quad 12 \\
 \qquad 9 \\
 \hline
 \qquad 17
 \end{array}$$

Hence, the transformed equation, in which $y = x - 3$, is
 $3y^4 + 17y^3 + 12y^2 - 32y + 24 = 0$.

$$\begin{array}{r}
 5. \qquad 1 \qquad 0 \quad -2.75 \quad .5 \quad 4.5 \quad | \quad 3 \\
 \qquad \qquad 3 \quad 9 \quad 18.75 \quad 57.75 \\
 \hline
 \qquad 3 \quad 6.25 \quad 19.25 \quad | \quad 62.25 \\
 \qquad 3 \quad 18 \quad 72.75 \\
 \hline
 \qquad 6 \quad 24.25 \quad | \quad 92 \\
 \qquad 3 \quad 27 \\
 \hline
 \qquad 9 \quad | \quad 51.25 \\
 \qquad 3 \\
 \hline
 \qquad 12
 \end{array}$$

Hence, the transformed equation, in which $y = x - 3$, is
 $y^4 + 12y^3 + 51.25y^2 + 92y + 62.25 = 0$.

$$\begin{array}{r}
 1. \qquad 1 \qquad 9 \quad 29 \quad 39 \quad 18 \quad | \quad -5 \\
 \qquad \qquad -5 \quad -20 \quad -45 \quad 30 \\
 \hline
 \qquad 4 \quad 9 \quad -6 \quad | \quad 48 \\
 \qquad -5 \quad 5 \quad -70 \\
 \hline
 \qquad -1 \quad 14 \quad | \quad -76 \\
 \qquad -5 \quad 30 \\
 \hline
 \qquad -6 \quad | \quad 44 \\
 \qquad -5 \\
 \hline
 \qquad -11
 \end{array}$$

Hence, the transformed equation, in which $y = x + 5$, is
 $y^4 - 11y^3 + 44y^2 - 76y + 48 = 0$.

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$$\begin{array}{r}
 2. \qquad 1 \quad 20 \quad 131 \quad 280 \quad \underline{-5} \\
 \quad \quad -5 \quad -75 \quad -280 \\
 \hline
 \quad \quad 15 \quad 56 \quad 0 \\
 \quad \quad -5 \quad -50 \\
 \hline
 \quad \quad 10 \quad 6 \\
 \quad \quad -5 \\
 \hline
 \quad \quad 5
 \end{array}$$

Hence, the transformed equation, in which $y = x + 5$, is
 $y^3 + 5y^2 + 6y = 0$.

$$\begin{array}{r}
 3. \qquad 1 \quad 6 \quad 10 \quad 9 \quad 4 \quad \underline{-5} \\
 \quad \quad -5 \quad -5 \quad -25 \quad 80 \\
 \hline
 \quad \quad 1 \quad 5 \quad -16 \quad 84 \\
 \quad \quad -5 \quad 20 \quad -100 \\
 \hline
 \quad \quad -4 \quad 25 \quad -116 \\
 \quad \quad -5 \quad 45 \\
 \hline
 \quad \quad -9 \quad 70 \\
 \quad \quad -5 \\
 \hline
 \quad \quad -14
 \end{array}$$

Hence, the transformed equation, in which $y = x + 5$, is
 $y^4 - 14y^3 + 70y^2 - 116y + 84 = 0$.

$$\begin{array}{r}
 4. \qquad 1 \quad 15 \quad 71 \quad 105 \quad \underline{-5} \\
 \quad \quad -5 \quad -50 \quad -105 \\
 \hline
 \quad \quad 10 \quad 21 \quad 0 \\
 \quad \quad -5 \quad -25 \\
 \hline
 \quad \quad 5 \quad -4 \\
 \quad \quad -5 \\
 \hline
 \quad \quad 0
 \end{array}$$

Hence, the transformed equation, in which $y = x + 5$, is
 $y^3 - 4y = 0$.

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3. Since the equation $x^3 - 12x^2 + 43x - 40 = 0$ is of the third degree, and the coefficient of the second term is -12 , decrease the roots by $\frac{12}{3}$, or by 4.

$$\begin{array}{r}
 1 \quad -12 \quad 43 \quad -40 \quad \underline{4} \\
 \quad 4 \quad -32 \quad 44 \\
 \hline
 \quad -8 \quad 11 \quad 4 \\
 \quad 4 \quad -16 \\
 \hline
 \quad -4 \quad -5 \\
 \quad 4 \\
 \hline
 \quad 0
 \end{array}$$

Hence,
 in which

$$\begin{array}{l}
 y^3 - 5y + 4 = 0, \\
 y = x - 4.
 \end{array}$$

4. Since the equation $x^3 - 3x^2 - 88x - 240 = 0$ is of the third degree, and the coefficient of the second term is -3 , decrease the roots by $\frac{2}{3}$, or by 1.

$$\begin{array}{r}
 1 \quad -3 \quad -88 \quad -240 \quad \underline{1} \\
 \quad 1 \quad -2 \quad -90 \\
 \hline
 -2 \quad -90 \quad -330 \\
 \quad 1 \quad -1 \\
 \hline
 -1 \quad -91 \\
 \quad 1 \\
 \hline
 0
 \end{array}$$

Hence,
in which

$$y^3 - 91y - 330 = 0, \\ y = x - 1.$$

5. Since the equation $x^3 + 15x^2 + 68x + 96 = 0$ is of the third degree, and the coefficient of the second term is 15, decrease the roots by $-\frac{15}{3}$, or by -5 .

$$\begin{array}{r}
 1 \quad 15 \quad 68 \quad 96 \quad \underline{-5} \\
 -5 \quad -50 \quad -90 \\
 \hline
 10 \quad 18 \quad 6 \\
 -5 \quad -25 \\
 \hline
 5 \quad -7 \\
 -5 \\
 \hline
 0
 \end{array}$$

Hence,
in which

$$y^3 - 7y + 6 = 0, \\ y = x - (-5) = x + 5.$$

6. Multiplying the roots by 3, $x^3 + x^2 - 2 = 0$,
in which $y^3 + 3y^2 - 54 = 0$, $y = 3x$.

Since the transformed equation is of the third degree, and the coefficient of the second term is 3, decrease the roots by $-\frac{3}{3}$, or by -1 .

$$\begin{array}{r}
 1 \quad 3 \quad 0 \quad -54 \quad \underline{-1} \\
 -1 \quad -2 \quad 2 \\
 \hline
 2 \quad -2 \quad -52 \\
 -1 \quad -1 \\
 \hline
 1 \quad -3 \\
 -1 \\
 \hline
 0
 \end{array}$$

Hence,
in which

$$z^3 - 3z - 52 = 0, \\ z = y - (-1) = y + 1 = 3x + 1.$$

7. Multiplying the roots by 3, $x^3 + x^2 + x + 2 = 0$,
in which $y^3 + 3y^2 + 9y + 54 = 0$, $y = 3x$.

Since the transformed equation is of the third degree, and the coefficient of the second term is 3, decrease the roots by $-\frac{3}{3}$, or by -1 .

$$\begin{array}{r}
 1 \quad 3 \quad 9 \quad 54 \quad \underline{-1} \\
 -1 \quad -2 \quad -7 \\
 \hline
 2 \quad 7 \quad 47 \\
 -1 \quad -1 \\
 \hline
 1 \quad 6 \\
 -1 \\
 \hline
 0
 \end{array}$$

Hence,
in which

$$z^3 + 6z + 47 = 0, \\ z = y - (-1) = y + 1 = 3x + 1.$$

8. $2x^3 - x^2 - 2x + 1 = 0.$

Dividing both members by 2 and multiplying the roots by 6,

$$y^3 - 3y^2 - 36y + 108 = 0,$$

in which $y = 6x.$

Since the transformed equation is of the third degree and the coefficient of the second term is -3 , decrease the roots by $\frac{3}{3}$, or by 1.

$$\begin{array}{rrrr} 1 & -3 & -36 & 108 & \underline{1} \\ & 1 & -2 & -38 & \\ \hline & -2 & -38 & 70 & \\ & 1 & -1 & & \\ \hline & -1 & & -39 & \\ & 1 & & & \\ \hline & 0 & & & \end{array}$$

Hence,
in which

$$z^3 - 39z + 70 = 0, \quad z = y - 1 = 6x - 1.$$

9. Since the equation $x^4 + 4x^3 - 7x^2 - 22x + 24 = 0$ is of the fourth degree and the coefficient of the second term is 4, decrease the roots by $-\frac{4}{4}$, or by -1 .

$$\begin{array}{rrrr} 1 & 4 & -7 & -22 & 24 & \underline{-1} \\ & -1 & -3 & 10 & 12 & \\ \hline & 3 & -10 & -12 & 36 & \\ & -1 & -2 & 12 & & \\ \hline & 2 & -12 & 0 & & \\ & -1 & -1 & & & \\ \hline & 1 & & -13 & & \\ & -1 & & & & \\ \hline & 0 & & & & \end{array}$$

Hence,
in which

$$y^4 - 13y^2 + 36 = 0, \quad y = x - (-1) = x + 1.$$

The transformed equation may be written

$$(y^2 - 4)(y^2 - 9) = (y - 2)(y + 2)(y - 3)(y + 3) = 0.$$

$$\therefore y = 2, -2, 3, -3.$$

$$\text{Since } x = y - 1, \quad x = 1, -3, 2, -4.$$

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1. The signs of $f(x)$ are $++++$.

Hence, the equation has no positive roots.

The signs of $f(-x)$ are $-+-+$.

Hence, the equation has four negative roots or less than four.

Since the equation has four roots, and the imaginary roots, if any, enter in pairs, all of the roots are negative, or two are negative and two are imaginary, or all are imaginary.

2. The signs of $f(x)$ are $+-+-$.

Hence, the equation has two positive roots or less than two.

The signs of $f(-x)$ are $++++$.

Hence, the equation has no negative roots.

Since the equation has four roots and the imaginary roots enter in pairs, two of the roots are positive and two are imaginary or all are imaginary.

3. The signs of $f(x)$ are $+- --$.

Hence, the equation has one positive root or less than one.

The signs of $f(-x)$ are $+- +-$.

Hence, the equation has three negative roots or less than three.

Since the equation has four roots and the imaginary roots, if any, enter in pairs, one root is positive and three are negative; or one is positive, one is negative, and two are imaginary; or two are negative and two are imaginary; or all are imaginary.

4. Removing the zero root, the depressed equation is

$$x^5 + x^3 + x + 1 = 0, \text{ say } f(x) = 0.$$

The signs of $f(x)$ are $++++$ and the signs of $-f(-x)$ are $++++$.

Hence, the equation $f(x) = 0$ has no positive roots and not more than one negative root; that is, the equation has not more than one real root.

Since the equation $f(x) = 0$ has five roots, and the imaginary roots enter in pairs, and since there is not more than one real root, the equation has one negative root and four imaginary roots.

Hence, the given equation has one zero root, one negative root, and four imaginary roots.

5. The signs of $f(x)$ are $+- +$.

Hence, there are two positive roots or less than two.

The signs of $f(-x)$ are $+++$.

Hence, there are no negative roots.

Since the equation has four roots and the imaginary roots enter in pairs, two of the roots are positive and two are imaginary, or all the roots are imaginary.

6. The signs of $f(x)$ are $+- -$ and the signs of $-f(-x)$ are $+- +$.

Hence, the equation has not more than one positive root and not more than two negative roots.

Since the equation has three roots and the imaginary roots, if any, enter in pairs, there are three real roots, one positive and two negative, or else only one real root and two imaginary roots.

In the latter case, to find the sign of the real root suppose that $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ represent the imaginary roots.

The product of these imaginary roots with their signs changed is $(-a - b\sqrt{-1})(-a + b\sqrt{-1})$, or $a^2 + b^2$, a *positive* number.

Since the product of the imaginary roots with their signs changed is positive, and the product of all the roots with their signs changed is the absolute term -2 , a negative number, the third or real root with its sign changed is negative. Hence, the real root is positive.

7. The signs of $f(x)$ are $+-$ and the signs of $f(-x)$ are $--$.

Hence, the equation has at most one positive root, no negative roots, and at least four imaginary roots.

Since the equation has five roots, not more than one of which can be real, and since the imaginary roots enter in pairs, there cannot be more than two pairs of them. Therefore the remaining root is real and positive.

Hence, the equation has one positive and four imaginary roots.

8. The signs of $f(x)$ are $++$ and the signs of $f(-x)$ are $-+$.

Hence, the equation has no positive roots, at most one negative root, and at least four imaginary roots.

Since the equation has five roots, not more than one of which can be real, and since the imaginary roots enter in pairs, there cannot be more than two pairs of them. Therefore the remaining root is real and negative.

Hence, the equation has one negative and four imaginary roots.

9. If the terms of $f(x)$ are all positive, $f(x)=0$ has no positive roots. If, in addition, each term of $f(x)$ involves an even power of x , the terms of $f(-x)$ likewise are all positive, and $f(x)=0$ has no negative root.

Hence, in this case all the roots of $f(x)=0$ are imaginary.

10. The signs of $f(x)$ are $+ -$.

If n is even, the signs of $f(-x)$ are $+ -$.

If n is odd, the signs of $f(-x)$ are $- -$.

Hence, by Descartes' rule, if n is even, $x^n - 1 = 0$ has one positive, one negative, and $n - 2$ imaginary roots; or else all the roots are imaginary.

But all the roots cannot be imaginary when n is even. For each quadratic factor corresponding to a pair of conjugate imaginary roots has the form

$$(x - a - b\sqrt{-1})(x - a + b\sqrt{-1}), \text{ or } x^2 - 2ax + a^2 + b^2,$$

in which the absolute term ($a^2 + b^2$) is essentially positive whatever the signs of a and b . Therefore, the product of all the factors corresponding to the imaginary roots has a *positive* absolute term, while the absolute term of $x^n - 1$ is negative.

Since all the roots of $x^n - 1$ cannot be imaginary, and since there cannot be more than one positive and one negative root, if n is even the equation $x^n - 1 = 0$ has two real roots, one positive and one negative.

Again, by Descartes' rule, if n is odd there are no negative roots, and not more than one positive root. But since the imaginary roots enter in pairs, there is at least one real root.

Hence, if n is odd, $x^n - 1$ has only one real root, and this root is positive.

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2. Let $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$.

Then, $f'(x) = 4x^3 - 3x^2 - 6x + 5$.

The H. C. D. of $f(x)$ and $f'(x)$ is found to be $\phi(x) = x^2 - 2x + 1$.

Put $x^2 - 2x + 1 = 0$.

Solving, $x = 1, 1$.

Therefore, 1 is a double root of $f'(x) = 0$.

Hence, Prin. 7, 1 is a triple root of $f(x) = 0$.

Removing the corresponding factor $(x-1)^3$ from $f(x)$, the other factor is found to be $x + 2$, which gives the root -2 .

Hence, the roots of the given equation are 1, 1, 1, -2 .

3. Let $f(x) = x^5 + 6x^4 + 14x^3 + 20x^2 + 24x + 16$.

Then, $f'(x) = 5x^4 + 24x^3 + 42x^2 + 40x + 24$.

The H. C. D. of $f(x)$ and $f'(x)$ is found to be $\phi(x) = x^2 + 4x + 4$.

Put $x^2 + 4x + 4 = 0$.

Solving, $x = -2, -2$.

Therefore, -2 is a double root of $f'(x) = 0$.

Hence, Prin. 7, -2 is a triple root of $f(x) = 0$.

Removing the corresponding factor $(x+2)^3$ from $f(x)$, the equation is depressed to the quadratic equation $x^2 + 2 = 0$, whose roots are $\pm\sqrt{-2}$.

Hence, the roots of the given equation are $-2, -2, -2, \pm\sqrt{-2}$.

4. Let $f(x) = x^5 - 7x^4 + 14x^3 - 2x^2 - 15x + 9$.

Then, $f'(x) = 5x^4 - 28x^3 + 42x^2 - 4x - 15$.

The H. C. D. of $f(x)$ and $f'(x)$ is found to be $\phi(x) = x^2 - 4x + 3$.

Put $x^2 - 4x + 3 = 0$.

Solving, $x = 1, 3$.

Therefore, 1 and 3 are roots of $f'(x) = 0$.

Hence, Prin. 7, 1 and 3 are double roots of $f(x) = 0$.

Removing the corresponding factors $(x-1)^2(x-3)^2$ from $f(x)$, the

remaining factor is found to be $x + 1$, which gives the root -1 . Hence, the roots of the given equation are 1, 1, 3, 3, -1 .

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$$\begin{array}{ll} 1. \text{ Given} & x^4 + x^3 + 7x^2 - 8x - 3 = 0, \\ \text{whence} & x^4 - x^3 + 7x^2 + 8x - 3 = 0 \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

is the equation formed by changing the signs of the roots.

In (1), $N = 8$ and $r = 3$. $\therefore \sqrt[3]{N} + 1 = 3$, a *superior* limit to the positive roots of the given equation.

In (2), $N = 3$ and $r = 1$. $\therefore \sqrt[3]{N} + 1 = 4$, a *superior* limit to the positive roots of (2), whence -4 is an *inferior* limit to the negative roots of the given equation.

$$\begin{array}{ll} 2. \text{ Given} & x^5 - 3x^3 - 6x^2 - 14x - 8 = 0, \\ \text{whence} & x^5 - 3x^3 + 6x^2 - 14x + 8 = 0 \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

is the equation formed by changing the signs of the roots.

In (1), $N = 14$ and $r = 2$. $\therefore \sqrt[2]{N} + 1 = \sqrt{14} + 1$, which is > 4 and < 5 . Hence, 5 is a *superior* limit to the positive roots of the given equation.

In (2), $N = 14$ and $r = 2$. $\therefore \sqrt[2]{N} + 1 = \sqrt{14} + 1$, which is > 4 and < 5 . Hence, 5 is a *superior* limit to the positive roots of (2), whence -5 is an *inferior* limit to the negative roots of the given equation.

$$\begin{array}{ll} 3. \text{ Given} & x^6 - 28x^3 - 49x^2 + 112x + 132 = 0, \\ \text{whence} & x^6 + 28x^3 - 49x^2 - 112x + 132 = 0 \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

is the equation formed by changing the signs of the roots.

In (1), $N = 49$ and $r = 3$. $\therefore \sqrt[3]{N} + 1 = \sqrt[3]{49} + 1$, which is > 4 and < 5 . Hence, 5 is a *superior* limit to the positive roots of the given equation.

In (2), $N = 112$ and $r = 4$. $\therefore \sqrt[4]{N} + 1 = \sqrt[4]{112} + 1$, which is > 4 and < 5 . Hence, 5 is a *superior* limit to the positive roots of (2), whence -5 is an *inferior* limit to the negative roots of the given equation.

$$\begin{array}{ll} 4. \text{ Given} & x^6 + x^3 - 4x^2 + 6x - 20 = 0, \\ \text{whence} & x^6 - x^3 - 4x^2 - 6x - 20 = 0 \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

is the equation formed by changing the signs of the roots.

In (1), $N = 20$ and $r = 4$. $\therefore \sqrt[4]{N} + 1 = \sqrt[4]{20} + 1$, which is > 3 and < 4 . Hence, 4 is a *superior* limit to the positive roots of the given equation.

In (2), $N = 20$ and $r = 3$. $\therefore \sqrt[3]{N} + 1 = \sqrt[3]{20} + 1$, which is > 3 and < 4 . Hence, 4 is a *superior* limit to the positive roots of (2), whence -4 is an *inferior* limit to the negative roots of the given equation.

$$\begin{array}{ll} 5. \text{ Given} & x^7 - x^5 + 3x^4 + 76x^3 - 243x^2 + 108 = 0, \\ \text{whence} & x^7 - x^5 - 3x^4 + 76x^3 + 243x^2 - 108 = 0 \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

is the equation formed by changing the signs of the roots.

In (1), $N = 243$ and $r = 2$. $\therefore \sqrt[2]{N} + 1 = \sqrt{243} + 1$, which is > 16 and < 17 . Hence, 17 is a *superior* limit to the positive roots of the given equation.

In (2), $N = 108$ and $r = 2$. $\therefore \sqrt[2]{N} + 1 = \sqrt{108} + 1$, which is > 11 and < 12 . Hence, 12 is a *superior* limit to the positive roots of (2), whence -12 is an *inferior* limit to the negative roots of the given equation.

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$$2. \text{ Let } f(x) = 2x^3 - 15x^2 + 32x - 21.$$

$$\text{Then, } \frac{1}{2}f'(x) = 3x^2 - 15x + 16.$$

NOTE.—It is legitimate to remove any positive numerical factor from $f'(x)$, or from any of Sturm's functions; for removing a *positive* factor does not affect the *sign* of any of the functions for any value of x .

The following table shows the signs of Sturm's functions when x takes various values.

| Sturm's functions | $-\infty$ | 0 | ∞ | 0 | 1 | 2 | 0 | -1 | -2 | -3 |
|---------------------------|-----------|---|----------|---|---|---|---|----|----|----|
| $16x^3 - 4x^2 - 80x + 75$ | - | + | + | + | + | + | + | + | + | - |
| $6x^2 - x - 10$ | + | - | + | - | - | + | - | - | + | + |
| $482x - 655$ | - | - | + | - | - | + | - | - | - | - |
| + | + | + | + | + | + | + | + | + | + | + |
| Variations | 3 | 2 | 0 | 2 | 2 | 0 | 2 | 2 | 2 | 3 |

Since Sturm's functions lose one variation when x increases from $-\infty$ to 0 and two variations when x increases from 0 to ∞ , the equation has one negative root and two positive roots.

Since the equation has only three roots and all of them are real, there are no imaginary roots.

To locate the positive roots between successive integers, substitute 0, 1, 2, ... successively for x in Sturm's functions until two variations are lost. Since two variations are lost when x increases from 1 to 2, both positive roots lie between 1 and 2.

To locate the negative root between two successive integers, substitute 0, -1, -2, ... until one variation is gained. Since one variation is gained when x decreases from -2 to -3, or lost when x increases from -3 to -2, the negative root lies between -2 and -3.

4. Let

$$f(x) = x^4 - 2x^3 - 7x^2 + 10x + 10.$$

Then,

$$\frac{1}{2}f'(x) = 2x^3 - 3x^2 - 7x + 5. \quad (\text{See Note, Ex. 2.})$$

| | | | |
|----|-------------------------------------|---|------|
| | $2 - 3 - 7 + 5$ | $1 - 2 - 7 + 10 + 10$ | |
| | <u>17</u> | <u>2</u> | |
| | | $2 - 4 - 14 + 20 + 20$ | |
| | | <u>$2 - 3 - 7 + 5$</u> | 1 |
| | | $-1 - 7 + 15 + 20$ | |
| | | <u>2</u> | |
| | | $-2 - 14 + 30 + 40$ | |
| | | <u>$-2 + 3 + 7 - 5$</u> | -1 |
| | | $-17 + 23 + 45$ | |
| 2 | $34 - 51 - 119 + 85$ | $\therefore F_2(x) = 17x^2 - 23x - 45$ | |
| | <u>$34 - 46 - 90$</u> | <u>152</u> | |
| | $-5 - 29 + 85$ | | |
| | <u>17</u> | | |
| | $-85 - 493 + 1445$ | | |
| -5 | <u>$-85 + 115 + 225$</u> | | |
| | $4) -608 + 1220$ | | |
| | <u>$-152 + 305$</u> | | |
| | $\therefore F_3(x) = 152x - 305$ | $17 \cdot 152 - 3496 - 6840$ | |
| | | <u>$17 \cdot 152 - 5185$</u> | 17 |
| | | $1689 - 6840$ | |
| | | <u>152</u> | |
| | | $152 \cdot 1689 - 1039680$ | |
| | | <u>$152 \cdot 1689 - 515145$</u> | 1689 |
| | | -524535 | |
| | | $\therefore F_4(x) = +$ | |

The following table shows the signs of Sturm's functions when x takes various values.

| Sturm's functions | $-\infty$ | 0 | ∞ | 0 | 1 | 2 | 3 | 0 | -1 | -2 | -3 |
|--------------------------------|-----------|---|----------|---|---|---|---|---|----|----|----|
| $x^4 - 2x^3 - 7x^2 + 10x + 10$ | + | + | + | + | + | + | + | + | - | - | + |
| $2x^3 - 3x^2 - 7x + 5$ | - | + | + | + | - | - | + | + | + | - | - |
| $17x^2 - 23x - 45$ | + | - | + | - | - | - | + | - | - | + | + |
| $152x - 305$ | - | - | + | - | - | - | + | - | - | - | - |
| + | + | + | + | + | + | + | + | + | + | + | + |
| Variations | 4 | 2 | 0 | 2 | 2 | 2 | 0 | 2 | 3 | 3 | 4 |

Since Sturm's functions lose two variations when x increases from $-\infty$ to 0, and two variations when x increases from 0 to ∞ , the equation has two negative roots, two positive roots, and no imaginary roots.

To locate the positive roots between successive integers, substitute 0, 1, 2, ... successively for x in Sturm's functions until two variations are lost. Since two variations are lost when x increases from 2 to 3, both positive roots lie between 2 and 3.

To locate the negative roots between successive integers, substitute 0, -1, -2, ... successively for x in Sturm's functions until two variations are gained. Since one variation is gained when x decreases from 0 to -1, or lost when x increases from -1 to 0, one negative root lies between 0 and -1. For a like reason, the other negative root lies between -2 and -3.

5. Let

$$f(x) = x^4 - x^2 - 10x - 4.$$

Then,

$$\frac{1}{2}f'(x) = 2x^3 - x - 5.$$

(See Note, Ex. 2.)

| | | | | | |
|-----|-----------------------------------|----|----|-------------------------------------|-------|
| | 2 + 0 | -1 | -5 | 1 + 0 - 1 - 10 - 4 | |
| | | | | 2 | |
| | | | | 2 + 0 - 2 - 20 - 8 | |
| | | | | 2 + 0 - 1 - 5 | 1 |
| | | | | -1 - 15 - 8 | |
| 2 | 2 + 30 + 16 | | | $\therefore F_2(x) = x^2 + 15x + 8$ | |
| | -30 - 17 - 5 | | | 433 | |
| -30 | -30 - 450 - 240 | | | | |
| | 433 + 235 | | | | |
| | $\therefore F_3(x) = -433x - 235$ | | | 433 + 6495 + 3464 | |
| | | | | 433 + 235 | |
| | | | | 6260 + 3464 | |
| | | | | 433 | |
| | | | | 433 · 6260 + 1499912 | |
| | | | | 433 · 6260 + 1471100 | -6260 |
| | | | | + 28812 | |
| | | | | $\therefore F_4(x) = -$ | |

The following table shows the signs of Sturm's functions when x takes various values.

| Sturm's functions | $-\infty$ | 0 | ∞ | 0 | 1 | 2 | 3 | 0 | -1 |
|-----------------------|-----------|---|----------|---|---|---|---|---|----|
| $x^4 - x^2 - 10x - 4$ | + | - | + | - | - | - | + | - | + |
| $2x^3 - x - 5$ | - | - | + | - | - | + | + | - | - |
| $x^2 + 15x + 8$ | + | + | + | + | + | + | + | + | - |
| $-433x - 235$ | + | - | - | - | - | - | - | - | + |
| - | - | - | - | - | - | - | - | - | - |
| Variations | 3 | 2 | 1 | 2 | 2 | 2 | 1 | 2 | 3 |

Since Sturm's functions lose one variation when x increases from $-\infty$ to 0 and another when x increases from 0 to ∞ , the equation has one negative root and one positive root. The other two roots, then, must be imaginary.

To locate the positive root between two successive integers, substitute 0, 1, 2, ... successively for x in Sturm's functions until one variation is lost. Since one variation is lost when x increases from 2 to 3, the positive root lies between 2 and 3.

To locate the negative root between two successive integers, substitute 0, -1, -2, ... successively for x in Sturm's functions until one variation is gained. Since one variation is gained when x decreases from 0 to -1, or lost when x increases from -1 to 0, the negative root lies between 0 and -1.

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| | | | | | |
|----|---|------|--------|-----------|-------------|
| 1. | 4 | - 3 | 1 | - 14 | <u>1.75</u> |
| | | 4 | 1 | 2 | |
| | | 1 | 2 | - 12000 | |
| | | 4 | 5 | 10682 | |
| | | 5 | 700 | - 1318000 | |
| | | 4 | 826 | 1318000 | |
| | | 90 | 1526 | 0 | |
| | | 28 | 1022 | | |
| | | 118 | 254800 | | |
| | | 28 | 8800 | | |
| | | 146 | 263600 | | |
| | | 28 | | | |
| | | 1740 | | | |
| | | 20 | | | |
| | | 1760 | | | |

Hence, the root is 1.75.

Next, to test the result, multiply the roots by 4 and try 4 times 1.75, or 7, for a root of the transformed equation.

$$\text{Given equation, } 4x^3 - 3x^2 + x - 14 = 0. \quad (1)$$

$$\text{Transformed equation, in which } y = 4x,$$

$$4y^3 - 12y^2 + 16y - 896 = 0; \quad (2)$$

$$\text{or, dividing by 4, } y^3 - 3y^2 + 4y - 224 = 0.$$

$$\text{Substitute 7 for } y \text{ in (2).} \quad (2)$$

$$\begin{array}{r} 1 - 3 + 4 - 224 \quad | 7 \\ 7 + 28 + 224 \\ \hline 1 + 4 + 32 \end{array}$$

Hence, 7 is a root of the transformed equation (2), and consequently $\frac{7}{4}$, or 1.75, is a root of the given equation (1).

The other two roots of the transformed equation (2) are the roots of

$$y^2 + 4y + 32 = 0.$$

$$\text{Solving, } y = -2 \pm 2\sqrt{-7}.$$

Therefore, since $y = 4x$, or $x = \frac{1}{4}y$, the imaginary roots of the given equation are

$$x = \frac{1}{4}(-1 \pm \sqrt{-7}).$$

NOTE. — It is not necessary to multiply the roots by 4 in order to find the imaginary roots. The imaginary roots, and also the real root 1.75, may be found by *completing* the last transformation in the process, giving $4 + 1800 + 272500 + 0 = 0$. The successive equations and the relations between their corresponding roots will be as follows:

Given equation, $4x^3 - 3x^2 + x - 14 = 0$. (1)

1st transformed eq., $4y^3 + 90y^2 + 700y - 12000 = 0$, (2)

in which $y = 10(x - 1)$.

2d transformed eq., $4z^3 + 1740z^2 + 254800z - 1318000 = 0$, (3)

in which $z = 10(y - 7)$.

3d transformed eq., $4v^3 + 1800v^2 + 272500v + 0 = 0$, (4)

in which $v = z - 5$.

Solving (4), $v = 0, -225 \pm 50\sqrt{-7}$. (5)

But $v = z - 5 = 10y - 75 = 100x - 175$.

$$\therefore x = \frac{v + 175}{100}$$

by (5), $= 1.75, \frac{1}{2}(-1 \pm \sqrt{-7})$.

| | | | | | |
|----|---|-------|----------|-------------|-------|
| 2. | 3 | - 29 | 29 | - 21 | 2.625 |
| | | 16 | - 26 | 6 | |
| | | - 13 | 3 | - 15000 | |
| | | 16 | 6 | 13968 | |
| | | 3 | 900 | - 1032000 | |
| | | 16 | 1428 | 822224 | |
| | | 190 | 2328 | - 209776000 | |
| | | 48 | 1716 | 209776000 | |
| | | 238 | 404400 | 0 | |
| | | 48 | 6712 | | |
| | | 286 | 411112 | | |
| | | 48 | 6744 | | |
| | | 3340 | 41785600 | | |
| | | 16 | 169600 | | |
| | | 3356 | 41955200 | | |
| | | 16 | | | |
| | | 3372 | | | |
| | | 16 | | | |
| | | 3380 | | | |
| | | 40 | | | |
| | | 33920 | | | |

Hence, the root is 2.625.

Since $2.625 = 2\frac{5}{8}$, to find the other roots multiply the roots by 8, remove the root $2\frac{5}{8} \times 8$, or 21, solve the depressed equation, and divide its roots by 8.

Multiplying the roots by 8 and dividing both members by 8,

$$y^3 - 29y^2 + 232y - 1344 = 0,$$

in which $y = 8x$.

Depressing the equation by removing the root 21, or the factor $y - 21$,

$$\begin{array}{r} 1 - 29 + 232 - 1344 \quad | \quad 21 \\ \underline{21 - 168 + 1344} \\ 1 - 8 + 64 \end{array}$$

$$y^2 - 8y + 64 = 0.$$

Solving,

$$y = 4 \pm 4\sqrt{-3}.$$

Since $y = 8x$,

$$x = \frac{1}{2}(1 \pm \sqrt{-3}).$$

Hence, the roots of the given equation are $2.625, \frac{1}{2}(1 \pm \sqrt{-3})$.

3. First find the less positive root.

| | | | | | |
|------|---------|------------|-------------|----------|-------|
| 1 | -1 | -4 | 3 | 3 | 1.618 |
| | 1 | 0 | -4 | -1 | |
| | 0 | -4 | -1 | 20000 | |
| | 1 | 1 | -3 | -19824 | |
| | 1 | -3 | -4000 | 1760000 | |
| | 1 | 2 | 696 | -1029859 | |
| 2 | -100 | -3304 | 7301410000 | | |
| 1 | 216 | 2208 | -7272372224 | | |
| 30 | 116 | -1096000 | 29037776 | | |
| 6 | 252 | 66141 | | | |
| 36 | 368 | -1029859 | | | |
| 6 | 288 | 66683 | | | |
| 42 | 65600 | -963176000 | | | |
| 6 | 541 | 54129472 | | | |
| 48 | 66141 | -909046528 | | | |
| 6 | 542 | | | | |
| 540 | 66683 | | | | |
| 1 | 543 | | | | |
| 541 | 6722600 | | | | |
| 1 | 43584 | | | | |
| 542 | 6766184 | | | | |
| 1 | | | | | |
| 543 | | | | | |
| 1 | | | | | |
| 5440 | | | | | |
| 8 | | | | | |
| 5448 | | | | | |

Next, find the greater positive root. Since this root also lies between 1 and 2, begin with the first transformed equation with its roots multiplied by 10. (See above process.)

| | | | | | |
|------|---------|-----------|-----------|-------------|-------|
| 1 | 30 | -100 | -4000 | 20000 | 1.732 |
| | 7 | 259 | 1113 | -20209 | |
| | 37 | 159 | -2887 | 2090000 | |
| | 7 | 308 | 3269 | 1903341 | |
| | 44 | 467 | 382000 | -1866590000 | |
| | 7 | 357 | 252447 | 1819452976 | |
| 51 | 82400 | 634447 | -47137024 | | |
| 7 | 1749 | 257721 | | | |
| 580 | 84149 | 892168000 | | | |
| 3 | 1758 | 17558488 | | | |
| 583 | 85907 | 909726488 | | | |
| 3 | 1767 | | | | |
| 586 | 8767400 | | | | |
| 3 | 11844 | | | | |
| 589 | 8779244 | | | | |
| 3 | | | | | |
| 5920 | | | | | |
| 2 | | | | | |
| 5922 | | | | | |

To find the negative roots of the given equation, change the signs of the roots and search for the corresponding positive roots of the transformed equation $y^4 + y^3 - 4y^2 - 3y + 3 = 0$. Since the smaller negative root of $f(x) = 0$ lies between 0 and

— 1, first multiply the roots of the transformed equation by 10 to avoid decimals.

| | | | | | |
|---|------|-------|--------------|---------------|-------------|
| 1 | 10 | — 400 | — 3000 | 30000 | <u>.618</u> |
| | 6 | 96 | — 1824 | — 28944 | |
| | 16 | — 304 | — 4824 | 10560000 | |
| | 6 | 132 | — 1032 | — 5856059 | |
| | 22 | — 172 | — 5856000 | — 47039410000 | |
| | 6 | 168 | — 59 | 46840436224 | |
| | 28 | — 400 | — 5856059 | — 198973776 | |
| | 6 | 341 | 283 | | |
| | 340 | — 59 | — 5855776000 | | |
| | 1 | 342 | 721472 | | |
| | 341 | 283 | — 5855054528 | | |
| | 1 | 343 | | | |
| | 342 | 62600 | | | |
| | 1 | 27584 | | | |
| | 343 | 90184 | | | |
| | 1 | | | | |
| | 3440 | | | | |
| | 8 | | | | |
| | 3448 | | | | |

Hence, the corresponding negative root of $f(x) = 0$ is $-.618$.

The larger negative root is found in a similar manner, thus :

| | | | | | |
|---|------|----------|-------------|---------------|--------------|
| 1 | 1 | — 4 | — 3 | 3 | <u>1.732</u> |
| | 1 | 2 | — 2 | — 5 | |
| | 2 | — 2 | — 5 | — 2000 | |
| | 1 | 3 | 1 | 16051 | |
| | 3 | 1 | — 4000 | — 39490000 | |
| | 1 | 4 | 6293 | 36846741 | |
| | 4 | 500 | 2293 | — 26432590000 | |
| | 1 | 399 | 9429 | 25775788976 | |
| | 50 | 899 | 11722000 | — 656801024 | |
| | 7 | 448 | 560247 | | |
| | 57 | 1347 | 12282247 | | |
| | 7 | 497 | 567321 | | |
| | 64 | 184400 | 12849568000 | | |
| | 7 | 2349 | 38326488 | | |
| | 71 | 186749 | 12887894488 | | |
| | 7 | 2358 | | | |
| | 780 | 189107 | | | |
| | 3 | 2367 | | | |
| | 783 | 19147400 | | | |
| | 3 | 15844 | | | |
| | 786 | 19163244 | | | |
| | 3 | | | | |
| | 789 | | | | |
| | 3 | | | | |
| | 7920 | | | | |
| | 2 | | | | |
| | 7922 | | | | |

Hence, the corresponding negative root of $f(x) = 0$ is -1.732 .

4. Trying the factors of 4 for roots, it is found that the equation has no commensurable roots.

By Descartes' rule, the equation has at most three positive roots and one negative root.

By §§ 688, 689, the real roots, if any, lie between the limits 7 and -4 .

To find the number and location of the real roots, use Sturm's Theorem.

16 Find the number and location of the real roots, use Sturm's theorem, of $f(x) = x^4 - 6x^3 + 5x^2 + 14x - 4$, and $f'(x) = 4x^3 - 18x^2 + 10x + 14$.

Instead of $f'(x)$, use $\frac{1}{2}f'(x) = 2x^3 - 9x^2 + 5x + 7$.

| | | | | | | | | | | | | | |
|-----|----|-------------------------|-------|------|-----|----------------------|----------------------------|-----|------|------|------|-----|---------------|
| | 2 | — | 9 + | 5 + | 7 | 1 | — | 6 + | 5 + | 14 — | 4 | | |
| | 17 | | | | | | 2 | | | | | | |
| | | | | | | | 2 | — | 12 + | 10 + | 28 — | 8 | |
| | | | | | | | 2 | — | 9 + | 5 + | 7 | | 1 |
| | | | | | | | | — | 3 + | 5 + | 21 — | 8 | |
| | | | | | | | 2 | | | | | | |
| | | | | | | | | — | 6 + | 10 + | 42 — | 16 | |
| | | | | | | | | — | 6 + | 27 — | 15 — | 21 | —3 |
| | | | | | | | | | | | | | — 17 + 57 + 5 |
| | 34 | — | 153 + | 85 + | 119 | ∴ | $F_2(x) = 17x^2 - 57x - 5$ | | | | | | |
| 2 | 34 | — | 114 — | 10 | | | 152 | | | | | | |
| | | — | 39 + | 95 + | 119 | | | | | | | | |
| | | | 17 | | | | | | | | | | |
| | | — 17 · 39 + 1615 + 2023 | | | | | | | | | | | |
| —39 | | — 17 · 39 + 2223 + 195 | | | | | | | | | | | |
| | | 4) — 608 + 1828 | | | | | | | | | | | |
| | | — 152 + 457 | | | | | | | | | | | |
| | | ∴ $F_3(x) = 152x - 457$ | | | | | | | | | | | |
| | | | | | | 152 · 17 — 8664 — | 760 | | | | | | |
| | | | | | | 152 · 17 — 7769 | | | | | | | 17 |
| | | | | | | — 895 — | 760 | | | | | | |
| | | | | | | 152 | | | | | | | |
| | | | | | | — 152 · 895 — 115520 | | | | | | | |
| | | | | | | — 152 · 895 + () | | | | | | 895 | |
| | | | | | | — | | | | | | | |
| | | | | | | ∴ $F_4(x) = +$ | | | | | | | |

| Sturm's functions | $-\infty$ | 0 | ∞ | 0 | 1 | 2 | 3 | 4 | 0 | -1 | -2 |
|-------------------------------|-----------|---|----------|---|---|---|---|---|---|----|----|
| $x^4 - 6x^3 + 5x^2 + 14x - 4$ | + | - | + | - | + | + | + | + | - | - | + |
| $2x^3 - 9x^2 + 5x + 7$ | - | + | + | + | + | - | - | + | + | - | + |
| $17x^2 - 57x - 5$ | + | - | + | - | - | - | - | + | - | + | - |
| $152x - 457$ | - | - | + | - | - | - | - | + | - | - | - |
| + | + | + | + | + | + | + | + | + | + | + | + |
| Variations | 4 | 3 | 0 | 3 | 2 | 2 | 2 | 0 | 3 | 3 | 4 |

Hence, there are three positive roots, one between 0 and 1, and two between 3 and 4; and one negative root, between -1 and -2.

To find the least positive root, which is a decimal, first multiply the roots of $f(x) = 0$ by 10.

| | | | | | |
|---|--------|---------|-------------|----------------|--------------|
| 1 | - 60 | 500 | 14000 | - 40000 ° | .268, nearly |
| | 2 | - 116 | 768 | 29536 | |
| | - 58 | 384 | 14768 | - 104640000 | |
| | 2 | - 112 | 544 | 92351376 | |
| | - 56 | 272 | 15312000 | - 122886240000 | |
| | 2 | - 108 | 79896 | 123671934976 | |
| | - 54 | 16400 | 15391896 | | |
| | 2 | - 3084 | 61608 | | |
| | - 520 | 13316 | 15453504000 | | |
| | 6 | - 3048 | 5487872 | | |
| | - 514 | 10268 | 15458991872 | | |
| | 6 | - 3012 | | | |
| | - 508 | 725600 | | | |
| | 6 | - 39616 | | | |
| | - 502 | 685984 | | | |
| | 6 | | | | |
| | - 4960 | | | | |
| | 8 | | | | |
| | - 4952 | | | | |

To find the other two positive roots between 3 and 4, decrease the roots of the given equation by 3.

| | | | | | |
|---|-----|-----|------|-----|---|
| 1 | - 6 | 5 | 14 | - 4 | 3 |
| | 3 | - 9 | - 12 | 6 | |
| | - 3 | - 4 | 2 | 2 | |
| | 3 | 0 | - 12 | | |
| | 0 | - 4 | - 10 | | |
| | 3 | 9 | | | |
| | 3 | 5 | | | |
| | 3 | | | | |
| | 6 | | | | |

By this transformation the equation has lost one variation, because the least positive root .618 has been passed. Substituting .1, .2, .3, ... in the transformed equation, it is found that .2 is the greatest number of tenths that can be substituted without causing the equation to lose another variation and that .7 is the greatest number of tenths

that can be substituted without causing the equation to lose two more variations. Hence, the roots between 3 and 4 are 3.2+ and 3.7+, respectively.

Multiply the roots of the above equation by 10 to avoid decimals.

| | | | | | |
|---|-----|-------|-----------|------------|-------|
| 1 | 60 | 500 | - 10000 | 20000 | 3.236 |
| | 2 | 124 | 1248 | - 17504 | |
| | 62 | 624 | - 8752 | 24960000 | |
| | 2 | 128 | 1504 | - 20929959 | |
| | 64 | 752 | - 7248000 | 4030041 | |
| | 2 | 132 | 271347 | - 3985278 | |
| | 66 | 88400 | - 6976653 | 44763 | |
| | 2 | 2049 | 277521 | | |
| | 680 | 90449 | - 6699132 | | |
| | 3 | 2058 | 5700 | | |
| | 683 | 92507 | - 664213 | | |
| | 3 | 2067 | | | |
| | 686 | 94574 | | | |
| | 3 | 4 | | | |
| | 689 | 950 | | | |
| | 3 | | | | |
| | 692 | | | | |

| | | | | | |
|---|----------------|--------|---------|------------|-------|
| 1 | 60 | 500 | - 10000 | 20000 | 3.732 |
| | 7 | 469 | 6783 | - 22519 | |
| | 67 | 969 | - 3217 | - 25190000 | |
| | 7 | 518 | 10409 | 23448441 | |
| | 74 | 1487 | 7192000 | - 1741559 | |
| | 7 | 567 | 624147 | 1698198 | |
| | 81 | 205400 | 7816147 | - 43361 | |
| | 7 | 2649 | 632121 | | |
| | 880 | 208049 | 8448268 | | |
| | 3 | 2658 | 4272 | | |
| | 883 | 210707 | 849099 | | |
| | 3 | 2667 | | | |
| | 886 | 213374 | | | |
| | 3 | 2 | | | |
| | 889 | 2136 | | | |
| | 3 | | | | |
| | 892 | | | | |

To find the negative root between -1 and -2 , change the signs of the roots of the given equation and search for the positive root, between 1 and 2 , of the equation $y^4 + 6y^3 + 5y^2 - 14y - 4 = 0$.

| | | | | | |
|---|-----------------|--------|----------|-------------|-------|
| 1 | 6 | 5 | - 14 | - 4 | 1.236 |
| | 1 | 7 | 12 | - 2 | |
| | 7 | 12 | - 2 | - 60000 | |
| | 1 | 8 | 20 | 48416 | |
| | 8 | 20 | 18000 | - 115840000 | |
| | 1 | 9 | 6208 | 95696841 | |
| | 9 | 2900 | 24208 | - 20143159 | |
| | 1 | 204 | 6624 | 19916010 | |
| | 100 | 3104 | 30832000 | - 227149 | |
| | 2 | 208 | 1066947 | | |
| | 102 | 3312 | 31898947 | | |
| | 2 | 212 | 1076721 | | |
| | 104 | 352400 | 32975668 | | |
| | 2 | 3249 | 21768 | | |
| | 106 | 355649 | 3319335 | | |
| | 2 | 3258 | | | |
| | 1080 | 358907 | | | |
| | 3 | 3267 | | | |
| | 1083 | 362174 | | | |
| | 3 | 6 | | | |
| | 1086 | 3628 | | | |
| | 3 | | | | |
| | 1089 | | | | |
| | 3 | | | | |
| | 1092 | | | | |

Hence, the negative root is -1.236 .

5. The fifth root of 330383.69407 is the real root of the equation $x^5 - 330383.69407 = 0$.

First divide the roots by 10 and then proceed in the usual manner.

| | | | | | | |
|---|----------|----------|----------|------------|-----------------|------|
| 1 | 0 | 0 | 0 | 0 | — 3.30383 69407 | 1.27 |
| | <u>1</u> | <u>1</u> | <u>1</u> | <u>1</u> | <u>1</u> | |
| | 1 | 1 | 1 | 1 | — 2 30383.69407 | |
| | 1 | 2 | 3 | 4 | 1 48832 | |
| | 2 | 3 | 4 | 50000 | — 81551 69407 | |
| | 1 | 3 | 6 | 24416 | 81551 69407 | |
| | 3 | 6 | 10000 | 74416 | | 0 |
| | 1 | 4 | 2208 | 29264 | | |
| | 4 | 1000 | 12208 | 1036800000 | | |
| | 1 | 104 | 2424 | 128224201 | | |
| | 50 | 1104 | 14632 | 1165024201 | | |
| | 2 | 108 | 2648 | | | |
| | 52 | 1212 | 17280000 | | | |
| | 2 | 112 | 1037743 | | | |
| | 54 | 1324 | 18317743 | | | |
| | 2 | 116 | | | | |
| | 56 | 144000 | | | | |
| | 2 | 4249 | | | | |
| | 58 | 148249 | | | | |
| | 2 | | | | | |
| | 600 | | | | | |
| | 7 | | | | | |
| | 607 | | | | | |

Since the roots were first divided by 10, the corresponding root of the given equation is 10 times 1.27, or 12.7. Hence, $\sqrt[5]{330383.69407} = 12.7$.

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4. $2x^5 - 15x^4 + 37x^3 - 37x^2 + 15x - 2 = 0.$

By grouping terms equally distant from the ends it is seen that the first member is divisible by $x - 1$. Depressing to the standard form,

$$\begin{array}{r} 2 - 15 + 37 - 37 + 15 - 2 \quad | 1 \\ \hline 2 - 13 + 24 - 13 + 2 \\ \hline 2 - 13 + 24 - 13 + 2 \end{array}$$

Grouping terms with like coefficients and dividing by x^2 ,

$$2 \left(x^2 + \frac{1}{x^2} \right) - 13 \left(x + \frac{1}{x} \right) + 24 = 0.$$

Substituting y for $x + \frac{1}{x}$ and $y^2 - 2$ for $x^2 + \frac{1}{x^2}$,

$$2y^2 - 4 - 13y + 24 = 0, \text{ or } 2y^2 - 13y + 20 = 0.$$

Factoring, $(2y - 5)(y - 4) = 0.$

$$\therefore y = \frac{5}{2} \text{ or } 4,$$

that is, $x + \frac{1}{x} = \frac{5}{2} \text{ or } 4.$

Solving, $x = 2, \frac{1}{2}, 2 + \sqrt{3}, 2 - \sqrt{3}.$

Hence, the roots of the given equation are $1, 2, \frac{1}{2}, 2 \pm \sqrt{3}.$

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$$5. \quad 12x^5 + 23x^4 - 135x^3 - 135x^2 + 23x + 12 = 0.$$

By grouping the terms with like coefficients it is seen that the first member is divisible by $x + 1$. Depressing to the standard form,

$$\begin{array}{r} 12 + 23 - 135 - 135 + 23 + 12 \quad | -1 \\ -12 - 11 + 146 - 11 - 12 \end{array}$$

$$12 + 11 - 146 + 11 + 12$$

Grouping terms with like coefficients and dividing by x^2 ,

$$12\left(x^2 + \frac{1}{x^2}\right) + 11\left(x + \frac{1}{x}\right) - 146 = 0.$$

Substituting y for $x + \frac{1}{x}$ and $y^2 - 2$ for $x^2 + \frac{1}{x^2}$,

$$12y^2 - 24 + 11y - 146 = 0, \text{ or } 12y^2 + 11y - 170 = 0.$$

Factoring,

$$(3y - 10)(4y + 17) = 0.$$

$$\therefore y = \frac{10}{3} \text{ or } -\frac{17}{4},$$

that is,

$$x + \frac{1}{x} = \frac{10}{3} \text{ or } -\frac{17}{4}.$$

Solving,

$$x = 3, \frac{1}{3}, -4, -\frac{1}{4}.$$

Hence, the roots of the given equation are $-1, 3, \frac{1}{3}, -4, -\frac{1}{4}$.

6.

$$x^6 - 2x^5 - 7x^4 + 7x^2 + 2x - 1 = 0.$$

By grouping terms with like coefficients it is seen that both $x - 1$ and $x + 1$ are factors of the first member. Depressing to the standard form,

$$\begin{array}{r} 1 - 2 - 7 + 0 + 7 + 2 - 1 \quad | 1 \\ 1 - 1 - 8 - 8 - 1 + 1 \end{array}$$

$$\begin{array}{r} 1 - 1 - 8 - 8 - 1 + 1 \quad | -1 \\ -1 + 2 + 6 + 2 - 1 \end{array}$$

$$1 - 2 - 6 - 2 + 1$$

Grouping terms with like coefficients and dividing by x^2 ,

$$x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) - 6 = 0.$$

Substituting y for $x + \frac{1}{x}$ and $y^2 - 2$ for $x^2 + \frac{1}{x^2}$,

$$y^2 - 2 - 2y - 6 = 0, \text{ or } y^2 - 2y - 8 = 0.$$

Factoring,

$$(y + 2)(y - 4) = 0.$$

$$\therefore y = -2 \text{ or } 4;$$

that is,

$$x + \frac{1}{x} = -2 \text{ or } 4.$$

Solving,

$$x = -1, -1, 2 + \sqrt{3}, 2 - \sqrt{3}.$$

Hence, the roots of the given equation are $1, -1, -1, -1, 2 \pm \sqrt{3}$.

7.

$$x^6 + x^5 - 14x^4 + 17x^3 - 14x^2 + x + 1 = 0.$$

The equation is a standard reciprocal equation. Hence, grouping terms equally distant from the middle term, and dividing by x^3 ,

$$\left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) - 14\left(x + \frac{1}{x}\right) + 17 = 0.$$

Substituting y for $x + \frac{1}{x}$, $y^2 - 2$ for $x^2 + \frac{1}{x^2}$, and $y^3 - 3y$ for $x^3 + \frac{1}{x^3}$,

$$(y^3 - 3y) + (y^2 - 2) - 14y + 17 = 0,$$

or

$$y^3 + y^2 - 17y + 15 = 0.$$

Solving by trial,

$$y = 1, 3, -5;$$

that is,

$$x + \frac{1}{x} = 1, 3, -5.$$

Solving,

$$x = \frac{1}{2}(1 \pm \sqrt{-3}), \frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-5 \pm \sqrt{21}).$$

$$8. \quad x^8 + x^7 + 5x^6 + 4x^5 + 9x^4 + 4x^3 + 5x^2 + x + 1 = 0.$$

The equation is a standard reciprocal equation. Hence, grouping terms equally distant from the middle term, and dividing by x^4 ,

$$\left(x^4 + \frac{1}{x^4}\right) + \left(x^3 + \frac{1}{x^3}\right) + 5\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) + 9 = 0.$$

Substituting y for $x + \frac{1}{x}$, $y^2 - 2$ for $x^2 + \frac{1}{x^2}$, etc.,

$$(y^4 - 4y^2 + 2) + (y^3 - 3y) + 5(y^2 - 2) + 4y + 9 = 0,$$

or $y^4 + y^3 + y^2 + y + 1 = 0.$

This equation also is a standard reciprocal equation. Hence, grouping terms equally distant from the middle term, and dividing by y^2 ,

$$\left(y^2 + \frac{1}{y^2}\right) + \left(y + \frac{1}{y}\right) + 1 = 0.$$

Substituting z for $y + \frac{1}{y}$ and $z^2 - 2$ for $y^2 + \frac{1}{y^2}$,

$$(z^2 - 2) + z + 1 = 0, \text{ or } z^2 + z - 1 = 0,$$

in which $z = y + \frac{1}{y} = \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = \frac{x^4 + 3x^2 + 1}{x^3 + x}.$

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$$2. \quad x^3 + 24x + 56 = x^3 + ax + b = 0.$$

Hence, $-\frac{b}{2} = -28$, $\frac{b^2}{4} = 784$, and $\frac{a^3}{27} = \left(\frac{a}{3}\right)^3 = 8^3 = 512.$

$$\therefore h = \sqrt[3]{-28 + \sqrt{512 + 784}} = \sqrt[3]{-28 + 36} = 2,$$

and $k = \sqrt[3]{-28 - \sqrt{512 + 784}} = \sqrt[3]{-28 - 36} = -4.$

Hence, the roots are

$$h + k = 2 - 4 = -2,$$

$$h\omega + k\omega^2 = -1 + \sqrt{-3} + 2 + 2\sqrt{-3} = 1 + 3\sqrt{-3},$$

and $h\omega^2 + k\omega = -1 - \sqrt{-3} + 2 - 2\sqrt{-3} = 1 - 3\sqrt{-3}.$

$$3. \quad 8x^3 - 20x^2 + 14x - 3 = 0.$$

Dividing by 8 and multiplying the roots by 2×3 , or 6,

$$y^3 - 15y^2 + 63y - 81 = 0,$$

in which $y = 6x$. Decreasing the roots by 5,

$$\begin{array}{rrrr} 1 & -15 & +63 & -81 \quad | \quad 5 \\ & 5 & -50 & +65 \\ \hline & -10 & +13 & -16 \\ & 5 & -25 & \\ \hline & -5 & -12 & \\ & 5 & & \\ \hline & 0 & & \end{array}$$

gives $z^3 - 12z - 16 = 0$, in which $z = 6x - 5$.

This equation has the form $z^3 + az + b = 0$.

Hence, $-\frac{b}{2} = 8$, $\frac{b^2}{4} = 64$, and $\frac{a^3}{27} = \left(\frac{a}{3}\right)^3 = (-4)^3 = -64.$

$$\therefore h = \sqrt[3]{8 + \sqrt{-64 + 64}} = 2,$$

and $k = \sqrt[3]{8 - \sqrt{-64 + 64}} = 2.$

Hence, the roots of $z^3 - 12z - 16 = 0$ are

$$h + k = 2 + 2 = 4,$$

$$h\omega + k\omega^2 = 2(\omega + \omega^2) = -2,$$

and

$$h\omega^2 + k\omega = 2(\omega^2 + \omega) = -2.$$

Since $z = 6x - 5$,

$$x = \frac{1}{6}(z + 5).$$

Hence, the roots of the given equation are $\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$.

4. $27x^3 - 117x^2 + 105x + 49 = 0.$

Dividing by 27, and multiplying the roots by 9,

$$y^3 - 39y^2 + 315y + 1323 = 0,$$

in which $y = 9x$. Decreasing the roots by 13,

| | | | | |
|---|------|-------|--------|-----------|
| 1 | - 39 | + 315 | + 1323 | <u>13</u> |
| | 13 | - 338 | - 299 | |
| | - 26 | - 23 | + 1024 | |
| | 13 | - 169 | | |
| | - 13 | - 192 | | |
| | 13 | | | |
| | 0 | | | |

gives $z^3 - 192z + 1024 = 0$, in which $z = 9x - 13$.

This equation has the form $z^3 + az + b = 0$.

Hence, $-\frac{b}{2} = -512$, $\frac{b^2}{4} = 262144$, and $\frac{a^3}{27} = \left(\frac{a}{3}\right)^3 = (-64)^3 = -262144$.

$$\therefore h = \sqrt[3]{-512} + \sqrt{-262144 + 262144} = \sqrt[3]{-512} = -8,$$

and

$$k = \sqrt[3]{-512} - \sqrt{-262144 + 262144} = \sqrt[3]{-512} = -8.$$

Hence, the roots of $z^3 - 192z + 1024 = 0$ are

$$h + k = -8 - 8 = -16,$$

$$h\omega + k\omega^2 = -8(\omega + \omega^2) = 8,$$

and

$$h\omega^2 + k\omega = -8(\omega^2 + \omega) = 8.$$

Since $z = 9x - 13$,

$$x = \frac{1}{9}(z + 13).$$

Hence, the roots of the given equation are $-\frac{1}{3}, \frac{7}{3}, \frac{7}{3}$.

5. $7x^3 + 15x^2 + 12x + 4 = 0.$

Dividing by 7, and multiplying the roots by 7,

$$y^3 + 15y^2 + 84y + 196 = 0,$$

in which $y = 7x$. Decreasing the roots by -5,

| | | | | |
|---|------|------|-------|------------|
| 1 | + 15 | + 84 | + 196 | <u>- 5</u> |
| | - 5 | - 50 | - 170 | |
| | 10 | + 34 | + 26 | |
| | - 5 | - 25 | | |
| | 5 | + 9 | | |
| | - 5 | | | |
| | 0 | | | |

gives $z^3 + 9z + 26 = 0$, in which $z = 7x - (-5) = 7x + 5$.

This equation has the form $z^3 + az + b = 0$.

Hence, $-\frac{b}{2} = -13$, $\frac{b^2}{4} = 169$, and $\frac{a^3}{27} = \left(\frac{a}{3}\right)^3 = 3^3 = 27$.

$$\therefore h = \sqrt[3]{-13} + \sqrt{27 + 169} = \sqrt[3]{-13} + 14 = 1,$$

and

$$k = \sqrt[3]{-13} - \sqrt{27 + 169} = \sqrt[3]{-13} - 14 = -3.$$

Hence, the roots of $z^3 + 9z + 26 = 0$ are

$$h + k = 1 - 3 = -2,$$

$$h\omega + k\omega^2 = \frac{1}{2}(-1 + \sqrt{-3}) - \frac{3}{2}(-1 - \sqrt{-3}) = 1 + 2\sqrt{-3},$$

and $h\omega^2 + k\omega = \frac{1}{2}(-1 - \sqrt{-3}) - \frac{3}{2}(-1 + \sqrt{-3}) = 1 - 2\sqrt{-3}.$

Since $z = 7x + 5,$ $x = \frac{1}{7}(z - 5).$

Hence, the roots of the given equation are $-1, \frac{2}{7}(-2 \pm \sqrt{-3}).$

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2. $x^4 - 2x^2 - 8x - 3 = x^4 + qx^2 + rx + s = 0.$

$\therefore q = -2, r = -8, s = -3,$ and (4) becomes

$$A^6 - 4A^4 + 16A^2 - 64 = 0.$$

Factoring, $A^4(A^2 - 4) + 16(A^2 - 4) = (A^4 + 16)(A + 2)(A - 2) = 0.$

Hence, $A = 2$ is one root of this equation.

Substituting 2 for $A,$ -2 for $q,$ -8 for $r,$ and -3 for s in (3),

$$B + C = 2, 2C - 2B = -8, \text{ and } BC = -3.$$

Solving, $B = 3,$ and $C = -1.$

$$\therefore x^4 - 2x^2 - 8x - 3 = (x^2 + 2x + 3)(x^2 - 2x - 1) = 0.$$

Equating the quadratic factors to zero and solving,

$$x = -1 \pm \sqrt{-2}, 1 \pm \sqrt{2}.$$

3. $x^4 - 4x^2 - 8x + 35 = x^4 + qx^2 + rx + s = 0.$

$\therefore q = -4, r = -8, s = 35,$ and (4) becomes

$$A^6 - 8A^4 - 124A^2 - 64 = 0.$$

Substitute A_1 for $A^2.$

Then, $A_1^3 - 8A_1^2 - 124A_1 - 64 = 0.$

By trial, $A_1 = 16$ is found to be one root of this equation.

$$\therefore A^2 = 16, \text{ whence } A = 4.$$

Substituting 4 for $A,$ -4 for $q,$ -8 for $r,$ and 35 for s in (3),

$$B + C = 12, 4C - 4B = -8, \text{ and } BC = 35.$$

Solving, $B = 7$ and $C = 5.$

$$\therefore x^4 - 4x^2 - 8x + 35 = (x^2 + 4x + 7)(x^2 - 4x + 5) = 0.$$

Equating the quadratic factors to zero and solving,

$$x = -2 \pm \sqrt{-3}, 2 \pm \sqrt{-1}.$$

4. $x^4 - 12x^3 + 55x^2 - 102x + 124 = 0.$

Decreasing the roots by $\frac{1}{4},$ or by 3,

$$y^4 + y^2 + 12y + 70 = 0, \text{ in which } y = x - 3.$$

Since this equation is in the form $y^4 + qy^2 + ry + s = 0,$

$$q = 1, r = 12, \text{ and } s = 70, \text{ and (4) becomes}$$

$$A^6 + 2A^4 - 279A^2 - 144 = 0.$$

Substitute A_1 for $A^2.$

Then, $A_1^3 + 2A_1^2 - 279A_1 - 144 = 0.$

By trial, $A_1 = 16$ is found to be a root of this equation.

$$\therefore A^2 = 16, \text{ whence } A = 4.$$

Substituting 4 for $A,$ 1 for $q,$ 12 for $r,$ and 70 for s in (3),

$$B + C = 17, 4C - 4B = 12, \text{ and } BC = 70.$$

Solving, $B = 7$ and $C = 10.$

$$\therefore y^4 + y^2 + 12y + 70 = (y^2 + 4y + 7)(y^2 - 4y + 10) = 0.$$

Equating the quadratic factors to zero and solving,

$$y = -2 \pm \sqrt{-3}, 2 \pm \sqrt{-6}.$$

Since $y = x - 3, x = y + 3.$

$$\therefore x = 1 \pm \sqrt{-3}, 5 \pm \sqrt{-6}.$$

5. $x^4 - 6x^3 + 11x^2 - 10x + 2 = 0.$

Multiplying the roots by 2,

$$y^4 - 12y^3 + 44y^2 - 80y + 32 = 0, \text{ in which } y = 2x.$$

Decreasing the roots of the last equation by $\frac{1}{2}$, or by 3,

$$z^4 - 10z^3 - 32z^2 - 55z = 0, \text{ in which } z = 2x - 3.$$

Since this equation is in the form $z^4 + qz^2 + rz + s = 0$,

$$q = -10, r = -32, \text{ and } s = -55, \text{ and (4) becomes}$$

$$A^6 - 20A^4 + 320A^2 - 1024 = 0.$$

Substitute A_1 for A^2 .

$$\text{Then, } A_1^3 - 20A_1^2 + 320A_1 - 1024 = 0.$$

By trial, $A_1 = 4$ is found to be a root of this equation.

$$\therefore A^2 = 4, \text{ whence } A = 2.$$

Substituting 2 for A , -10 for q , -32 for r , and -55 for s in (3),

$$B + C = -6, 2C - 2B = -32, BC = -55.$$

Solving,

$$B = 5 \text{ and } C = -11.$$

$$\therefore z^4 - 10z^3 - 32z^2 - 55z = (z^2 + 2z + 5)(z^2 - 2z - 11) = 0.$$

Equating the quadratic factors to zero and solving,

$$z = -1 \pm 2\sqrt{-1}, 1 \pm 2\sqrt{3}.$$

$$\text{Since } z = 2x - 3, \quad x = \frac{1}{2}(z + 3).$$

$$\therefore x = 1 \pm \sqrt{-1}, 2 \pm \sqrt{3}.$$

6. $x^4 - 2x^3 - 15x^2 + 16x + 14 = 0.$

Multiplying the roots by 2,

$$y^4 - 4y^3 - 60y^2 + 128y + 224 = 0, \text{ in which } y = 2x.$$

Decreasing the roots of the last equation by 1,

$$z^4 - 66z^2 + 289 = 0, \text{ in which } z = 2x - 1.$$

If

$$z^4 - 66z^2 + 289 = z^4 + qz^2 + rz + s,$$

$$q = -66, r = 0, \text{ and } s = 289,$$

and (4) becomes

$$A^6 - 132A^4 + 3200A^2 = 0.$$

Solving for A^2 ,

$$A^2 = 0, 100, 32.$$

It is most convenient to use the second root, giving $A = 10$.

Substituting 10 for A , -66 for q , 0 for r , and 289 for s in (3),

$$B + C = 34, 10C - 10B = 0, BC = 289.$$

Solving,

$$B = 17 \text{ and } C = 17.$$

$$\therefore z^4 - 66z^2 + 289 = (z^2 + 10z + 17)(z^2 - 10z + 17) = 0.$$

Equating the quadratic factors to zero and solving,

$$z = -5 \pm 2\sqrt{2}, 5 \pm 2\sqrt{2}.$$

$$\text{Since } z = 2x - 1, \quad x = \frac{1}{2}(z + 1).$$

$$\therefore x = -2 \pm \sqrt{2}, 3 \pm \sqrt{2}.$$

NOTE. — The solution may be abbreviated, if desired, by solving the transformed equation,

$$z^4 - 66z^2 + 289 = 0,$$

by *quadratic* methods instead of by Descartes' method.

Solving for z^2 ,

$$z^2 = 33 \pm \sqrt{800}.$$

$$\therefore z = \pm(5 \pm 2\sqrt{2}).$$

$$\therefore x = \frac{1}{2}(z + 1)$$

$$= 3 \pm \sqrt{2}, -2 \pm \sqrt{2}.$$

